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Calculus of Variations II

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Solution Localization for Nonlinear Problems: A Variational Approach

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Lecture-series mainly based on the following **References:**

1. M. Schechter, *Linking Methods in Critical Point Theory*, Birkhäuser, 1999.

2. M. Struwe, *Variational Methods*, Springer, 1990.

3. R. Precup, *Methods in Nonlinear Integral Equations*, Springer, 2002.

4. R. Precup, The Leray-Schauder boundary condition in critical point theory, *Nonlinear Anal.* 71 (2009), 3218-3228.

Goal of the lecture-series:

Consider an operator (of fixed point type) equation

$$Tu = u$$

in a Banach space X. One says that the equation has a variational form if it is equivalent (has the same solutions) to an equation of the form

$$E'(u) = 0,$$

where $E : X \to \mathbf{R}$ is a C^1-functional and $E' : X \to X^*$ is its Fréchet derivative. Hence the solutions of the operator equation are the critical points of the functional E. The task is to *find critical points* of extremum and saddle points and to *localize* such points in bounded sets, for instance in *balls*.

The basic results are the abstract *Fermat's Theorem* (for extremum points), *Ekeland's Variational Principle* and the *Ambrossetti-Rabinowitz Mountain-Pass Theorem* (for saddle points) and their localization versions due to Schechter. We shall explain the role for the localization of critical points of the *Leray-Schauder boundary condition* from the fixed point theory. Applications are given to *elliptic problems*.

1 The Fréchet Derivative and Fermat's Theorem

Let $(X, |.|)$ be a real Banach space, $U \subset X$ an open set, $E : U \to \mathbf{R}$ a functional and $u \in U$ a given point.

Definition 1.1 *(a) The derivative of E in direction $v \in X$ at u is defined as*

$$\lim_{t \to 0+} t^{-1} (E(u + tv) - E(u))$$

if this limit exists.

(b) E is said to be Gâteaux differentiable at u if there exists an $E'(u) \in X^*$ such that

$$(E'(u), v) = \lim_{t \to 0+} t^{-1} (E(u + tv) - E(u))$$

for all $v \in X$. The element $E'(u)$ is called the Gâteaux derivative of E at u.

(c) E is said to be Fréchet differentiable at u if there exists an $E'(u) \in X^*$ such that

$$E(u + v) - E(u) = (E'(u), v) + \omega(u, v)$$

and

$$\omega(u, v) = o(|v|), \ i.e., \ \frac{\omega(u, v)}{|v|} \to 0 \ as \ v \to 0.$$

The element $E'(u)$ is called the Fréchet derivative of E at u.

Proposition 1.2 *(a) (Exercise) If E is Fréchet differentiable at u, then E is Gâteaux differentiable at u, the two derivatives coincide and E is continuous at u.*

(b) If E is Gâteaux differentiable in an open neighborhood V of u and $E' : V \to X^*$ is continuous at u, then E is Fréchet differentiable at u.

Proof. (b) Let $r \in (0,1]$ be such that $B_r(u) \subset V$. Since $E' : V \to X^*$ is continuous at u, for every $\varepsilon > 0$ there exists a $\delta \in (0,r]$ with

$$|E'(u+tv) - E'(u)| < \varepsilon \quad \text{for} \quad |v| < \delta, \ |t| \leq 1.$$

On the other hand, for each $v \in B_r(0)$ the function

$$t \in [0,1] \mapsto (E'(u+tv), v)$$

is the derivative of the function

$$g(t) = E(u+tv) \qquad (t \in [0,1]).$$

Consequently

$$\int_0^1 (E'(u+tv), v)\, dt = g(1) - g(0) = E(u+v) - E(u).$$

Hence

$$E(u+v) - E(u) - (E'(u), v) = \int_0^1 (E'(u+tv) - E'(u), v)\, dt. \quad (1)$$

Let

$$\omega(u,v) = \int_0^1 (E'(u+tv) - E'(u), v)\, dt.$$

For all $v \in B_\delta(0)$, one has

$$|\omega(u,v)| \leq \int_0^1 |E'(u+tv) - E'(u)|\, |v|\, dt < \varepsilon |v|.$$

Hence $\omega(u,v) = o(|v|)$ as $v \to 0$. Finally (1) shows that $E'(u)$ is the Fréchet derivative of E at u. ■

We say that $E \in C^1(U)$ if E is Fréchet differentiable in U and its Fréchet derivative $E' : U \to X^*$ is continuous, equivalently, if E is Gâteaux differentiable in U and its Gâteaux differentiable derivative $E' : U \to X^*$ is continuous. We say that $E \in C^1(\overline{U})$ if $E \in C^1(U)$ and E, E' can be extended continuously to $\overline{U}$.

Example 1.3 *(Exercise) Let $X = \mathbf{R}^N$, $\Omega \subset \mathbf{R}^N$ open, $E : \Omega \to \mathbf{R}$. If E is differentiable in a neighborhood of a point $x \in \Omega$ in the classical sense and its partial derivatives $\partial E / \partial x_j$, $j = 1, 2, ..., N$ are continuous at x, then E is Fréchet differentiable at x and its Fréchet derivative coincides with its gradient, i.e., $E'(x) = \nabla E(x)$.*

Example 1.4 *(Exercise) Let X be a Hilbert space and*

$$E(u) = \frac{1}{2}|u|^2, \quad u \in X.$$

Then $E \in C^1(X)$ and

$$(E'(u), v) = (u, v), \quad v \in X.$$

Example 1.5 *Let $\Omega \subset \mathbf{R}^N$ be a bounded open set, $p \in (1, \infty)$, q the conjugate exponent of p, and $F : \Omega \times \mathbf{R}^n \to \mathbf{R}$ a function with $F(x, 0) = 0$ on Ω. Assume that $F(., z)$ is measurable for all $z \in \mathbf{R}^n$ and $F(x, .)$ is continuously differentiable for a.e. $x \in \Omega$. Let $f : \Omega \times \mathbf{R}^n \to \mathbf{R}^n$ be defined as*

$$f(x, .) = \nabla F(x, .) \quad (f \text{ is of potential type})$$

and assume that f is (p, q)-Carathéodory. Then the functional $E : L^p(\Omega, \mathbf{R}^n) \to \mathbf{R}$ given by

$$E(u) = \int_\Omega F(x, u(x))\, dx$$

belongs to $C^1(L^p(\Omega, \mathbf{R}^n))$ and

$$E' = N_f, \quad i.e.,$$

$$(E'(u), v) = \int_\Omega (f(x, u(x)),\ v(x))\, dx, \quad v \in L^p(\Omega, \mathbf{R}^n).$$

Proof. For $u, v \in L^p(\Omega, \mathbf{R}^n)$ define the measurable function $\theta_{u,v} : \Omega \to [0, 1]$, by

$$\begin{aligned} \theta_{u,v}(x) &= \inf\{t \in [0,1] : F(x, v(x) + u(x)) - F(x, v(x)) = \\ &\quad (u(x), f(x, v(x) + tu(x)))\}. \end{aligned}$$

Since $F(x, 0) = 0$,

$$\begin{aligned} |E(u)| &\le \int_\Omega |F(x, u(x))|\, dx \le \int_\Omega |u(x)|\, |f(x, \theta_{u,0}(x)\, u(x))|\, dx \\ &\le |u|_{L^p}\, |N_f(\theta_{u,0} u)|_{L^q} < \infty. \end{aligned}$$

Thus E is well defined. On the other hand

$$\begin{aligned}
E(u+v)-E(u) &= \int_\Omega (v(x), N_f(u+\theta_{u,v}v))\,dx = \int_\Omega (v(x), N_f(u)(x))\,dx \\
&\quad + \int_\Omega (v(x), N_f(u+\theta_{u,v}v)(x) - N_f(u)(x))\,dx \\
&= (N_f(u), v) + \omega(u,v),
\end{aligned}$$

where

$$\omega(u,v) = \int_\Omega (v(x), N_f(u+\theta_{u,v}v)(x) - N_f(u)(x))\,dx.$$

One has

$$|\omega(u,v)| \le |v|_{L^p} |N_f(u+\theta_{u,v}v) - N_f(u)|_{L^q},$$

whence, using the continuity of the superposition operator,

$$\frac{\omega(u,v)}{|v|_{L^p}} \to 0 \quad \text{as } v \to 0.$$

Thus $E'(u) = N_f(u)$. ■

Example 1.6 *(Exercise) If f, F are as in Example 5 with $p \in (1, 2^*]$ and $E : H_0^1(\Omega, \mathbf{R}^n) \to \mathbf{R}$,*

$$E(u) = \frac{1}{2}|u|^2_{H_0^1} - \int_\Omega F(x, u(x))\,dx,$$

then

$$E'(u) = u - (-\Delta)^{-1} N_f(u).$$

Hence the weak solutions of the Dirichlet problem $-\Delta u = f(x,u)$ in Ω; $u = 0$ on $\partial\Omega$ are the critical points of the above functional.

Theorem 1.7 (Fermat) *Let $E : D \subset X \to \mathbf{R}$ be any functional. If $u_0 \in$ int D is a point of local extremum of E and E is Fréchet differentiable at u_0, then $E'(u_0) = 0$.*

Proof. Assume that u_0 is a point of local minimum, i.e., $E(u_0) \le E(u)$ for all u in a neighborhood of u_0. Since E is differentiable at u_0, one has

$$0 \le E(u_0 + v) - E(u_0) = (E'(u_0), v) + \omega(u_0, v)$$

and

$$\frac{\omega(u_0, v)}{|v|} \to 0 \quad \text{as} \quad v \to 0.$$

Setting $v = tw$, where $t > 0$ and $w \in X$, $|w| = 1$, dividing by t and then letting $t \to 0^+$ we obtain

$$(E'(u_0), w) \geq 0.$$

Similarly, replacing w by $-w$, we obtain

$$(E'(u_0), w) \leq 0$$

for all $w \in X$, $|w| = 1$. As a result $E'(u_0) = 0$. ■

Proposition 1.8 *Let X be a reflexive Banach space, $D \subset X$ a closed convex set and $E : D \to \mathbf{R}$ a weakly l.s.c. functional on D. If either D is bounded or E is coercive, then E is bounded from below and attains its infimum.*

Corollary 1.9 *If X is a reflexive Banach space and $E : X \to \mathbf{R}$ is coercive, weakly l.s.c. on X and Fréchet differentiable in X, then there exists $u_0 \in X$ with*

$$E(u_0) = \inf_X E, \qquad E'(u_0) = 0.$$

Exercise 1.10 *(a) If the function F from Example 6 is concave in its second variable, then E is strictly convex.*

(b) Give a sufficient condition on the growth of F such that E is coercive.

Hint. (b) $|F(x, z)| \leq a|z|^2 + b(x)$ for sufficiently small $a > 0$. Use Poincaré's inequality.

2 Ekeland's Variational Principle

Theorem 2.1 (Ekeland) *Let (X, d) be a complete metric space and let $E : X \to \mathbf{R}$ be a lower semicontinuous function bounded from below. Then given $\varepsilon > 0$ and $u_0 \in X$, there exists a point $u \in X$ such that*

$$E(v) - E(u) + \varepsilon d(u, v) \geq 0 \quad \text{for all} \;\; v \in X, \tag{2}$$

$$E(u) \leq E(u_0) - \varepsilon d(u, u_0). \tag{3}$$

Corollary 2.2 *Under the assumptions of Theorem 2.1, for each $\varepsilon > 0$, there exists an element $u \in X$ such that (2) holds and*

$$E(u) \leq \inf_X E + \varepsilon.$$

Proof. Apply Theorem 2.1 to an element $u_0 \in X$ with $E(u_0) \leq \inf_X E + \varepsilon$. ■

Corollary 2.3 *Under the assumptions of Theorem 2.1, if X is a Banach space with norm $|.|$, and E is a C^1 functional, there exists a sequence (u_k) with*

$$E(u_k) \to \inf_X E \quad \text{and} \quad E'(u_k) \to 0. \tag{4}$$

Proof. For $\varepsilon = \frac{1}{k}$, by Corollary 2.2 there is an element u_k such that

$$\begin{aligned} E(v) - E(u_k) + \frac{1}{k}|u_k - v| &\geq 0, \quad v \in X \\ E(u_k) &\leq \inf_X E + \frac{1}{k}. \end{aligned} \tag{5}$$

Take any $w \in X$ and let $v = u_k + tw$, $t \in \mathbf{R}$. It follows that

$$E(u_k + tw) - E(u_k) + \frac{1}{k}|t||w| \geq 0$$

Then for $|t|$ small enough,

$$t(E'(u_k), w) + o(|t|) + \frac{1}{k}|t||w| \geq 0.$$

For $t > 0, t \to 0^+$, we deduce

$$(E'(u_k), w) \geq -\frac{1}{k}|w|,$$

while for $t < 0, t \to 0^-$, we obtain

$$(E'(u_k), w) \leq \frac{1}{k}|w|.$$

Hence $|(E'(u_k), w)| \leq \frac{1}{k}|w|$. Therefore $|E'(u_k)| \leq \frac{1}{k}$. ■

Definition 2.4 *We say that functional E satisfies the Palais-Smale (compactness) condition if any sequence satisfying (4) has a convergent subsequence.*

Corollary 2.5 *Under the assumptions of Corollary 2.3, if in addition E satisfies the Palais-Smale condition, then there is a point $u \in X$ with*

$$E(u) = \inf_X E \quad \text{and} \quad E'(u) = 0.$$

Theorem 2.6 (Schechter's first theorem) *If X is a Hilbert space with inner product $(.,.)$ and norm $|.|$, $R > 0$ and $E : \overline{B}_R \to \mathbf{R}$ is a C^1 functional bounded from bellow with $(E'(u), u) \geq a > -\infty$ for every $u \in X, |u| = R$, then there exists a sequence (u_k) such that either*

$$|u_k| < R, \quad E(u_k) \to \inf_X E \quad \text{and} \quad E'(u_k) \to 0, \tag{6}$$

or

$$\begin{aligned} |u_k| &= R, \quad (E'(u_k), u_k) \to b \leq 0, \qquad (7)\\ E(u_k) &\to \inf_X E \quad \text{and} \quad E'(u_k) - \frac{(E'(u_k), u_k)}{R^2} u_k \to 0. \end{aligned}$$

Proof. As in the proof of the previous Corollary, there exists a sequence (u_k) satisfying (5). If for a given k, $|u_k| < R$, then the reasoning which follows in the proof of Corollary 2.3 remains true. Hence if at least for a subsequence $|u_k| < R$, we are done. Assume that $|u_k| = R$ for all k (except eventually a finite number).

(a) We first prove that (passing eventually to a subsequence) $(E'(u_k), u_k) \to b \leq 0$. To this end take $v = (1-t) u_k$ with $0 < t < 1$ and apply (5). We obtain

$$-t(E'(u_k), u_k) + \omega(u_k, -tu_k) + \frac{t}{k}|u_k| \geq 0,$$

equivalently

$$-(E'(u_k), u_k) + \frac{\omega(u_k, -tu_k)}{t} + \frac{1}{k}R \geq 0.$$

Letting $t \to 0^+$ we derive

$$(E'(u_k), u_k) \leq \frac{1}{k}R,$$

whence the conclusion (recall the assumption $(E'(u), u) \geq a > -\infty$ for every $u \in X, |u| = R$).

(b) Next we prove that

$$(E'(u_k), w) \geq -\frac{1}{k} \text{ for every } w \in X \text{ with } |w| = 1 \text{ and } (u_k, w) \leq 0. \quad (8)$$

Apply (5) with $v = u_k + tw$, where $w \in X$, $|w| = 1$, $(u_k, w) \leq 0$ and $t > 0$. We need $|v| \leq R$, i.e., $|u_k + tw|^2 \leq R^2$. This gives $R^2 + 2t(u_k, w) + t^2 \leq R^2$, that is $2t(u_k, w) + t^2 \leq 0$. This is true for $t \in (0, -2(u_k, w)]$. Now (5) implies

$$t(E'(u_k), w) + \omega(u_k, tw) + \frac{t}{k} \geq 0,$$

or equivalently

$$(E'(u_k), w) + \frac{\omega(u_k, tw)}{t} + \frac{1}{k} \geq 0.$$

Letting $t \to 0^+$ we obtain the desired conclusion.

(c) Finally we prove that $\overline{w}_k := E'(u_k) - \frac{(E'(u_k), u_k)}{R^2} u_k \to 0$. We apply (8) with $w = -\frac{\overline{w}_k}{|\overline{w}_k|}$. It is easy to check that $(u_k, \overline{w}_k) = 0$. Hence $(E'(u_k), w) \geq -\frac{1}{k}$. This gives

$$|\overline{w}_k|^2 = |E'(u_k)|^2 - \frac{(E'(u_k), u_k)^2}{R^2} \leq \frac{1}{k} |\overline{w}_k|.$$

Thus $|\overline{w}_k| \leq \frac{1}{k}$ and so $\overline{w}_k \to 0$ as desired. ■

Definition 2.7 *We say that functional E satisfies the Palais-Smale-Schechter (compactness) condition in $\overline{B}_R$ if any sequence satisfying (6) or (7) has a convergent subsequence.*

Corollary 2.8 *Under the assumptions of Theorem 2.6, if in addition E satisfies the Palais-Smale-Schechter condition and the Leray-Schauder boundary condition*

$$E'(u) + \mu u \neq 0 \text{ for } |u| = R \text{ and } \mu > 0, \quad (9)$$

then there is a point $u \in \overline{B}_R$ with

$$E(u) = \inf_{\overline{B}_R} E \quad \text{and} \quad E'(u) = 0.$$

Proof of Ekeland's Theorem. We may assume that $\varepsilon = 1$. For $u \in X$ consider the set

$$X(u) = \{v \in X : E(v) - E(u) + d(u,v) \leq 0\}.$$

One has $u \in X(u)$ and if $v \in X(u)$, then $X(v) \subset X(u)$. Let $\varepsilon_k > 0$, $\varepsilon_k \to 0$ and let u_k be such that

$$u_{k+1} \in X(u_k), \quad E(u_{k+1}) \leq \inf_{X(u_k)} E + \varepsilon_{k+1}.$$

Since $u_{k+1} \in X(u_k)$ we have

$$E(u_{k+1}) - E(u_k) + d(u_{k+1}, u_k) \leq 0.$$

Hence the sequence $E(u_k)$ is decreasing. It is also bounded since E is bounded from below. Thus it converges. On the other hand, from the last inequality we deduce that

$$E(u_m) - E(u_k) + d(u_m, u_k) \leq 0 \quad \text{for } k < m.$$

It follows that the sequence (u_k) is Cauchy. Let u be its limit. From $u_k \in X(u_0)$ we obtain

$$E(u_k) - E(u_0) + d(u_k, u_0) \leq 0,$$

whence for $k \to \infty$ we derive (3). To prove (2) take any $v \in X$. Two cases are possible: (a) $v \in \cap X(u_k)$. Then

$$E(u_{k+1}) \leq \inf_{X(u_k)} E + \varepsilon_{k+1} \leq E(v) + \varepsilon_{k+1} \quad \text{for all } k.$$

Using the lower semicontinuity of E we deduce $E(u) \leq E(v)$ and so (2). (b) $v \notin \cap X(u_k)$. Then $v \notin X(u_k)$ for every $k \geq m$ and some m. Consequently

$$E(v) - E(u_k) + d(v, u_k) > 0, \quad k \geq m.$$

Letting $k \to \infty$ we obtain (2). ■

3 The Mountain Pass Theorem

Theorem 3.1 (Ambrosetti-Rabinowitz) *Let X be a Banach space and $E \in C^1(X)$. Assume that there exist $u_0, u_1 \in X$ and $r > 0$ with $|u_0| < r < |u_1|$ such that*

$$\max\{E(u_0), E(u_1)\} < \inf\{E(u) : u \in X, |u| = r\}.$$

Let

$$\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = u_0, \gamma(1) = u_1\}$$

and

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} E(\gamma(t)).$$

Then there exists a sequence of elements $u_k \in X$ with

$$E(u_k) \to c, \qquad E'(u_k) \to 0.$$

We note that (Exercise) if $v \in X^*$, then as a simple consequence of the definition of the norm of v, for each $\varepsilon > 0$ there exists $u \in X$ with

$$|u| \leq 1 \quad \text{and} \quad (v, u) > |v| - \varepsilon.$$

The next Lemma guarantees that if v depends continuously on a parameter t, then the corresponding element u can be chosen so that it depends continuously on t as well.

Lemma 3.2 *Let X be a Banach space and $f \in C([0,1], X^*)$. Then for each $\varepsilon > 0$ there exists a function $\varphi \in C([0,1], X)$ such that*

$$|\varphi(t)| \leq 1, \quad (f(t), \varphi(t)) > |f(t)| - \varepsilon$$

for all $t \in [0,1]$.

Proof. Let $t_0 \in [0,1]$. According to the above remark there is $u_0 \in X$ with $|u_0| \leq 1$ and $(f(t_0), u_0) > |f(t_0)| - \varepsilon$. Let

$$U(t_0) = \{t \in [0,1] : (f(t), u_0) > |f(t)| - \varepsilon\}.$$

Clearly $t_0 \in U(t_0)$ and $U(t_0)$ is open in $[0,1]$. Since $[0,1] = \cup_{t\in[0,1]} U(t)$, there is a finite open covering of $[0,1] : U(t_1), U(t_2), ..., U(t_n)$. Let u_i, $i = 1, 2, ..., n$ be the corresponding elements, i.e.,

$$U(t_i) = \{t \in [0,1] : (f(t), u_i) > |f(t)| - \varepsilon\}.$$

Let $\rho_i(t) = \text{dist}(t, [0,1] \setminus U(t_i))$ and $\zeta_i(t) = \rho_i(t) / \sum_{j=1}^n \rho_j(t)$. Notice that $\zeta_i : [0,1] \to [0,1]$ is continuous, $\zeta_i(t) \neq 0$ if and only if $t \in U(t_i)$ and $\sum_{j=1}^n \zeta_i(t) = 1$ for all t. Finally the desired function is

$$\varphi(t) = \sum_{j=1}^n \zeta_i(t) u_i.$$

■

Proof of Theorem 3.1. Apply Corollary 2.2 to the metric space Γ and to the functional $\psi : \Gamma \to \mathbf{R}$,

$$\psi(\gamma) = \max_{t\in[0,1]} E(\gamma(t)).$$

The functional is lower semicontinuous and bounded from below by c. It follows that for every natural number $k \geq 1$, there exists a $\gamma_k \in \Gamma$ with

$$\psi(\eta) - \psi(\gamma_k) + \frac{1}{k} d(\eta, \gamma_k) \geq 0, \quad \eta \in \Gamma, \tag{10}$$

$$c \leq \psi(\gamma_k) \leq \inf_{\Gamma} \psi + \frac{1}{k} = c + \frac{1}{k}. \tag{11}$$

Let

$$\Lambda_k = \{t \in [0,1] : E(\gamma_k(t)) = \psi(\gamma_k)\}.$$

For concluding the proof it is sufficient that there exists a $t_k \in \Lambda_k$ with $|E'(\gamma_k(t_k))| < \frac{2}{k}$. To this end we apply the above Lemma to the function

$$f(t) = E'(\gamma_k(t)).$$

Hence there exists a function $\varphi \in C([0,1], X)$ with $|\varphi(t)| \leq 1$ and

$$(E'(\gamma_k(t)), \varphi(t)) > |E'(\gamma_k(t))| - \frac{1}{k} \quad \text{on } [0,1].$$

In (10) take $\eta = \gamma_k - \lambda w$ with $\lambda > 0$ and

$$w(t) = \zeta(t)\varphi(t),$$

where $\zeta : [0,1] \to [0,1]$ is continuous, $\zeta(t) = 1$ on Λ_k and $\zeta(0) = \zeta(1) = 0$. We have $d(\eta, \gamma_k) = \lambda |w| \leq \lambda$,

$$\psi(\eta) = \max_{t\in[0,1]} E(\eta(t)) = E(\eta(t_\lambda))$$

for some $t_\lambda \in [0,1]$. Hence

$$E(\eta(t_\lambda)) - \max_{t\in[0,1]} E(\gamma_k(t)) + \frac{\lambda}{k} \geq 0.$$

Since

$$E(\eta(t_\lambda)) - E(\gamma_k(t_\lambda)) = -\lambda (E'(\gamma_k(t_\lambda)), w(t_\lambda)) + o(\lambda)$$

we deduce that

$$-(E'(\gamma_k(t_\lambda)), w(t_\lambda)) + \frac{1}{k} + \frac{1}{\lambda} o(\lambda) \geq 0.$$

We may assume that $t_\lambda \to t_k \in \Lambda_k$ as $\lambda \to 0$. Then

$$-(E'(\gamma_k(t_k)), w(t_k)) + \frac{1}{k} \geq 0.$$

Thus

$$|E'(\gamma_k(t_k))| - \frac{1}{k} < (E'(\gamma_k(t_k)), w(t_k)) \leq \frac{1}{k},$$

whence $|E'(\gamma_k(t_k))| < \frac{2}{k}$. ■

The analogue of Theorem 2.4 for critical points of mountain pass type is the following result that we state without proof.

Theorem 3.3 (Schechter's second theorem) *Let X be a Hilbert space, $R > 0$ and $E : \overline{B}_R \to \mathbf{R}$ is a C^1 functional with $(E'(u), u) \geq -a > -\infty$ for every $u \in X, |u| = R$. Assume that there exist $u_0, u_1 \in \overline{B}_R$ and $r > 0$ with $|u_0| < r < |u_1|$ such that*

$$\max\{E(u_0), E(u_1)\} < \inf\{E(u) : u \in \overline{B}_R,\ |u| = r\}. \tag{12}$$

Let

$$\Gamma_R = \{\gamma \in C([0,1], \overline{B}_R) : \gamma(0) = u_0, \gamma(1) = u_1\}$$

and

$$c_R = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} E(\gamma(t)).$$

Then there exists a sequence of elements $u_k \in \overline{B}_R$ with

$$E(u_k) \to c_R, \qquad E'(u_k) \to 0,$$

or

$$\begin{aligned} |u_k| &= R, \quad (E'(u_k), u_k) \to b \leq 0, \\ E(u_k) &\to c_R \quad \text{and} \quad E'(u_k) - \frac{(E'(u_k), u_k)}{R^2} u_k \to 0. \end{aligned}$$

Corollary 3.4 *(i) Under the assumptions of Theorem 3.1, if in addition E satisfies the Palais-Smale condition, then there is a point $u \in X \setminus \{u_0, u_1\}$ with*

$$E(u) = c \quad and \quad E'(u) = 0.$$

(ii) Under the assumptions of Theorem 3.3, if in addition E satisfies the Palais-Smale-Schechter condition and the Leray-Schauder boundary condition

$$E'(u) + \mu u \neq 0 \text{ for } |u| = R \text{ and } \mu > 0,$$

then there is a point $u \in \overline{B}_R$ with

$$E(u) = c_R \quad and \quad E'(u) = 0.$$

4 Applications to Elliptic Problems

Consider the elliptic problem

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \Omega. \end{cases} \tag{13}$$

Here $\Omega \subset \mathbf{R}^n$ $(n \geq 3)$ is a bounded open set and $f : \mathbf{R} \to \mathbf{R}_+$ is a continuous function.

The energy functional is $E : H_0^1(\Omega) \to \mathbf{R}$, $E(u) = \int_\Omega \left(\frac{1}{2}|\nabla u|^2 - F(u)\right) dx$ with $E'(u) = u - (-\Delta)^{-1} N_f(u)$.

Proposition 4.1 *If there are $a, b \in \mathbf{R}_+$ and $p \in [1, 2^*)$ with*

$$f(\tau) \leq a|\tau|^{p-1} + b \quad for\ all \ \ \tau \in \mathbf{R}, \tag{14}$$

then E satisfies the Palais-Smale-Schechter condition in any ball $\overline{B}_R$.

Proof. The operator N_f sends bounded sets from $L^p(\Omega)$ into bounded sets of $L^q(\Omega)$ $\left(\frac{1}{p} + \frac{1}{q} = 1\right)$. Also the embedding $H_0^1(\Omega) \subset L^p(\Omega)$ is compact since $p < 2^*$. It follows that the operator $(-\Delta)^{-1} N_f$ is compact from $H_0^1(\Omega)$ to itself. ∎

Proposition 4.2 *Under the same assumption, the functional E is bounded from below on any ball $\overline{B}_R$.*

Proof. One has $F(\tau) \le \frac{a}{p}|\tau|^p + b|\tau|$. Hence

$$E(u) \ge -\int_\Omega F(u)\,dx \ge -\int_\Omega \left(\frac{a}{p}|u|^p + b|u|\right)dx.$$

Let c_1, c_2 be the embedding constants for $H_0^1(\Omega) \subset L^p(\Omega)$ and $H_0^1(\Omega) \subset L^1(\Omega)$, i.e., $|u|_{L^p} \le c_1 |u|_{H_0^1}$ and $|u|_{L^1} \le c_1 |u|_{H_0^1}$ for all $u \in H_0^1(\Omega)$. Then

$$E(u) \ge -\frac{a}{p}c_1^p R^p - bc_2 R > -\infty.$$

■

Proposition 4.3 *Under the same assumption, if $p \in [1,2)$, then the Leray-Schauder boundary condition holds for any ball $\overline{B}_R$ of a sufficiently large radius R.*

Proof. Assume the contrary. Then $E'(u) + \mu u = 0$, or equivalently $(-\Delta)^{-1} N_f(u) = (1+\mu)u$, for some $|u|_{H_0^1} = R$ and $\mu > 0$. It follows that

$$\begin{aligned} R^2 &= |u|_{H_0^1}^2 = \frac{1}{1+\mu}(N_f(u), u) \le \frac{1}{1+\mu}|f(u)|_{H^{-1}}|u|_{H_0^1} \\ &\le \frac{cR}{1+\mu}|f(u)|_{L^q} \le \frac{cR}{1+\mu}\left(a|u|_{L^p}^{p-1} + \widetilde{b}\right) \le cR\left(ac_1^{p-1}R^{p-1} + \widetilde{b}\right), \end{aligned}$$

where $\widetilde{b} = b|1|_{L^q}$. This is a contradiction provided that $R > 0$ is chosen large enough that $c\left(ac_1^{p-1}R^{p-1} + \widetilde{b}\right) < R$. ■

Theorem 4.4 *If f satisfies condition (14) for some $p \in [1,2)$, then in any ball $\overline{B}_R$ of $H_0^1(\Omega)$ of a sufficiently large radius, problem (13) has a solution minimizing the energy functional in $\overline{B}_R$.*

Proof. Use Corollary 2.8. ■

Theorem 4.5 *Assume that f satisfies condition (14) for some $p \in [1,2)$ and that*

$$f(\tau) \ge c\tau^\alpha \quad \textit{for} \ \ 0 \le \tau \le \tau_0 \tag{15}$$

and some $c > 0$. In addition assume that

$$\limsup_{\tau \to 0^-} \frac{f(\tau)}{\tau} < \lambda_1 \tag{16}$$

and that for some $\alpha > 1$ one has

$$\frac{1}{2} - \frac{c}{\alpha+1}\tau_0^{\alpha-1}\int_{(\phi\leq 1)}\phi(x)^{\alpha+1}\,dx \leq 0. \tag{17}$$

Here λ_1 and ϕ are the first eigenvalue and the corresponding positive eigenfunction of the Dirichlet problem for $-\Delta$.

Then there exist r (small enough) and R (large enough) such that the mountain pass condition (12) holds. As a result, problem (13) has at least two solutions.

Proof. Fix any number $\beta \in (2, 2^*]$ and choose a d with

$$\frac{1}{2}\limsup_{\tau\to 0^-}\frac{f(\tau)}{\tau} < d < \frac{\lambda_1}{2}.$$

From (14) and (16) we find that there exists a constant $c_d > 0$ with

$$F(\tau) \leq d\tau^2 + c_d\tau^\beta \quad \text{for all } \ \tau \in \mathbf{R}_+.$$

Then, for every $u \in H_0^1(\Omega)$, we have

$$\begin{aligned}
E(u) &= \frac{1}{2}|u|^2_{H_0^1} - \int_\Omega F(u)\,dx \geq \frac{1}{2}|u|^2_{H_0^1} - \int_{(u\geq 0)}F(u)\,dx \\
&\geq \frac{1}{2}|u|^2_{H_0^1} - \int_{(u\geq 0)}\left(du^2 + c_d u^\beta\right)dx \geq \frac{1}{2}|u|^2_{H_0^1} - \int_\Omega\left(du^2 + c_d|u|^\beta\right)dx \\
&\geq \frac{1}{2}|u|^2_{H_0^1} - \frac{d}{\lambda_1}|u|^2_{H_0^1} - c_d c_3^\beta|u|^\beta_{H_0^1} = |u|^2_{H_0^1}\left(\frac{1}{2} - \frac{d}{\lambda_1} - c_d c_3^\beta|u|^{\beta-2}_{H_0^1}\right).
\end{aligned}$$

Here c_3 is the embedding constant for $H_0^1(\Omega) \subset L^\beta(\Omega)$. Since $\frac{1}{2} - \frac{d}{\lambda_1} > 0$ and $\beta > 2$, we can find a small enough $r \in (0, \tau_0)$ such that $E(u) \geq \gamma > 0$ for all $u \in H_0^1(\Omega)$ with $|u|_{H_0^1} = r$ and some $\gamma > 0$.

Let $u_0 = 0$ and $u_1 = \tau_0\phi$. Clearly $E(0) = 0$. From (15) we have

$$F(\tau) \geq \frac{c}{\alpha+1}\tau^{\alpha+1} \quad \text{for } \ 0 \leq \tau \leq \tau_0.$$

This together with (17) gives

$$E(\tau_0\phi) = \frac{\tau_0^2}{2} - \int_\Omega F(\tau_0\phi)\,dx \leq \frac{\tau_0^2}{2} - \frac{c}{\alpha+1}\tau_0^{\alpha+1}\int_{(\phi\leq 1)}\phi(x)^{\alpha+1}\,dx \leq 0.$$

Hence $\max\{E(u_0), E(u_1)\} \leq 0$ and thus (12) holds. Therefore Theorem 3.3 applies. ■

Example 4.6 *Let $f : \mathbf{R} \to \mathbf{R}_+$ be defined by*

$$f(\tau) = \begin{cases} f(-\tau) & \text{for } \tau < 0 \\ c\tau^2 & \text{for } 0 \le \tau \le \tau_0 \\ a\sqrt{\tau - \tau_0} + c\tau_0^2 & \text{for } \tau > \tau_0 \end{cases}$$

for some $a, c, \tau_0 > 0$. If $c\tau_0$ is sufficiently large, then all the above conditions are satisfied with $p = \frac{3}{2}$ and $\alpha = 2$.

5 Appendix: The Leray-Schauder Boundary Condition

Let us first recall the well known Schauder's Fixed Point Theorem.

Theorem 5.1 (Schauder) *Let X be a Banach space, $D \subset X$ a nonempty closed convex bounded set and $T : D \to D$ a compact operator (i.e., continuous, with $T(D)$ relatively compact). Then T has at least one fixed point in D.*

The main drawback in applying Schauder's fixed point theorem is the "invariance condition" $T(D) \subset D$. It can be overcome if instead a "boundary condition" is required as shown by the next result which is known as the Leray-Schauder Principle or Schaefer's Fixed Point Theorem:

Theorem 5.2 (Leray-Schauder) *Let $(X, |.|)$ be a Banach space and $T : \overline{B}_R \to X$ a compact operator. If*

$$Tu \neq \lambda u \quad \text{for all } u \in \partial B_R \text{ and } \lambda > 1, \tag{18}$$

then T has at least one fixed point in $\overline{B}_R$.

Proof. Since T is compact, there is $\widetilde{R} \ge R$ with $T\left(\overline{B}_R\right) \subset \overline{B}_{\widetilde{R}}$. Define $\widetilde{T} : \overline{B}_{\widetilde{R}} \to \overline{B}_{\widetilde{R}}$,

$$\widetilde{T}u = \begin{cases} Tu & \text{if } |u| \le R \\ T\left(\frac{R}{|u|}u\right) & \text{if } |u| > R. \end{cases}$$

Schauder's fixed point theorem applied to $\widetilde{T}$ in $\overline{B}_{\widetilde{R}}$ guarantees the existence of an element $u_0 \in \overline{B}_{\widetilde{R}}$ with $\widetilde{T}u_0 = u_0$. If $|u_0| > R$, then $T\left(\frac{R}{|u_0|}u_0\right) = u_0$

and if we let $v = \frac{R}{|u_0|}u_0$, then we can see that $|v| = R$ and $Tv = \lambda v$, where $\lambda = \frac{|u_0|}{R} > 1$. This contradiction to the Leray-Schauder boundary condition (18) shows that $|u_0| \leq R$. Hence $Tu_0 = u_0$ as we wished. ■

Notice that in most applications, the Leray-Schauder condition is obtained for a given operator $T : X \to X$, by means of the so called "a priori" bounds technique. This consists in proving that the set of all possible solutions in X of the equations $Tu = \lambda u$ for $\lambda > 1$, is bounded, i.e., $|u| < R$, for some $R > 0$ independent on λ.

Finally note that if X is a Hilbert space identified to its dual and $E'(u) = u - Tu$, then condition (9) coincides with (18).

6 Projects

Project 1 (The Gâteaux and Fréchet derivatives).

(a) Prove that if $E : X \to \mathbf{R}$ is Fréchet differentiable at a point u, then E is Gâteaux differentiable at u, the two derivatives coincide and E is continuous at u.

(b) Prove that if X is a Hilbert space endowed with inner product $(.,.)$ and norm $|.|$, and $E : X \to \mathbf{R}$ is the functional

$$E(u) = \frac{1}{2}|u|^2, \quad u \in X,$$

then $E \in C^1(X)$ and

$$(E'(u), v) = (u, v) \qquad \text{for all } u,\ v \in X.$$

Project 2 (The Leray-Schauder boundary condition). Consider the initial value problem

$$\begin{cases} u' = f(t,u), & t \in [0,1] \\ u(0) = u_0. \end{cases}$$

Here $f : [0,1] \times \mathbf{R} \to \mathbf{R}$ is continuous and $u_0 \in \mathbf{R}$. The problem is equivalent to the integral equation

$$u(t) = u_0 + \int_0^t f(s, u(s))\, ds \quad (t \in [0,1])$$

in $C[0,1]$, and thus, to the operator equation $u = Tu$, where $T : C[0,1] \to C[0,1]$ is given by the right hand side of the integral equation. Show

(a) If there is a function $g \in C([0,1], \mathbf{R}_+)$ with $f(t,u)\,u \le g(t)\,u$ for all $t \in [0,1]$ and $u \in \mathbf{R}$, then the set of all solutions of the problems

$$\begin{cases} u' = \lambda f(t,u), & t \in [0,1] \\ u(0) = \lambda u_0 \end{cases}$$

for $\lambda \in [0,1]$ is bounded in $C[0,1]$, that is, the Leray-Schauder condition holds for T and each ball $\overline{B_R}$ in $C[0,1]$ of a sufficiently large radius R.

(b) The same conclusion holds if there are two functions $g \in C([0,1], \mathbf{R}_+)$, $h \in C(\mathbf{R}_+, (0,\infty))$ such that

$$|f(t,u)| \le g(t)\,h(|u|) \qquad \text{for all } t \in [0,1],\ u \in \mathbf{R},$$

and

$$\int_{|u_0|}^{\infty} \frac{d\tau}{h(\tau)} > \int_0^1 g(s)\,ds.$$

Project 3 (Ekeland's variational principle). Let (X,d) be a complete metric space and let $E : X \to \mathbf{R}$ be a lower semicontinuous function bounded from below. (a) Show that for each $\varepsilon > 0$ and for each $u_0 \in X$ such that $E(u_0) \le \inf_X E + \varepsilon$, there exists $u \in X$ such that

$$\begin{aligned} E(u) &\le E(u_0) \\ d(u_0,u) &\le 1 \\ E(v) - E(u) + \varepsilon d(u,v) &\ge 0 \text{ for all } v \in X. \end{aligned}$$

Hint: apply Theorem 2.1 in the ball $\overline{B_1(u_0)}$.

(b) Changing the metric d by $\frac{1}{\sqrt{\varepsilon}}d$, prove the existence of u such that

$$\begin{aligned} E(u) &\le E(u_0) \\ d(u_0,u) &\le \sqrt{\varepsilon} \\ E(v) - E(u) + \sqrt{\varepsilon} d(u,v) &\ge 0 \text{ for all } v \in X. \end{aligned}$$

(c) Assuming in addition that X is a Banach space with norm $|.|$ and $E \in C^1(X)$ and using the result in (b) by taking $v = u + tw$, $t > 0$, $|w| = 1$, show that there exists $u \in X$ with

$$\begin{aligned} E(u) &\leq E(u_0) \\ |u - u_0| &\leq \sqrt{\varepsilon} \\ |E'(u)| &\leq \sqrt{\varepsilon}. \end{aligned}$$

Project 4 (Schechter's first critical point theorem). Consider the two point boundary value problem

$$\begin{cases} -u'' = g(u) + h(t), & t \in (0,1) \\ u(0) = u(1) = 0, \end{cases}$$

where $h \in L^2(0,1)$ and $g : \mathbf{R} \to \mathbf{R}$ is a continuous function with $g(u)\,u \leq 0$ for every $u \in \mathbf{R}$. Denote $G(\tau) = \int_0^\tau g(s)\,ds$.

(a) Show that the functional $E : H_0^1(0,1) \to \mathbf{R}$,

$$E(u) = \int_0^1 \left(\frac{1}{2} u'^2 - G(u) - hu \right) dt$$

is C^1 and its critical points are the solutions of the problem.

(b) Prove that E is bounded from below.

(c) Show that the solutions of the problem

$$\begin{cases} -u'' = \lambda \left(g(u) + h(t) \right), & t \in (0,1) \\ u(0) = u(1) = 0, \end{cases}$$

for $\lambda \in [0,1]$ satisfy $|u|_{H_0^1(0,1)} \leq R := |h|_{L^2(0,1)}$.

(d) Deduce the existence of a function $u \in H_0^1(0,1)$ with $|u|_{H_0^1(0,1)} \leq R$, $E(u) = \inf_{\overline{B_R}} E$ and $E'(u) = 0$.