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Dedicated to Maria S. Pop on her 60^{th} anniversary

ON THE CHEBYSHEV METHOD FOR APPROXIMATING THE SOLUTIONS OF POLYNOMIAL OPERATOR EQUATIONS OF DEGREE 2

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Abstract: In this paper, the Chebyshev method for approximating the solutions of polinomial operator equations of degree 2 is presented. The convergence of the Chebyshev method is studied

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1.INTRODUCTION

The polynomial operator equations represent an important class of operator equations [1]. Among them, the polynomial operator equations of degree 2 have a special importance because the convergence hypotheses for the usual methods (Newton method, chord method, Steffensen method, Chebyshev method, etc.) are much simplified compared to the general case [6]. In this note we shall study the convergence of the Chebyshev method for the mentioned equations.

Let X be a Banach space and consider a mapping $f : X \to X$. We remind that the mapping f is a polynomial operator of degree two if

a) f is three times differentiable;

b) $f'''(x) = \theta_3$, $\forall x \in X$, where θ_3 is the trilinear null operator.

2.THE CONVERGENCE OF THE CHEBYSHEV METHOD.

Consider the equation

$$f(x) = \theta, \tag{2.1}$$

where $\theta \in X$ is the zero element. Given an initial approximation $u_0 \in X$ of a solution $\overline{u}$ of the above equation, the Chebyshev method generates the sequence $(u_n)_{n\geq 0}$ by

$$u_{n+1} = u_n - \Gamma_n f(u_n) - \tfrac{1}{2}\Gamma_n f''(u_n)(\Gamma_n f(u_n))^2, \quad u_0 \in X,\ n = 0, 1, 2, ...,$$

where $\Gamma_n = f'(u_n)^{-1}$.

Consider $r > 0$ and denote $S = \{u \in X : \|u - u_0\| \leq r\}$. Since $f'''(x) = \theta_3, \forall x \in X$, it is clear that $f''(x)$ does not depend on x, so we may take $m_2 = \|f''(x)\|$. From the Taylor formula we obtain

$$f(u) = f(u_0) + f'(u_0)(u - u_0) + \tfrac{1}{2}f''(u_0)(u - u_0)^2 \tag{2.2}$$

$$f'(u) = f'(u_0) + f''(u_0)(u - u_0). \tag{2.3}$$

By (1) it follows

$$\|f'(u)\| \leq \|f'(u_0)\| + m_2 r, \quad \forall u \in S,$$

which implies

$$\sup_{u\in S} \|f'(u)\| \leq \|f'(u_0)\| + m_2 r. \tag{2.4}$$

Similarly (1), leads to

$$\sup_{u\in S} \|f(u)\| \leq \|f(u_0)\| + r\|f'(u_0)\| + \tfrac{1}{2}m_2 r^2. \tag{2.5}$$

We shall make the following notations:

$$m_0 = \|f(u_0)\| + r\|f'(u_0)\| + \tfrac{1}{2}m_2 r^2, \tag{2.6}$$

$$\mu = \tfrac{1}{2}m_2^2 b^4 \left(1 + \tfrac{1}{4}m_2 m_0 b^2\right), \tag{2.7}$$

$$\nu = b\left(1 + \tfrac{1}{2}m_2 m_0 b^2\right), \tag{2.8}$$

where

$$b = \frac{b_0}{1 - m_2 b_0 r} \text{ and } b_0 = \left\|f'(u_0)^{-1}\right\|. \tag{2.9}$$

With the above notations, the following result holds:

Theorem 1 *If for some $u_0 \in X$ and $r > 0$, the mapping f satisfies*

i. $\exists f'(u_0)^{-1}$;

ii. $m_2 b_0 r < 1$;

iii. the numbers μ and ν given by (1) *and* (1) *verify*

$$\rho_0 = \sqrt{\mu} \cdot \|f(u_0)\| < 1 \text{ and}$$
$$\frac{\nu \cdot \rho_0}{\sqrt{\mu}(1-\rho_0)} \leq r,$$

then the following relations hold:

j. the sequence $(u_n)_{n \geq 0}$ generated by the Chebyshev method converges;

jj. denoting $\overline{u} = \lim_{n \to \infty} u_n$, then $\overline{u} \in S$ and $f(\overline{u}) = \theta$;

jjj. $\|u_{n+1} - u_n\| \leq \frac{\nu \rho_0^{3^n}}{\sqrt{\mu}}, \quad n = 0, 1, \ldots;$

jv. $\|\overline{u} - u_n\| \leq \frac{\nu \rho_0^{3^n}}{\sqrt{\mu}\left(1-\rho_0^{3^n}\right)}, \quad n = 0, 1, \ldots$.

Proof. First we shall show that hypothesis ii) implies the existence of the application $f'(u)^{-1}$ for all $u \in S$ and, moreover, $\left\|f'(u)^{-1}\right\| \leq b$. Indeed, one has

$$\left\|f'(u_0)^{-1}(f'(u_0) - f'(u))\right\| \leq m_2 b_0 r, \quad \forall u \in S.$$

Applying the Banach Lemma and taking into account relation ii) it follows the existence of $f'(u)^{-1}$ for all $u \in S$ and, moreover,

$$\left\|f'(u)^{-1}\right\| \leq \tfrac{b_0}{1-m_2 b_0 r} = b. \tag{2.10}$$

Denote by g the mapping $g : S \to X$ given by

$$g(u) = -\Gamma(u) f(u) - \tfrac{1}{2}\Gamma(u) f''(u) (\Gamma(u) f(u))^2 \tag{2.11}$$

where $\Gamma(u) = f'(u)^{-1}$.

It can be easily seen that for all $u \in S$, the following identity holds:

$$\begin{aligned} f(u) + f'(u) g(u) + \tfrac{1}{2} f''(u) g^2(u) = \\ = \tfrac{1}{2} f''(u) \left(f'(u)^{-1} f(u), f'(u)^{-1} f''(u) \left(f'(u)^{-1} f(u)\right)^2\right) + \\ + \tfrac{1}{8} f''(u) \left(f'(u)^{-1} f''(u) \left(f'(u)^{-1} f(u)\right)^2\right)^2, \end{aligned}$$

whence

$$\left\|f(u) + f'(u) g(u) + \tfrac{1}{2} f''(u) g^2(u)\right\| \leq \mu \|f(u)\|^3, \quad \forall u \in S. \tag{2.12}$$

Since $u_{n+1} = u_n + g(u_n)$, from the Taylor formula we get

$$\begin{aligned} f(u_{n+1}) &= f(u_n) + f'(u_n)(u_{n+1} - u_n) + \tfrac{1}{2} f''(u_n)(u_{n+1} - u_n)^2 \\ &= f(u_n) + f'(u_n) g(u_n) + \tfrac{1}{2} f''(u_n) g^2(u_n), \end{aligned}$$

and, by (1),

$$\|f(u_{n+1})\| \le \mu \|f(u_n)\|^3, \tag{2.13}$$

provided that $u_n \in S$. Since $u_0 \in S$ one obtains

$$\|u_1 - u_0\| = \|g(u_0)\| \le \nu \|f(u_0)\| \le \frac{\nu\sqrt{\mu}\|f(u_0)\|}{\sqrt{\mu}(1-\rho_0)} = \frac{\nu\rho_0}{\sqrt{\mu}(1-\rho_0)} \le r,$$

i.e. $u_1 \in S$. Suppose now that the following relations hold:

α) $u_i \in S$, $i = \overline{0,k}$;

β) $\|f(u_i)\| \le \mu \|f(u_{i-1})\|^3$, $i = \overline{1,k}$.

From $u_k \in S$ and (1) it results

$$\|f(u_{k+1})\| \le \mu \|f(u_k)\|^3 \text{ and} \tag{2.14}$$

$$\|u_{k+1} - u_k\| \le \nu \|f(u_k)\|. \tag{2.15}$$

Inequality β) and (1) lead to

$$\|f(u_i)\| \le \frac{1}{\sqrt{\mu}} \left(\sqrt{\mu}\|f(u_0)\|\right)^{3^i}, \quad i = \overline{1, k+1}.$$

By (1) one gets

$$\begin{aligned} \|u_{k+1} - u_0\| &\le \sum_{i=1}^{k+1} \|u_i - u_{i-1}\| \le \sum_{i=1}^{k+1} \nu \|f(u_{i-1})\| \le \\ &\le \frac{\nu}{\sqrt{\mu}} \sum_{i=1}^{k+1} \rho_0^{3^{i-1}} \le \frac{\nu\rho_0}{\sqrt{\mu}(1-\rho_0)}, \end{aligned}$$

i.e., $u_{k+1} \in S$.

It is easy to show that

$$\|u_{n+m} - u_n\| \le \frac{\nu\rho_0^{3^n}}{\sqrt{\mu}\left(1 - \rho_0^{3^n}\right)}, \quad n = 0, 1, ..., \ m \in \mathbb{N} \tag{2.16}$$

and, since $\rho_0 < 1$, it follows that the sequence $(u_n)_{n \ge 0}$ is Cauchy, so it converges. Denoting $\overline{u} = \lim u_n$, it is clear that $f(\overline{u}) = \theta$. Letting $m \to \infty$ in (1) leads us to jv. ■

The Chebyshev method may be applied with the aid of the following algorithm:

Let u_n be an arbitrary approximation of the solution of (1), and which satisfies the hypotheses of Theorem 1. The next approximation u_{n+1} may be obtained by

1. Solve the linear operator equation

$$f'(u_n)\,p_n = f(u_n),$$

2. Solve the linear operator equation

$$f'(u_n)\,q_n = f''(u_n)\,p_n^2$$

3. Compute

$$u_{n+1} = u_n - p_n - \tfrac{1}{2}q_n.$$

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