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# Solution of a Polylocal Problem with a Pseudospectral Method

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Abstract. Consider the problem: $y''(x) + f(x,y) = 0$, $x \in [0,1]$, $y(a) = \alpha$, $y(b) = \beta$, $a, b \in (0,1)$. This is not a two-point boundary value problem since $a, b \in (0,1)$. It is possible to solve this problem by dividing it into the three problems: a two-point boundary value problem (BVP) on $[a,b]$ and two initial-value problems (IVP), on $[0,a]$ and $[b,1]$. The aim of this work is to present a solution procedure based on pseudospectral collocation with Chebyshev extreme points combined with a Runge Kutta method. Finally, some numerical examples are given.

Key Words: spectral methods, boundary-value problem, collocation, centrosymmetric matrix

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## 1 Introduction

Consider the problem (PVP):

$$y''(x) + f(x, y) = 0, \qquad x \in [0, 1] \tag{1.1}$$

$$y(a) = \alpha \tag{1.2}$$

$$y(b) = \beta, \qquad a, b \in (0, 1), a < b. \tag{1.3}$$

where $a, b, \alpha, \beta \in \mathbb{R}$. *This is not a two-point boundary value problem*, since $a, b \in (0, 1)$.

We try to solve the problem using a pseudospectral collocation method with Chebyshev extrema combined with a Runge Kutta method.

Then, we compare them in terms of error and cost.

Our choice to use this method is based on the following reasons :

1. We write the code using the functions in MATLAB `dmsuite` [1].
2. The accuracy of spectral method is superior to finite elements methods (FEM) and finite difference methods (FDM) (the rate of convergence associated with problems with smooth solutions are $O(\exp(-cN))$ or $O(\exp(c\sqrt{N}))$, where $N$ is the number of degrees of freedom in the expansion).
3. There exists elegant theoretical results on the convergence of collocation method (see, for example, [2]).

As drawbacks, we mention:

1. the matrices involved are full, not sparse;
2. the condition number is larger than those of FEM and FDM;
3. symmetric matrices are replaced by nonsymmetric matrices.

We also consider the BVP :

$$y''(x) + f(x, y) = 0, \qquad x \in [c, d] \tag{1.4}$$

$$y(c) = \alpha \tag{1.5}$$

$$y(d) = \beta, \tag{1.6}$$

To apply the collocation theory we need to have an isolated solution $y(x)$ of the problem (1.4)+(1.5)+(1.6), and this occurs if the above linearized problem for $y(x)$ is uniquely solvable. R.D Russel and L.F.Shampine [3] study the existence and the uniqueness of the isolated solution.

**Theorem 1.1** *[3] Suppose that $y(x)$ is a solution of the boundary value problem (1.4)+(1.5)+(1.6), that the functions :*

$$f(x,y) \text{ and } \frac{\partial f(x,y)}{\partial y}$$

*are defined and continuous for $a \le x \le b$, and $\left|y_k - y^{(k)}(x)\right| \le \delta$, $k = 0,1$; $\delta > 0$, and the homogeneous equations $y''(x) = 0$ subject to the homogeneous boundary conditions (1.5)+(1.6) has only the trivial solution. If the linear homogeneous equations:*

$$y''(x) + \sum_{k=0}^{1} \frac{\partial f(x,y)}{\partial y_k} y^{(k)}(x) = 0$$

*has only trivial solution, then this is sufficient to guarantee that there exists a $\sigma > 0$ such $y(x)$ is the unique solution of problem $BVP$ in the sphere:*

$$\left\| w - u'' \right\| \le \sigma.$$

For the existence and uniqueness of an IVP, we recall the following:

**Theorem 1.2** *[4, pp. 112-113] Suppose that $D = \{a \le x \le b, -\infty < y < \infty\}$ and $f(x,y)$ is continuous on $D$. If $f$ satisfies a Lipschitz condition on $D$ in the variable $y$, then the initial value problem (IVP)*

$$\text{(1.7)} \qquad \begin{cases} y' = z, \\ z' = -f(x,y),\ a \le x \le b,\ y(a) = \alpha,\ y'(a) = \varsigma, \end{cases}$$

*has a unique solution $y(x)$ for $a \le x \le b$.*

If the problem BVP has the unique solution, the requirement $y(x) \in C^2[0,1]$ ensure the existence and the uniqueness of the solution of PVP.

## 2 The description of the method

Our second method consists into decomposition of the the problem (1.1)+(1.2)+(1.3) into three problems:

1. A BVP on $[a, b]$
2. Two IVPs on $[0, a]$ and $[b, 1]$.

Also we suppose that the problem (1.4)+(1.5)+(1.6) satisfy Theorem **1.1**, which assures a sufficient condition to guarantee that there exists a $\sigma > 0$ such $y(x)$ is the unique solution of problem BVP in the sphere:

$$\left\| w - u^{(")} \right\| \leq \sigma$$

For $0 \leq x \leq a$ and $b \leq x \leq 1$ we have two initial value problems.

Due to conditions in Theorems 1.1 and 1.2, the problem (1.1)+(1.2)+(1.3) has a unique solution.

Consider the grid

$$\begin{aligned} \Delta : a = x_{-q} < \cdots < x_{-1} < c = x_0 < x_1 < \cdots < x_N = d < \\ x_{N+1} < \cdots < x_{N+p} = b. \end{aligned} \tag{2.1}$$

We shall use a pseudospectral method for the solution of (1.4) + (1.5) + (1.6) and a Runge-Kutta method for the two IVPs.

The solution of the two-point boundary value problems follow the ideas in [6] Let $y(x)$ be the solution of (1.4)+(1.5)+(1.6). Considering the Lagrange basis $(\ell_k)$, we have

$$y(x) = \sum_{k=0}^{N} \ell_k(x) y(x_k) + (R_N y)(x), \qquad x \in [c, d], \tag{2.2}$$

where

$$(R_N y)(x) = \frac{y^{(N+1)}(\xi)}{(N+1)!} (x - x_0) \cdots (x - x_N)$$

is the remainder of Lagrange interpolation and

$$\ell_k(x) = \prod_{\substack{j=0 \\ j\neq k}}^{n} \frac{x - x_j}{x_k - x_j}.$$

Since $y(x)$ fulfills the differential equation (1.4), we obtain

$$\sum_{k=0}^{N} \ell_k''(x_i)y(x_k) + (R_N y)''(x_i) = -f(x_i, y(x_i)), \qquad i = 1, \dots, N-1. \tag{2.3}$$

Setting $y(x_k) = y_k$ and ignoring the rest, one obtains the nonlinear system

$$\sum_{k=0}^{N} \ell_k''(x_i)y_k = -f(x_i, y(x_i)), \qquad i = 1, \dots, N-1, \tag{2.4}$$

with unknowns $y_k$, $k = 1, \dots, N-1$; here $y_0 = y(x_0) = \alpha$ and $y_N = y(x_N) = \beta$. The approximate solution (that is the collocation polynomial for (1.4)+(1.5)+(1.6)) is the Lagrange interpolation polynomial at nodes $\{x_k\}$

$$y_N(x) = \sum_{k=0}^{N} \ell_k(x)y_k. \tag{2.5}$$

The nonlinear system (2.4) can be rewritten as

$$AY_N = F(Y_N) + b_N\,,$$

where

$$A = [a_{ik}]\,, \quad a_{ik} = \ell_k''(x_i),\ k, i = 1, \dots, N-1,$$

$$F(Y_N) = \begin{bmatrix} -f(x_1, y_1) \\ \vdots \\ -f(x_{N-1}, y_{N-1}) \end{bmatrix}, \ b_N = \begin{bmatrix} -\alpha\ell_0''(x_1) - \beta\ell_N''(x_1) \\ \vdots \\ -\alpha\ell_0''(x_{N-1}) - \beta\ell_N''(x_{N-1}) \end{bmatrix}.$$

If the nodes $(x_k)$, $k = 0, \ldots, N$, are symmetric with respect of $(c+d)/2$, then $A$ is centrosymmetric (see [6] for the proof), so nonsingular. We recall the definition [7]: an $m \times m$ matrix $A$ is *centrosymmetric* if

$$a_{i,j} = a_{m-i+1,m-j+1}, \qquad i, j = 1, \ldots, m.$$

Examples of such nodes are given by

$$x_i = c + \frac{d-c}{N} i \qquad \text{or} \qquad x_i = \frac{(d-c)\cos\frac{\pi i}{N} + d + c}{2}, \qquad i = 1, \ldots, N$$

i.e. the equispaced and the Chebyshev-Lobatto nodes.

We introduce

$$G(Y) = A^{-1}F(Y) + A^{-1}b_N. \tag{2.6}$$

To solve numerically (1.1)+ (1.2)+(1.3) on $\Delta$ given by (2.1), we apply pseodospectral collocation at points in $[c, d]$ and then a Runge-Kutta to the other points. To apply the Runge-Kutta method for the solution of two IVP on $[a, c]$ and $[d, b]$ we need the derivatives $y'(c)$ and $y'(d)$. This can be computed by deriving the formula (2.5).

In [5] the authors prove the following theorems.

**Theorem 2.1** *If $f$ verifies a Lipschitz condition with respect to the second variable*

$$|F(x, u_1) - F(x, u_2)| \leq L\, |u_1 - u_2|$$

*and $\left\|A^{-1}\right\| L < 1$, then the system (2.4) has a unique solution which can be calculated by the successive approximation method*

$$Y^{(n+1)} = G\left(Y^{(n)}\right), \quad n \in \mathbb{N}^*, \tag{2.7}$$

*with $Y^{(0)}$ fixed and $G$ given by (2.6).*

**Theorem 2.2** *If $Y = [y(x_1), \ldots, y(x_{N-1})]^T$ where $y(x)$ is the solution of problem (1.4)+(1.5)+(1.6); $Y_N = [y_1, \ldots, y_{N-1}]^T$ where $y_i$ are the values of approximated solution at $x_i$ computed by (2.4) and $R =$*

$[-(R_N y)''(x_1), \ldots, -(R_N y)''(x_{N-1})]^T$ *then for the error* $\|Y - Y_N\|$ *it holds*

$$\|Y - Y_N\| \leq \frac{\|A^{-1}\| \, \|R\|}{1 - \|A^{-1}\| \, L}. \tag{2.8}$$

Combining the results of Theorems 2.1 and 2.2 with the stability and convergence of Runge-Kutta methods we have:

**Theorem 2.3** *If*

$$\frac{\|A^{-1}\| \, \|R\|}{1 - \|A^{-1}\| \, L} = O(h^k)$$

*and*

$$\begin{aligned} \left|y_N(c) - y'(c)\right| &= O(h^k) \\ \left|y_N(d) - y'(d)\right| &= O(h^k), \end{aligned}$$

*then for each point* $x_i$ *in* $\Delta$, $i = -q, \ldots, N+p$, $|y(x_i) - y_i| = O(h^k)$.

**Proof.** The condition $\|A^{-1}\| \, L < 1$ assures us that $G$ is a contraction. From Banach's fixpoint theorem it follows that $(Y^{(n)})$ given by (2.7) is convergent to the exact solution $\overline{Y}$ of (2.4) and the following estimation holds

$$\left\|\overline{Y} - Y^{(n)}\right\| \leq \frac{(\|A^{-1}\| \, L)^n}{1 - \|A^{-1}\| \, L} \left\|Y^{(1)} - Y^{(0)}\right\|.$$

If the accuracy of the collocation method for the BVP is $O(h^k)$ (that is, the approximate solution $y$ and its derivative $y'$ at $c$ and $d$ are within this accuracy limit), and if the Runge-Kutta method is stable and has the order $k$, then the final solution has the same accuracy. The stability and convergence of Runge-Kutta method are guaranteed by Theorems 5.3.1, page 285 and 5.3.2, page 288 in [8]. ■

## 3 Numerical examples

Our combined method was implemented in MATLAB, using the functions `cebdif`, `cebint` and `cebdifft` contained in `dmsuite` and described in [1]. We chose $\{x_k, k = 0, N\}$ as extreme Chebysev points and the other of $\Delta$ were computed using the MATLAB solver `ode45`. Since the successive approximation method is slow, we solve the nonlinear system (2.4) by Newton's method. The MATLAB function `solvepolylocalceb` solve the nonlinear system and call the Runge-Kutta solver. The derivatives at $c$ and $d$ were computed by calling `cebdifft`.

Let us consider two nonlinear examples.

**Example 3.1** *[9, page 491]Bratu's equation for $\lambda = 1$:*

$$\begin{aligned} y'' + e^y = 0, \quad x \in (0,1) \\ y(0.2) = y(0.8) = 0.08918993462883. \end{aligned}$$

*We took $N = 128$ and the tolerance $\varepsilon = 10^{-10}$. The starting value is $y^{(0)}(x) = \frac{x(1-x)}{4}$. The solution of nonlinear system is obtained after 4 iterations. The graph of the solution is given in Figure 1.*

**Example 3.2** *Consider the problem*

$$\begin{aligned} u'' + u^p = 0; \qquad 0 < x < 1; \\ u(0) = 0, \ u(1) = 0. \end{aligned} \tag{3.1}$$

*The positive solution of this problem represents the average temperature in a reaction-diffusion process. In [10], the authors proved that, for $p = 3$, the problem (3.1) has a unique positive solution. We consider here the "variant"*

$$\begin{aligned} u'' + u^3 = 0; \qquad 0 < x < 1; \\ u(0.2) = u(0.8) = 1.929990320692795. \end{aligned}$$

*In this example, $N = 128$, $\varepsilon = 10^{-8}$, $y^{(0)} = \frac{6\pi}{\sqrt{2}}x(1-x)$. The desired tolerance was obtained after 8 iterations. Figure 2 shows the graph of the solution.*

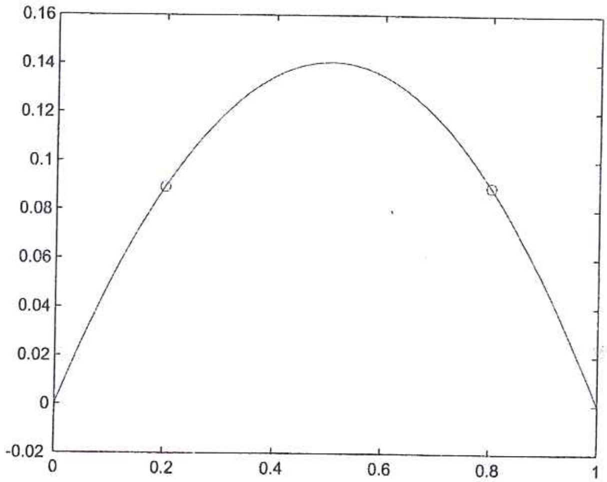


Figure 1: The solution to Bratu's problem for $\lambda = 1$.

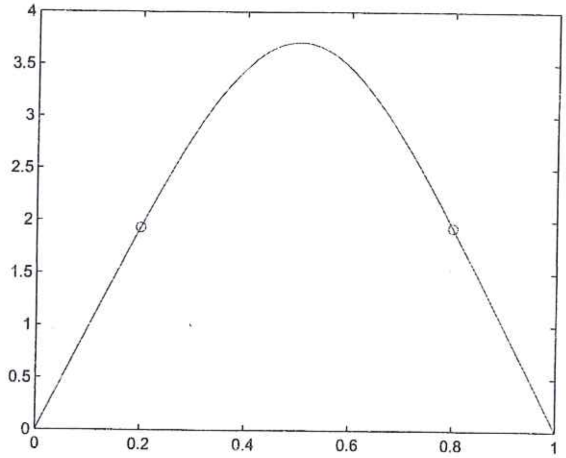


Figure 2: The positive solution to average temperature in a reaction-diffusion process.

## References


[1] J. A. C. Weideman, S. C. Reddy, A MATLAB Differentiation Matrix Suite, *ACM Transactions on Mathematical Software*, 26(4), 2000, pp. 465–519.

[2] W. Heinrichs, Strong convergence estimates for pseudospectral methods, *Applications of Mathematics*, 37(6), 1992, pp. 401–417.

[3] R.D. Russel, L.F.Shampine, A Collocation Method for boundary Value Problems, *Numer. Math.* 19, pp. 1-28, Springer Verlag, 1972.

[4] Henrici, P., *Discrete variable methods in ordinary differential equations*, John Willey, New York, pp. 345-384, 1962.

[5] F. Costabile, A. Napoli, A Collocation Method for Global Approximation of General Second Order BVPs, *Computing Letters*, 3(1), 2007, pp. 23–34.

[6] F. A. Costabile, F. Dell'Accio, On the nonsingularity of a special class of centrosymmetric matrices arising in spectral methods in BVPs, *Applied Mathematics and Computation*, 206, 2008, pp. 991–993.

[7] A. L. Andrew, Centrosymmetric Matrices, SIAM review, 40(3), 1998, pp. 697–698.

[8] W. Gautschi, *Numerical Analysis. An Introduction*, Birkhäuser, Boston, Basel, Berlin, 1997.

[9] U. Ascher, R.M. Mattheij, and R.D. Russel, *Numerical Solution of Boundary Value Problems for Ordinary Differential Equations*, SIAM, 1997.

[10] C. I. Gheorghiu, D. Trif, The Numerical Approximation to Positive Solution for some Reaction-Diffusion Problems, *Pu.M.A.*, 11, 2000, pp. 243–253.