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ON THE MONOTONICITY OF THE SEQUENCES OF APPROXIMATIONS OBTAINED BY STEFFENSEN'S METHOD

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In a recent paper [1], M. Bálazs studied the conditions in which the sequence $(x_n)_{n\geqslant 0}$ generated by Steffensen's method is monotonic and converges to the solution of equation

$$f(x) = 0 \tag{1}$$

where $f : I \to \mathbb{R}$ is a given function, and $I \subset \mathbb{R}$ is an interval of the real axix. Paper [1] considers the simple case of the sequence generated by the recurrence relation

$$x_{n+1} = x_n - \frac{f(x_n)}{[x_n, g(x_n); f]}, x_0 \in I, n = 0, 1, \ldots, \tag{2}$$

in which $g : I \to \mathbb{R}$ is given by $g(x) = x - f(x)$, and $[u, v; f]$ is the first order divided difference of the function f.

The following theorem is proved in [1]:

THEOREM 1, [1]. *Let $f : I \to \mathbb{R}$ be a continuous function on I, and define $g(x) = x - f(x)$. If the following conditions:*

(i) *The function $g : I \to \mathbb{R}$ is strictly decreasing and convex on I;*

(ii) *there exists a point $x_0 \in I$ such that $f(x_0) < 0$;*

(iii) *$I_0 = [x_0 - d, x_0 + d] \subset I$, where $d = \max\{|f(x_0)|, |f(g(x_0))|\}$* –) 22
hold, then all elements of the sequence $(x_n)_{n\geqslant 0}$ generated by (2) *belong to I_0; in addition, the following properties hold:*

(j) *the sequence $(x_n)_{n\geqslant 1}$ is decreasing and convergent;*

(jj) *the sequence $(g(x_n))_{n\geqslant 1}$ is increasing and convergent;*

(jjj) $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} g(x_n) = x^*$, *where x^* is the unique solution of equation* (1) *in I.*

We shall show further down that the properties resulting from Theorem 1 hold for more general Steffensen-type methods, while for the method (2), if hypothesis (ii) is replaced by:

(ii_1) there exists $x_0 \in I$ for which $f(x_0) > 0$ and $g(x_0) \in I$, then hypothesis (iii) can be dropped, and the conclusions of the theorem remain valid putting $I_0 = [g(x_0), x_0]$.

Consider an arbitrary function $g : I \to \mathbb{R}$ whose fixed points coincide with the real roots of equation (1), and reciprocally.

The following theorem holds:

THEOREM 2. *If the functions $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ are continuous on I, and if the following conditions:*

(i_2) *the function g is strictly decreasing on I;*

(ii_2) *the function f is strictly increasing and concave on I;*

(iii_2) *there exists $x_0 \in I$ such that $f(x_0) > 0$, $g(x_0) \in I$ and $x_0 - g(x_0) > 0$*

(iv_2) *the equations $f(x) = 0$ and $g(x) = x$ are equivalent, are fulfilled, then equation* (1) *has a unique solution $x^* \in [g(x_0), x_0]$ and the following properties hold:*

(j_2) *the sequence $(x_n)_{n \geqslant 0}$ is decreasing and convergent;*

(jj_2) *the sequence $(g(x_n))_{n \geqslant 0}$ is increasing and convergent;*

(jjj_2) $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} g(x_n) = x^*$, *where x^* is the unique solution of equation* (1), *therefore fixed point of g in the interval I.*

Proof. From (2), for $n = 1$, we get

$$x_1 - x_0 = -\frac{f(x_0)}{[x_0, g(x_0); f]} < 0,$$

that is, $x_1 < x_0$.

Writing $h(x) = x - g(x)$, we have $h(x_0) > 0$ by hypothesis; moreover:

$$h(g(x_0)) = g(x_0) - g(g(x_0)) = [g(x_0), x_0; g]\,(x_0 - g(x_0)) < 0$$

hence the equation $h(x) = 0$ has unique solution in the interval $[g(x_0), x_0]$, i.e. g has a unique fixed point x^* in this interval. Since the fixed points of g coincide with the roots of equation (1) and reciprocally, there follows that x^* is unique solution for equation (1) within the same interval, and $f(g(x_0)) < 0$. Now we shall show that $x_1 > x^*$. First show that $x_1 > g(x_0)$.

It is easy to verify that the terms of the sequence $(y_n)_{n \geqslant 0}$ provided by the relations

$$y_{n+1} = g(y_n) - \frac{f(g(y_n))}{[y_n, g(y_n); f]} \cdot \quad y_0 = x_0, \quad n = 0, 1, \ldots$$

coincide with those of the sequence $(x_n)_{n \geqslant 0}$ generated by (2). In other words the equalities

$$x_n - \frac{f(x_n)}{[x_n, g(x_n); f]} = g(x_n) - \frac{f(g(x_n))}{[x_n, g(x_n); f]} \cdot \quad n = 0, 1, \ldots$$

hold; for $n = 0$, it follows that:

$$x_1 - g(x_0) = -\frac{f(g(x_0))}{[x_0, g(x_0; f]} > 0$$

from which it results that $x_1 \in [g(x_0), x_0]$.

From Lagrange's interpolating polynomial we get

$$f(x_1) = f(x_0) + [x_0, g(x_0); f]\,(x_1 - x_0) + [x_1, x_0, g(x_0); f]\,(x_1 - x_0)(x_1 - g(x_0))$$

and, since $[x_1, x_0, g(x_0); f] < 0$, taking into account (2), it follows that $f(x_1) > 0$, that is, $x_1 > x^*$. Since the function h is increasing, it results that $h(x_1) > 0$, i.e. $x_1 - g(x_1) > 0$. Let us show that $g(x_1) \in [g(x_0), x_0]$. We have $g(x_1) - g(x_0) = [x_0, x_1; g](x_1 - x_0) > 0$ that is, $g(x_0) < g(x_1) < x^* < x_1 < x_0$.

Repeating the above argument, putting $x_k = x_0$, $k \in \mathbb{N}$, and supposing that the hypotheses of Theorem 2 are fulfilled for x_k, we obtain:

$$g(x_k) < g(x_{k+1}) < x^* < x_{k+1} < x_k.$$

It results subsequently that the sequences $(x_n)_{n \geqslant 0}$ and $(g(x_n))_{n \geqslant 0}$ fulfil the properties (j_2) and (jj_2) of Theorem 2 and, in addition, are bounded.

Write $\bar{u} = \lim\limits_{n \to \infty} g(x_n)$ and $\bar{v} = \lim\limits_{n \to \infty} x_n$; we obtain $\bar{u} = g(\bar{v})$ and $\bar{u} \leqslant x^* \leqslant \bar{v}$. Suppose that $\bar{u} < \bar{v}$, therefore $\bar{u} - \bar{v} < 0$. But $\bar{u} - \bar{v} = g(\bar{v}) - \bar{v} = -h(\bar{v}) \geqslant 0$, since $\bar{v} \geqslant x^*$; this shows that $\bar{u} = \bar{v} = x^*$

At limit (for $n \to \infty$), equalities (2) yield $f(x^*) = 0$, where the continuousness of f was also taken into account.

Remark 1. If we put in Theorem 2, $g(x) = x - f(x)$, since g is decreasing, it follows that $f(x) = x - g(x)$ is increasing; since g is convex, it follows that $[x, y, z; g] > 0$ for every $x, y, z \in I$, hence $[x, y, z; f] = -[x, y, z; g] < 0$, that is, f is concave. From $f(x_0) > 0$ it follows $x_0 - g(x_0) > 0$, e.i. $x_0 > g(x_0)$. In this case, Theorem 1 in which hypotheses (ii) and (iii) are replaced by (iii) is a consequence of Theorem 2.

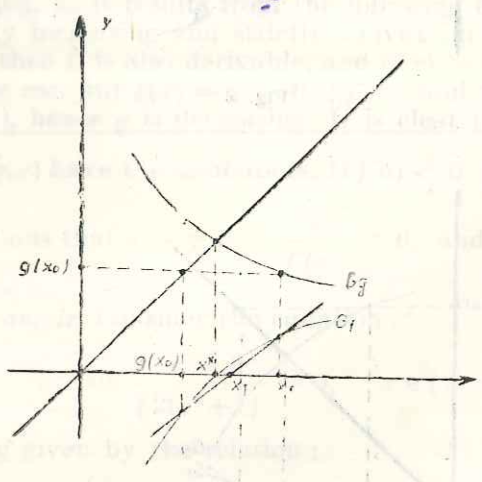

Fig. 1

The results of Theorem 2 are graphically illustrated in Figure 1.

In what follows we shall present, without proof, other cases in which properties of monotonicity analogous to those given by Theorem 2 hold.

THEOREM 3. *If the functions* $f : I \to \mathbb{R}$ *and* $g : I \to \mathbb{R}$ *are continuous on* I, *and if the following conditions are fulfilled* :

(i_3) *the function* g *is strictly decreasing on* I ;

(ii_3) *the function* f *is strictly increasing and convex on* I ;

(iii_3) *there exists* $x_0 \in I$ *for which* $f(x_0) < 0$, $g(x_0) \in I$ *and* $x_0 - g(x_0) < 0$;

(iv_3) *the equations* $f(x) = 0$ *and* $x = g(x)$ *are equivalent, then the sequence* $(x_n)_{n \geqslant 0}$ *generated by* (2) *is increasing and convergent, the sequence* $(g(x_n))_{n \geqslant 0}$ *is decreasing and convergent, and* $x^* = \lim\limits_{n \to \infty} x_n = \lim\limits_{n \to \infty} g(x_n)$ *is the solution of equation* (1).

Figure 2 plots the results of Theorem 3.

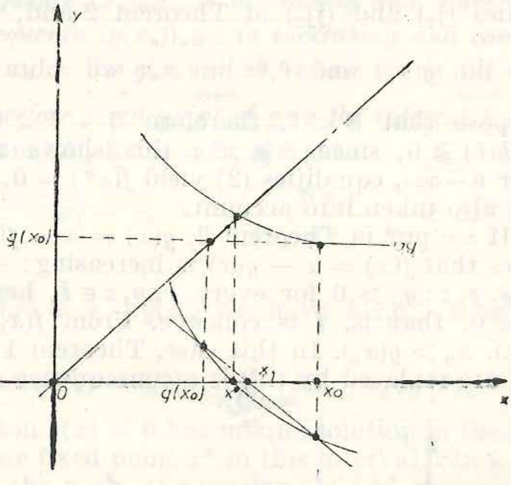

Fig. 2

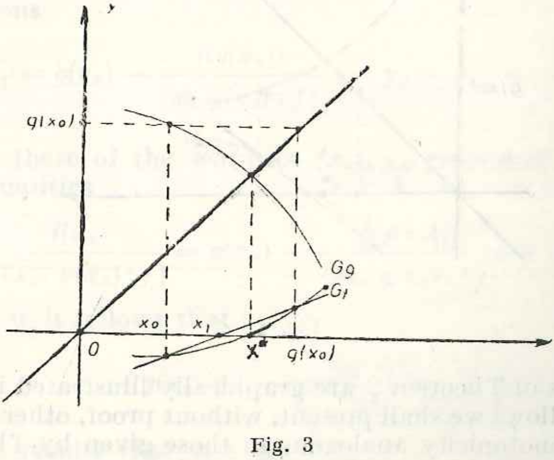

Fig. 3

THEOREM 4. *If the functions* $f: I \to \mathbb{R}$ *and* $g: I \to \mathbb{R}$ *are continuous on* I, *and if the following conditions are fulfilled*:

(i_4) *the function* g *is strictly decreasing on* I;

(ii_4) *the function* f *is strictly decreasing and convex on* I;

(iii_4) *there exists* $x_0 \in I$ *for which* $f(x_0) < 0$, $g(x_0) \in I$ *and* $x_0 - g(x_0) > 0$;

(iv_4) *the equations* $f(x) = 0$ *and* $x = g(x)$ *are equivalent, then the sequence* $(x_n)_{n \geqslant 0}$ *generated by* (2) *is decreasing and convergent, the sequence* $g(x_n))_{n \geqslant 0}$ *is increasing and convergent, and* $x^* = \lim_{n \to \infty} x_n = \lim_{n \to \infty} g(x_n)$, $f(x^*) = 0$.

The results of this theorem are illustrated by Figure 3. 2 ??

THEOREM 5. *If the functions* $f: I \to \mathbb{R}$ *and* $g: I \to \mathbb{R}$ *are continuous on* I, *and if the following conditions are fulfilled*:

(i_5) *the function* g *is strictly decreasing on* I;

(ii_5) *the function* f *is strictly decreasing and concave on* I;

(iii_5) *there exists* $x_0 \in I$ *such that* $f(x_0) > 0$, $g(x_0) \in I$ *and* $x_0 - g(x_0) < 0$;

(iv_5) *the equations* $f(x) = 0$ *and* $x = g(x)$ *are equivalent, then the sequence* $(x_n)_{n \geqslant 0}$ *generated by* (2) *is increasing and convergent, the sequence* $(g(x_n))_{n \geqslant 0}$ *is decreasing and convergent, and* $\lim_{n \to \infty} x_n = \lim_{n \to \infty} g(x_n) = x^*$, $f(x^*) = 0$.

Remark 2. The fact that the functions f and g from the above theorems are related only by the equivalence of equations $f(x) = 0$ and $x = g(x)$ offers large possibilities to choose these functions (i.e. to choose g when f is known, and conversely).

It is clear that if f keeps the same monotonicity and convexity on I, then we can moot the question of determining a real number λ such that $g(x) = x - \lambda f(x)$ be a decreasing function. Under certain conditions λ can be determined, as it results from the following example:

If f is strictly increasing and strictly convex on $I = [a, b]$, and if f is differentiable, then f' is also derivable, and $f'(x) > f'(a) > 0$ for every $x \in [a, b]$. Then we can put $g(x) = x - f(x)/f'(a)$, and we have $g'(x) \leqq 0$ for every $x \in [a, b]$, hence g is decreasing. It is clear that the equations $f(x) = 0$ and $x = g(x)$ have the same roots. If $f(a) < 0$ and $a - \frac{f(a)}{f'(a)} < b$, then it is obvious that $a - g(a) = \frac{f(a)}{f'(a)} < 0$, and Theorem 3 can be applied for $x_0 = a$.

Numerical example. Consider the equation:

$$f(x) = x - \arcsin \frac{x-1}{\sqrt{2(x^2+1)}} = 0, \quad x \in (-\infty, -1]$$

and the function g given by the relation:

$$g(x) = \arcsin \frac{x-1}{\sqrt{2(x^2+1)}}$$

Since $g'(x) = -\frac{1}{x^2+1}$ and $g''(x) = \frac{2x}{(x^2+1)^2}$ it follows that g is decreasing on $(-\infty, -1]$, and f is increasing and convex. One shows by direct calculation that $f(-2) \simeq -0.75 < 0$ and $g(-2) \simeq -1.25$, hence f fulfils the hypotheses of Theorem 3. The table below lists the results of the calculations for $x_0 = -2$.

n	x_n	$g(x_n)$
0	−2.000000000000000000	−1.249045772398254430
1	−1.414047729532868260	−1.400933154002817630
2	−1.404227441155695550	−1.404222310683232820
3	−1.404223602392559510	−1.404223602391771120
4	−1.404223602391969620	−1.404223602391969620

$f(x_n)$
−0.750954227601745574
−0.013114575530050630
−0.000005130472462734
−0.000000000000788388
−0.000000000000000000

The numerical results agree with the conclusions of Theorem 3; as one can see, after four iteration steps a solution approximation with 18 decimals is obtained (obviously, if troncation and rounding errors are neglected).

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