Construction of 3D potentials from a preassigned two-parametric family of orbits

Mira-Cristiana Anisiu[1] **and Thomas A Kotoulas**[2]

[1] T Popoviciu Institute of Numerical Analysis, Romanian Academy, PO Box 68,
400 110 Cluj-Napoca, Romania
[2] Department of Physics, Aristotle University of Thessaloniki, GR-541 24 Thessaloniki, Greece

E-mail: mira@math.ubbcluj.ro and tkoto@skiathos.physics.auth.gr

Abstract
One of the main problems of astrophysics is to determine the mean field potential of galaxies. The astronomical observations, as well as the numerical simulations, lead to the determination of families of star orbits in a galaxy. On this basis, using the tools of the inverse problem of dynamics, it is possible to find the gravitational potential which gives rise to such motions. The problem can be formulated in various ways; we consider here that the particle trajectories are given by a spatial two-parameter family of curves. From these trajectories we obtain certain functions known as orbital functions, and look for potentials of a special form. If the orbital functions satisfy some differential conditions, a step-by-step procedure offers the expression of the potential. Similar problems arise in thermodynamics and nuclear physics, where axially symmetric potentials are used as models for deformed nuclei.

1. Introduction

The mean field approach is important to the many-body systems, due to the fact that the N-body problem is practically impossible to handle for large values of N. The motion of a single particle may be studied in the field generated by many bodies, under the hypothesis that it does not perturb significantly the external field. This can be done, for example, in astrophysics (a star in a galaxy, a galaxy in a large galactic cluster) or in nuclear physics (nucleons in nuclei, atoms in metallic clusters).

In such models, the potential is not known in advance, but it is supposed to have some properties as spherical or axial symmetry. The knowledge of a family of orbits makes possible the determination of the potential, using the tools of the inverse problem of dynamics.

In this paper, we study the possibility of finding 3D potentials which give rise to a family of orbits known in advance. Among others, we construct a homogeneous potential which can produce a family of elliptic orbits (which are frequent in galaxy models), and an axially symmetric potential compatible with a family of hyperbolae. A related 2D problem was first studied by Szebehely (1974), with the aim to use the results for determining the Earth's potential on the basis of satellites' orbits: find all the potentials $V(x, y)$ which can give rise to a given mono-parametric family of curves $f(x, y) = c$ traced in the xy-Cartesian plane by a material point of unit mass. These potentials satisfy a linear first-order equation in which the total energy appears, known as Szebehely's equation. A linear second-order partial differential equation in the unknown function V, involving only the family of orbits and the potential—not the energy dependence, was produced by Bozis (1983). Anisiu (2004a) derived in a unified manner the two basic equations of the inverse problem of dynamics, and the region where real motion of the particle takes place (Bozis and Ichtiaroglou 1994).

Three-dimensional versions of the inverse problem were studied by Érdi (1982) for a monoparametric family of spatial orbits, and, for a two-parametric family, by Bozis (1983) (for general force fields) and by Váradi and Érdi (1983). Puel (1984) presented the intrinsic (independent of the coordinate frame) equations of the 3D inverse problem. Other results have been obtained by Bozis and Nakhla (1986), and Shorokov (1988). These results are summarized in the review paper of Bozis (1995). Recently, Bozis and Kotoulas (2004) have studied the case of two-parametric families of straight lines (FSL) produced by genuine three-dimensional potentials. Moreover, the same authors produced the free-of-energy equations and derived 3D homogeneous potentials which give rise to two-parametric families of homogeneous orbits in space (Bozis and Kotoulas 2005). At the same time, Anisiu (2004b, 2005) obtained in a direct way the *two* free-of-energy PDEs of the three-dimensional inverse problem and the region where real motion is allowed, presenting also several families of orbits compatible with 3D potentials.

The free-of-energy equation for the planar inverse problem was used to find the richest set of potentials which can generate a given family of ellipses (Bozis and Caranicolas 1997). The same equation enabled Caranicolas (1998) to produce potentials which allow for figure-eight families of orbits; such families appear in astrophysics, being detected in the N-body problem simulation in barred galaxies. More recently, Bozis and Anisiu (2005) have dealt with a solvable version of the planar inverse problem and found potentials of special type $V = v(\gamma)$ where $\gamma = f_y/f_x$. This approach was motivated by the fact that potentials of the form $v(\gamma)$ had already appeared as solutions of the planar inverse problem. Another reason was that, when potentials $v(\gamma)$ compatible with the given family γ do exist, they can be calculated by quadrature.

The present work extends to three dimensions the results concerning the planar potentials of the form $V = v(\gamma)$ compatible with a pre-assigned family of orbits $f(x, y) = c$. We deal with the following version of the 3D inverse problem of dynamics: given a two-parametric family of regular curves $f(x, y, z) = c_1$, $g(x, y, z) = c_2$ (with orbital functions α, β defined in (6)), find the potentials of the form $V = v(\alpha)$ or $V = F(\alpha, \beta)$ producing these families of curves as trajectories. For example, it is proved that the family (73) of elliptic orbits can be traced by a material point under the action of the potential $V(\alpha) = -\alpha^2(2 + \alpha^2)/4$. Working with the orbital functions α and β from (62), we obtain the potential $V = F(\alpha, \beta)$ from (64), which is an axially symmetric one. As mentioned by Boccaletti and Puccaco (1996, vol 1, p 349), axially symmetric potentials are important for astrophysical applications, since they are generated by mass distributions provided with rotational motions around the symmetry axis, typical of solid celestial bodies, stars, disc galaxies and elliptical galaxies. Besides the applications in galactic dynamics (Contopoulos 1960, Binney and Tremaine 1987, p 121; Caranicolas 2004), they are used in models in thermodynamics (e.g. Olaya-Castro and Quiroga (2000)) and in quantum mechanics (e.g. Heiss *et al* (1994)).

The problem under consideration does not have solutions for arbitrary families of curves. We determine the necessary and sufficient conditions on α and β which guarantee the existence of such potentials. A promising feature is that, when the conditions are fulfilled, we can construct the potentials by quadrature. We focus on genuine three-dimensional potentials, i.e. potentials whose analytical expression involves *all* the variables x, y, z. It follows that we take into account only potentials for which no partial derivative is identically zero. All over the paper we shall regard V as the potential energy function, hence the total energy is given by $\mathcal{E} = (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)/2 + V(x, y, z)$. In section 2 we present the basic equations of the 3D inverse problem of dynamics. In section 3 we study the case $V = v(\alpha)$. The case $V = F(\alpha, \beta)$ is examined in section 4. In both sections 3 and 4 we give step-by-step procedures, which lead to the expression of the potential when the orbital functions satisfy certain differential conditions. The mathematics are used as a tool, and the method can be applied whenever a family of orbits will be observed, or generated by a simulation. We refer to the case of straight lines in section 5 and conclude in section 6.

2. The 3D inverse problem of dynamics: basic facts

The three-dimensional inverse problem of dynamics, as formulated by Bozis and Nakhla (1986), seeks all the potentials $V = V(x, y, z)$ of C^2-class (continuous with continuous derivatives up to second order) which are compatible with a pre-assigned family of regular orbits

$$f(x, y, z) = c_1, \qquad g(x, y, z) = c_2. \tag{1}$$

By regular orbits we mean that the functions f, g are of C^3-class on a domain $D \subset \mathbb{R}^3$ and such that $\nabla f \neq 0$ and $\nabla g \neq 0$, and, in addition $\nabla f \times \nabla g \neq 0$ on D, so that $\delta_0 \neq 0$ in (3). The total energy is constant on each curve of the family, namely

$$\mathcal{E} = \mathcal{E}(f, g). \tag{2}$$

Differentiating equations (1) with respect to time t and taking into account the conservation of the energy one can express the velocity vector; differentiating again (1) and using the equations of motion one obtains the two linear PDEs satisfied by the potential V. These equations, in which the energy dependence function $\mathcal{E} = \mathcal{E}(f, g)$ appears, are

$$\begin{aligned} \delta_2 V_x - \delta_1 V_y &= \frac{2(\delta_1 a_2 - \delta_2 a_1)}{\delta_0}(\mathcal{E} - V), \\ \delta_3 V_x - \delta_1 V_z &= \frac{2(\delta_1 a_3 - \delta_3 a_1)}{\delta_0}(\mathcal{E} - V), \end{aligned} \tag{3}$$

where

$$\bar{\delta} = (\delta_1, \delta_2, \delta_3) = \nabla f \times \nabla g, \qquad \delta_0 = |\bar{\delta}|^2 \tag{4}$$

and

$$a_i = \bar{\delta} \cdot \nabla \delta_i, \qquad i = 1, 2, 3. \tag{5}$$

The vectorial and scalar products in $\mathbb{R}^3$ are denoted by '$\times$', respectively '$\cdot$', and the subscripts x, y, z denote the corresponding partial derivatives. Equations (3) in a vectorial form have been derived by Puel (1984) using Frenet's reference frame.

The orbital functions

$$\alpha = \frac{\delta_2}{\delta_1}, \qquad \beta = \frac{\delta_3}{\delta_1} \tag{6}$$

have been defined by Bozis and Kotoulas (2004), in a domain where $\delta_1 \neq 0$. With the use of notations

$$\begin{aligned} &\bar{\epsilon} = (1, \alpha, \beta), \qquad \Theta = 1 + \alpha^2 + \beta^2, \\ &\alpha_0 = \bar{\epsilon} \cdot \nabla \alpha = \alpha_x + \alpha \alpha_y + \beta \alpha_z, \\ &\beta_0 = \bar{\epsilon} \cdot \nabla \beta = \beta_x + \alpha \beta_y + \beta \beta_z, \end{aligned} \tag{7}$$

equations (3) take the simpler form (Bozis and Kotoulas 2005)

$$\alpha V_x - V_y = \frac{2\alpha_0}{\Theta}(\mathcal{E} - V), \qquad \beta V_x - V_z = \frac{2\beta_0}{\Theta}(\mathcal{E} - V). \tag{8}$$

The region where real motion exists (Shorokhov 1988, Anisiu 2004b) is

$$\frac{\alpha V_x - V_y}{\alpha_0} \geqslant 0. \tag{9}$$

Furthermore, Bozis and Kotoulas (2005) eliminated the energy dependence function from equations (8) and obtained two energy-free PDEs for the unknown potential function $V = V(x, y, z)$, one of first order and the other of second order. These equations relate the potential to the given family of curves, supposing that the last one does not consist of straight lines. It was proved by Bozis and Kotoulas (2004) that $\alpha_0 = 0$ and $\beta_0 = 0$ only when the family (1) consists of straight lines; we shall treat this special case in section 5.

Remark 1. As was mentioned by Bozis and Kotoulas (2005), the transformation $x \to x$, $y \to z$, $z \to y$ brings $\alpha \to \beta$ and $\beta \to \alpha$ as well as $\alpha_0 \to \beta_0$ and $\beta_0 \to \alpha_0$.

From now on we shall consider $\alpha_0 \neq 0$. (The case $\beta_0 \neq 0$ can be treated directly using instead of equation (12) the equation obtained from the second-order equation (25) in the paper of Anisiu (2004b).) The first-order equation is

$$k_1 V_x + k_2 V_y + k_3 V_z = 0, \tag{10}$$

where

$$k_1 = \alpha\beta_0 - \beta\alpha_0, \qquad k_2 = -\beta_0, \qquad k_3 = \alpha_0. \tag{11}$$

The second-order PDE of the 3D inverse problem reads

$$k_{11} V_{xx} + k_{12} V_{xy} + k_{13} V_{xz} + k_{22} V_{yy} + k_{23} V_{yz} + k_{01} V_x + k_{02} V_y + k_{03} V_z = 0, \tag{12}$$

where

$$\begin{aligned} &k_{11} = \alpha\Theta\alpha_0, \qquad k_{12} = (\alpha^2 - 1)\Theta\alpha_0, \qquad k_{13} = \alpha\beta\Theta\alpha_0, \\ &k_{22} = -\alpha\Theta\alpha_0, \qquad k_{23} = -\beta\Theta\alpha_0, \\ &k_{01} = (\Theta + 2)\alpha_0^2 + \alpha K, \qquad k_{02} = 2\alpha\alpha_0^2 - K, \qquad k_{03} = 2\beta\alpha_0^2, \\ &K = 2(\alpha\alpha_0 + \beta\beta_0)\alpha_0 - \Theta(\alpha_{0x} + \alpha\alpha_{0y} + \beta\alpha_{0z}). \end{aligned} \tag{13}$$

We note here that the previous equations (10) and (12) have also been produced on first grounds by Anisiu (2004b). These equations are equivalent to system (3), as was proved by Anisiu (2005), and have the advantage that it is not necessary to give the energy in advance. From (10) and (12) it is easy to check that if V is a solution, then $\tilde{V} = c_1 V + c_2$ is a solution too (c_1, c_2 are constants). So, without loss of generality, we shall omit these constants below.

In the present paper we extend the 2D results of Bozis and Anisiu (2005) and study the following version of the 3D inverse problem of dynamics: given a two-parametric family of regular orbits (1), we try to find the three-dimensional potentials of the form $V = v(\alpha)$ or $V = F(\alpha, \beta)$ which are compatible with that family. Having no *a priori* information on the energy, we deal with the two linear PDEs (10) and (12).

3. Potentials of the form $V = v(\alpha)$

In this section we shall try to find solutions of the form $V = v(\alpha)$ for the above two equations (10) and (12). In view of remark 1, we shall not consider here the case of $V = v(\beta)$.

3.1. Conditions on the orbital functions

We suppose that $V = v(\alpha)$ with v of C^2-class and prepare the derivatives of first order of the potential function V with respect to x, y, z respectively $V_x = v'\alpha_x$, $V_y = v'\alpha_y$, $V_z = v'\alpha_z$, where '$'$' denotes the derivative of v with respect to its unique variable α. We insert V_x, V_y and V_z into (10) and get

$$v'(\alpha_0(\beta\alpha_x - \alpha_z) - \beta_0(\alpha\alpha_x - \alpha_y)) = 0. \tag{14}$$

Since we do not admit constant potentials, $v'(\alpha)$ will not be zero on nonvoid open sets. It follows a first necessary condition on the orbital function α, namely

$$\alpha_0(\beta\alpha_x - \alpha_z) - \beta_0(\alpha\alpha_x - \alpha_y) = 0, \tag{15}$$

with α_0 and β_0 given in (7). Now, we shall work with the second-order PDE (12). Firstly, we prepare the second-order derivatives of the potential function V with respect to x, y, z,

$$\begin{aligned} &V_{xx} = v''\alpha_x^2 + v'\alpha_{xx}, \qquad &&V_{yy} = v''\alpha_y^2 + v'\alpha_{yy},\\ &V_{xz} = v''\alpha_x\alpha_z + v'\alpha_{xz}, \qquad &&V_{yz} = v''\alpha_y\alpha_z + v'\alpha_{yz},\\ &V_{xy} = v''\alpha_x\alpha_y + v'\alpha_{xy} \end{aligned} \tag{16}$$

and insert them into equation (12). Thus, we end up with the following relation,

$$\mathcal{A}v'' + \mathcal{B}v' = 0, \tag{17}$$

where

$$\begin{aligned} \mathcal{A} &= \Theta\alpha_0^2(\alpha\alpha_x - \alpha_y),\\ \mathcal{B} &= \Theta\alpha_0\sigma + 2\alpha_0^3 + (\alpha\alpha_x - \alpha_y)K, \end{aligned} \tag{18}$$

and

$$\sigma = \bar{\epsilon}\cdot\nabla(\alpha\alpha_x - \alpha_y). \tag{19}$$

Remark 2. Keeping in mind our assumption that $\alpha_0 \neq 0$, it follows that if $\mathcal{A} = 0$, then $\alpha\alpha_x - \alpha_y = 0$. Consequently, $\mathcal{B} = 2\alpha_0^3 \neq 0$ and equation (17) has no nonconstant solution.

From (17) we obtain for $\mathcal{A} \neq 0$

$$\frac{v''}{v'} = \mathcal{C}, \tag{20}$$

where

$$\mathcal{C} = -\frac{\mathcal{B}}{\mathcal{A}} = -\left[\frac{\sigma}{(\alpha\alpha_x - \alpha_y)\,\alpha_0} + \frac{2\alpha_0}{\Theta(\alpha\alpha_x - \alpha_y)} + \frac{K}{\Theta\alpha_0^2}\right], \tag{21}$$

with α_0 and Θ given in (7), K in (13) and σ in (19). Since the ratio v''/v' depends only on the orbital function α, the expression of $\mathcal{C}$ must have the same property. Thus, it must be

$$\mathcal{C} = \mathcal{C}(\alpha) \tag{22}$$

or, equivalently,

$$\frac{\mathcal{C}_x}{\alpha_x} = \frac{\mathcal{C}_y}{\alpha_y} = \frac{\mathcal{C}_z}{\alpha_z}. \tag{23}$$

This happens if and only if α and β satisfy

$$(i)\ \mathcal{C}_x\alpha_y - \mathcal{C}_y\alpha_x = 0; \qquad (ii)\ \mathcal{C}_y\alpha_z - \mathcal{C}_z\alpha_y = 0. \tag{24}$$

Conversely, let us suppose that conditions (15) and (24i, ii) are satisfied by the functions α and β, and that $\mathcal{A} \neq 0$. Then $\mathcal{C} = \mathcal{C}(\alpha)$ and we integrate (20) two times with respect to α and obtain

$$V = v(\alpha) = d_1 \int \mathrm{e}^{\int \mathcal{C}(\alpha)\mathrm{d}\alpha}\, \mathrm{d}\alpha + d_2, \qquad d_1, d_2 \quad \text{const.} \tag{25}$$

We shall omit the constants d_1, d_2 in the following as we explained above. The meaning of the solution (25) is that we can find the potential function V by quadratures.

Now, we can formulate the following

Proposition 1. *If $\mathcal{A} \neq 0$ and the three conditions (15) and (24i, ii) are satisfied for the given functions α and β, then a potential of the form $V = v(\alpha)$ does exist and it is determined uniquely from (25) up to two arbitrary constants.*

Synthesis of the problem

(1) We start with the functions α and β (or with f and g, and then obtain α, β from (6)).
(2) We calculate the expression in (15); if it is different from zero, the problem has no solution.
(3) If (15) is satisfied, we estimate $\mathcal{A}$ from (18); if $\mathcal{A} = 0$, we have no solution.
(4) If $\mathcal{A} \neq 0$, with $\mathcal{B}$ from (18) we calculate $\mathcal{C}$ from (21) and verify conditions (24i, ii). If at least one of them is not satisfied, there is no solution.
(5) If (24i, ii) are verified, we obtain the potential $V = v(\alpha)$ from (25).

3.2. *Suitable pairs of orbital functions and examples*

We look for suitable pairs (α, β), with α a function of three variables satisfying $\alpha_0 \neq 0$, for which conditions (15) and (24i, ii) are fulfilled.

A natural trial is for $\alpha = \beta$; condition (15) becomes $\alpha_y - \alpha_z = 0$, with the solution

$$\alpha = A(x, y + z), \tag{26}$$

where A is a two-variable function. The function α from (26) satisfies condition (24ii) too. Condition (24i) is fulfilled for several special forms of the function A in (26) mentioned below (M is an arbitrary one-variable function):

- $\alpha = \beta = M(x + y + z)$;
- $\alpha = \beta = M(x - y - z)$;
- $\alpha = \beta = M((y + z)/x)$.

We shall present in detail an example from the first class, for which the potential function $V = v(\alpha)$ is given explicitly.

Example 1. We consider the family

$$f(x, y, z) = \left(x + y + z + \tfrac{3}{2}\right)\mathrm{e}^{-2x}, \qquad g(x, y, z) = \tfrac{1}{4}(2y - 2z - 3)$$

which gives

$$\alpha = x + y + z + 1, \qquad \beta = x + y + z + 1. \tag{27}$$

For this family we have $\alpha_0 = \beta_0 = 2(x + y + z) + 3 = 2\alpha + 1$ and the three conditions (15) and (24i, ii) are satisfied. From (25) we calculate the potential function $v(\alpha) = (1 - 4\alpha)/(2(\alpha - 1)^2)$; hence

$$V(x, y, z) = -\frac{4(x + y + z) + 3}{2(x + y + z)^2}. \tag{28}$$

From (8) we calculate the energy $\mathcal{E} = 1$ and from (9) the allowed region of the motion of the test particle $1/(x + y + z)^2 \geqslant 0$, which means that the motion of the test particle can take place everywhere in the space, except for the plane $x + y + z = 0$.

Counter-examples. We consider the two-parametric family of curves (1) with

$$f(x, y, z) = \mathrm{e}^{y-z}, \qquad g(x, y, z) = -\tfrac{1}{2}(x^2\,\mathrm{e}^{y-z} + \mathrm{e}^{-2z}),$$

for which

$$\alpha = x\,\mathrm{e}^{y+z}, \qquad \beta = \alpha \tag{29}$$

and $\alpha_0 = \beta_0 = \mathrm{e}^{y+z}(1 + 2x^2\,\mathrm{e}^{y+z})$. We checked conditions (15) and (24i, ii) for this pair of orbits and ascertained that conditions (15) and (24ii) are verified but condition (24i) is *not* satisfied for this family. So, this family is not compatible with any potential $V = v(\alpha)$. The same conclusion holds for the orbital functions $\alpha = z\mathrm{e}^{x+y}$, $\beta = \alpha$, for which (24i) is satisfied, but (15) and (24ii) are not.

Another choice for the pair (α, β) is to make $\beta_0 = 0$. This happens, for example, for $\beta =$ constant, let it be $\beta = k$. Condition (15) reduces to $k\alpha_x - \alpha_z = 0$, with the solution

$$\alpha = A(y, x + kz) \tag{30}$$

where A is again a two-variable function. Generally, the functions α given by (30) and $\beta = k$ do not satisfy conditions (24i, ii). Nevertheless, we can find classes of functions which satisfy these conditions too, like:

- $\alpha = M(x + e_1 y + z)$, $\beta = k$ (e_1 free constant);
- $\alpha = M((f_1(x + kz) + f_2 y + f_3)/y)$, $\beta = k$ (f_1, f_2, f_3 free constants).

Example 2. For the two-parametric family of curves (1) with

$$f(x, y, z) = \mathrm{e}^x(x - y + kz - k^2 - 1), \qquad g(x, y, z) = kx - z,$$

we have

$$\alpha = x - y + kz, \qquad \beta = k, \tag{31}$$

and $\alpha_0 = 1 - x + y - kz + k^2$, $\beta_0 = 0$. From (25) we get $v(\alpha) = (2\alpha - k^2)/(\alpha + 1)^2$, and from (8) and (9) the energy $\mathcal{E} = 1$ and the allowed region $2/(x - y + kz + 1)^2 \geqslant 0$; hence the motion can take place everywhere in space except for the plane $x - y + kz + 1 = 0$.

It is not necessary that α and β are equal, or β is constant. For example, from the class $\alpha = x + e_1 y + e_2 z$, $\beta = x + e_1 y + e_2 z + e_3$, the functions which satisfy condition (15) are those with $e_3 = e_2 - e_1$; these functions satisfy also conditions (24i, ii) and the potential can be determined by quadratures.

4. Potentials of the form $V = F(\alpha, \beta)$

In this section we shall find potentials of the form $V = F(\alpha, \beta)$, with α, β functionally independent and F of C^2-class. The condition $\alpha_0 \neq 0$ is maintained. We impose necessary conditions on the orbital functions α, β and then we shall determine, when possible, the potential function $V = F(\alpha, \beta)$ so that it satisfies the two linear PDEs (10) and (12).

4.1. Conditions on the orbital functions α, β

We look for potentials which are functions of α and β. To this end, the necessary and sufficient condition for V, α, β is

$$\begin{vmatrix} V_x & \alpha_x & \beta_x \\ V_y & \alpha_y & \beta_y \\ V_z & \alpha_z & \beta_z \end{vmatrix} = 0$$

or, equivalently,

$$l_1 V_x + l_2 V_y + l_3 V_z = 0, \tag{32}$$

where

$$l_1 = \alpha_y \beta_z - \alpha_z \beta_y, \qquad l_2 = \alpha_z \beta_x - \alpha_x \beta_z, \qquad l_3 = \alpha_x \beta_y - \alpha_y \beta_x. \tag{33}$$

Equation (32) must be compatible with (10). Thus, we have another non-trivial PDE to ensure the compatibility of these two PDEs (Favard 1963, Smirnov 1964). This new PDE reads

$$m_1 V_x + m_2 V_y + m_3 V_z = 0, \tag{34}$$

where

$$m_i = k_1 \frac{\partial l_i}{\partial x} + k_2 \frac{\partial l_i}{\partial y} + k_3 \frac{\partial l_i}{\partial z} - l_1 \frac{\partial k_i}{\partial x} - l_2 \frac{\partial k_i}{\partial y} - l_3 \frac{\partial k_i}{\partial z}, \qquad i = 1, 2, 3. \tag{35}$$

Thus, the necessary and sufficient condition for finding the nonzero derivatives V_x, V_y, V_z from (10), (32) and (34) is

$$\begin{vmatrix} k_1 & l_1 & m_1 \\ k_2 & l_2 & m_2 \\ k_3 & l_3 & m_3 \end{vmatrix} = 0,$$

or, equivalently,

$$k_1(l_2 m_3 - l_3 m_2) - k_2(l_1 m_3 - l_3 m_1) + k_3(l_1 m_2 - l_2 m_1) = 0. \tag{36}$$

Let us suppose that condition (36) is fulfilled and $V = F(\alpha, \beta)$. We prepare the first-order derivatives of V with respect to x, y, z,

$$V_x = F_\alpha \alpha_x + F_\beta \beta_x, \qquad V_y = F_\alpha \alpha_y + F_\beta \beta_y, \qquad V_z = F_\alpha \alpha_z + F_\beta \beta_z, \tag{37}$$

and replace them into (10). We obtain

$$\mathcal{P} F_\alpha + \mathcal{Q} F_\beta = 0, \tag{38}$$

where

$$\begin{aligned} \mathcal{P} &= \alpha_0(\beta \alpha_x - \alpha_z) - \beta_0(\alpha \alpha_x - \alpha_y), \\ \mathcal{Q} &= \alpha_0(\beta \beta_x - \beta_z) - \beta_0(\alpha \beta_x - \beta_y). \end{aligned} \tag{39}$$

Remark 3. If $\mathcal{P} = 0$ and $\mathcal{Q} \neq 0$, we have $F_\beta = 0$ and the potential V will be a function $V = v(\alpha)$, case studied in section 3 (condition (15) is precisely $\mathcal{P} = 0$). For $\mathcal{Q} = 0$ and $\mathcal{P} \neq 0$, it follows that $F_\alpha = 0$ and V will depend merely on β. This case is similar to that studied in section 3 (see remark 1). We shall examine separately the cases $\mathcal{P} = 0$ and $\mathcal{Q} = 0$.

From (38) it follows for $\mathcal{P} \neq 0$ and $\mathcal{Q} \neq 0$ that

$$\frac{F_\alpha}{F_\beta} = -\frac{\mathcal{Q}}{\mathcal{P}} = \mathcal{D}. \tag{40}$$

Since $V = F(\alpha, \beta)$, the ratio F_α/F_β in (40) must depend only on α, β, i.e. $\mathcal{D} = \mathcal{D}(\alpha, \beta)$. To this end, the necessary and sufficient condition is

$$l_1\mathcal{D}_x + l_2\mathcal{D}_y + l_3\mathcal{D}_z = 0, \tag{41}$$

where l_1, l_2, l_3 were defined in (33). So, condition (41) is the second necessary one for α, β in order that the problem has a solution $V = F(\alpha, \beta)$.

Now, we proceed further and prepare the second-order derivatives of V with respect to x, y, z. We replace them into (12) and obtain

$$\Gamma_{11}F_{\alpha\alpha} + \Gamma_{12}F_{\alpha\beta} + \Gamma_{22}F_{\beta\beta} + \Gamma_{10}F_\alpha + \Gamma_{01}F_\beta = 0, \tag{42}$$

where

$$\begin{aligned}
&\Gamma_{11} = \alpha_0(\alpha\alpha_x - \alpha_y), \qquad \Gamma_{12} = \beta_0(\alpha\alpha_x - \alpha_y) + \alpha_0(\alpha\beta_x - \beta_y),\\
&\Gamma_{22} = \beta_0(\alpha\beta_x - \beta_y),\\
&\Gamma_{10} = \sigma + 2\alpha_0^2/\Theta + (\alpha\alpha_x - \alpha_y)K/(\Theta\alpha_0),\\
&\Gamma_{01} = \rho + 2\alpha_0\beta_0/\Theta + (\alpha\beta_x - \beta_y)K/(\Theta\alpha_0),
\end{aligned} \tag{43}$$

with K given in (13), σ in (19) and

$$\rho = \bar{\epsilon} \cdot \nabla(\alpha\beta_x - \beta_y). \tag{44}$$

If for the given α and β we have $\Gamma_{11} = \Gamma_{12} = \Gamma_{22} = 0$, equation (42) is a first-order one. We shall consider this case in subsection 4.1.2.

We suppose now that equation (42) contains indeed some of the second-order derivatives of F. From (40) we have

$$F_\alpha = \mathcal{D}F_\beta \tag{45}$$

and estimate the derivatives $F_{\alpha\alpha}$ and $F_{\alpha\beta}$,

$$F_{\alpha\alpha} = \mathcal{D}_\alpha F_\beta + \mathcal{D}F_{\alpha\beta}, \qquad F_{\alpha\beta} = \mathcal{D}_\beta F_\beta + \mathcal{D}F_{\beta\beta}. \tag{46}$$

We replace (45) and (46) into (42) and get

$$\mathcal{R}F_{\beta\beta} + \mathcal{S}F_\beta = 0, \tag{47}$$

where

$$\begin{aligned}
\mathcal{R} &= (\alpha_0\mathcal{D} + \beta_0)((\alpha\alpha_x - \alpha_y)\mathcal{D} + \alpha\beta_x - \beta_y),\\
\mathcal{S} &= ((\alpha\alpha_x - \alpha_y)\mathcal{D} + \alpha\beta_x - \beta_y)(\alpha_0\mathcal{D}_\beta + K/(\Theta\alpha_0))\\
&\quad + (\alpha\alpha_x - \alpha_y)(\alpha_0\mathcal{D}_\alpha + \beta_0\mathcal{D}_\beta) + \left(\sigma + 2\alpha_0^2/\Theta\right)\mathcal{D} + \rho + 2\alpha_0\beta_0/\Theta.
\end{aligned} \tag{48}$$

If $\mathcal{R} = 0$ and $\mathcal{S} \neq 0$, the potential V will depend only on α, which is excluded. The case $\mathcal{R} = \mathcal{S} = 0$ will be considered in subsection 4.1.3.

From (47) we obtain for $\mathcal{R} \neq 0$

$$\frac{F_{\beta\beta}}{F_\beta} = -\frac{\mathcal{S}}{\mathcal{R}} = H. \tag{49}$$

Since $V = F(\alpha, \beta)$, the ratio $F_{\beta\beta}/F_\beta$ in (49) must depend only on α, β. Thus, we have $H = H(\alpha, \beta)$. To this end, the necessary and sufficient condition is

$$l_1H_x + l_2H_y + l_3H_z = 0, \tag{50}$$

where l_1, l_2, l_3 were defined in (33). Condition (50) is the third necessary one for α, β so that the problem has such a solution.

Now, we integrate (49) with respect to β and get

$$F_\beta = w(\alpha)\exp\left[\int^\beta H(\alpha, b)\,\mathrm{d}b\right], \tag{51}$$

where w is an arbitrary function of one argument. We replace (51) into (45) and obtain

$$F_\alpha = \mathcal{D}(\alpha, \beta) w(\alpha)\exp\left[\int^\beta H(\alpha, b)\,\mathrm{d}b\right]. \tag{52}$$

The compatibility condition $F_{\alpha\beta} = F_{\beta\alpha}$ becomes

$$\frac{w'(\alpha)}{w(\alpha)} = \mathcal{D}H + \mathcal{D}_\beta - \int^\beta H_\alpha(\alpha, b)\,\mathrm{d}b = u. \tag{53}$$

Since the lhs of (53) depends only on α, the expression of u must depend only on α too. Thus, we have another condition on the orbital functions α, β, namely $\partial u/\partial\beta = 0$, or

$$\mathcal{D}_{\beta\beta} + \mathcal{D}_\beta H + \mathcal{D}H_\beta - H_\alpha = 0. \tag{54}$$

The differential condition (54) is the fourth one which must be satisfied by the pre-assigned set of functions α, β. Then the arbitrary function $w(\alpha)$ is determined as

$$w(\alpha) = d_1\,\mathrm{e}^{\int u(\alpha)\mathrm{d}\alpha}, \qquad d_1 = \text{const}. \tag{55}$$

At last, the potential function will be calculated from (51) and (52).

We shall examine now some special cases which can arise for the orbital functions α and β.

4.1.1. $\mathcal{P} = 0$ and $\mathcal{Q} = 0$. We have to deal with equation (42) alone and try to find its suitable solutions.

Keeping in mind that $\alpha_0 \neq 0$, we have $\mathcal{P} = 0$ and $\mathcal{Q} = 0$ if and only if

$$\beta\alpha_x - \alpha_z = \frac{\beta_0}{\alpha_0}(\alpha\alpha_x - \alpha_y), \qquad \beta\beta_x - \beta_z = \frac{\beta_0}{\alpha_0}(\alpha\beta_x - \beta_y). \tag{56}$$

Not all the pairs α and β which satisfy (56) are suitable for our problem. For example, let α and β satisfy $\beta\alpha_x - \alpha_z = 0$, $\alpha\alpha_x - \alpha_y = 0$, $\beta\beta_x - \beta_z = 0$ and $\alpha\beta_x - \beta_y = 0$. The matrix

$$\begin{pmatrix} \alpha_x & \alpha_y & \alpha_z \\ \beta_x & \beta_y & \beta_z \end{pmatrix} = \begin{pmatrix} \alpha_x & \alpha\alpha_x & \beta\alpha_x \\ \beta_x & \alpha\beta_x & \beta\beta_x \end{pmatrix}$$

does not have the rank equal to two, so α and β are not functionally independent.

Nevertheless, there are functionally independent functions with $\alpha_0 \neq 0$ which satisfy (56), as for example

$$\alpha = \frac{y((yz-1)^2 + x^2y^2)}{x(yz-1)}, \qquad \beta = \frac{xy}{1-yz}.$$

4.1.2. $\Gamma_{11} = 0$, $\Gamma_{12} = 0$ and $\Gamma_{22} = 0$. We have $\Gamma_{11} = 0$, $\Gamma_{12} = 0$ and $\Gamma_{22} = 0$ if and only if

$$\alpha\alpha_x - \alpha_y = 0 \qquad \text{and} \qquad \alpha\beta_x - \beta_y = 0. \tag{57}$$

It follows that σ and ρ, defined in (19), respectively in (44), are both zero. Equation (42) is now of first order, namely

$$\alpha_0 F_\alpha + \beta_0 F_\beta = 0. \tag{58}$$

Equation (45) reads

$$(\beta\alpha_x - \alpha_z)F_\alpha + (\beta\beta_x - \beta_z)F_\beta = 0. \tag{59}$$

In view of (57), we have $\alpha_0 = (1+\alpha^2)\alpha_x + \beta\alpha_z$ and $\beta_0 = (1+\alpha^2)\beta_x + \beta\beta_z$. We substitute these values into (60) and obtain that $\Theta(\alpha_z\beta_x - \alpha_x\beta_z) = 0$; hence

$$\alpha_z\beta_x - \alpha_x\beta_z = 0. \tag{61}$$

From (57) we obtain that $\alpha_x\beta_y = \alpha\alpha_x\beta_x = \alpha_y\beta_x$. This together with (61) implies that

$$\operatorname{rank}\begin{pmatrix}\alpha_x & \alpha_y & \alpha_z\\ \beta_x & \beta_y & \beta_z\end{pmatrix} = 1$$

and α and β are functionally dependent. Therefore in the case $\Gamma_{11} = 0$, $\Gamma_{12} = 0$ and $\Gamma_{22} = 0$ we do not obtain acceptable solutions for our problem.

4.1.3. $\mathcal{R} = 0$ and $\mathcal{S} = 0$. For α and β with $\mathcal{R} = 0$ and $\mathcal{S} = 0$, the function F has to satisfy only the first-order equation (45). On the other hand, α and β are bound to fulfil several differential conditions.

From (48) we have $\mathcal{R} = 0$ if and only if either $\alpha\alpha_x - \alpha_y = \alpha\beta_x - \beta_y = 0$, or $\mathcal{D} = -(\alpha\beta_x - \beta_y)/(\alpha\alpha_x - \alpha_y)$, or $\mathcal{D} = -\beta_0/\alpha_0$. For the first situation we obtain $\Gamma_{11} = \Gamma_{12} = \Gamma_{22} = 0$ (see section 4.1.2), which contradicts the actual supposition that equation (42) is of second order.

It follows that if $\mathcal{R} = 0$ and $\mathcal{S} = 0$, the functions α and β must satisfy in addition to (36) and (41), two relations obtained from (40) and $\mathcal{S} = 0$ ($\mathcal{S}$ is given in (48)):

$$\mathcal{P}(\alpha\beta_x - \beta_y) - \mathcal{Q}(\alpha\alpha_x - \alpha_y) = 0 \qquad \text{and}$$
$$(\alpha\alpha_x - \alpha_y)(\alpha_0\mathcal{D}_\alpha + \beta_0\mathcal{D}_\beta) + \left(\sigma + 2\alpha_0^2/\Theta\right)\mathcal{D} + \rho + 2\alpha_0\beta_0/\Theta = 0;$$

or

$$\mathcal{P}\beta_0 - \mathcal{Q}\alpha_0 = 0 \qquad \text{and}$$
$$((\alpha\alpha_x - \alpha_y)\mathcal{D} + \alpha\beta_x - \beta_y)(\alpha_0\mathcal{D}_\beta + K/(\Theta\alpha_0)) + (\alpha\alpha_x - \alpha_y)(\alpha_0\mathcal{D}_\alpha + \beta_0\mathcal{D}_\beta) + \sigma\mathcal{D} + \rho = 0.$$

Now, we can formulate the following

Proposition 2. *If $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$ are different from zero and the four conditions (36), (41), (50) and (54) are satisfied for the given orbital functions α and β, then a potential of the form $V = F(\alpha, \beta)$ always exists and it is determined uniquely from (51) and (52) up to two arbitrary constants.*

Synthesis of the problem

(1) We start with the functions α and β (functionally independent), or with f and g, and then obtain α, β from (6).
(2) We calculate the expression in (36); if it is different from zero, the problem has no solution.
(3) If (36) is satisfied, we calculate $\mathcal{P}$ and $\mathcal{Q}$ from (39). If $\mathcal{P} = 0$ and $\mathcal{Q} \neq 0$, or $\mathcal{P} \neq 0$ and $\mathcal{Q} = 0$, there is no solution V depending on both α and β. If $\mathcal{P} = 0$ and $\mathcal{Q} = 0$, we have at our disposal only equation (42) and we must consider it directly (see subsection 4.1.1).
(4) If $\mathcal{P} \neq 0$ and $\mathcal{Q} \neq 0$, we calculate $\mathcal{D}$ and verify (41). If it is not satisfied, we have no solution.
(5) If (41) is satisfied, we consider equation (42). If $\Gamma_{11} = 0$, $\Gamma_{12} = 0$ and $\Gamma_{22} = 0$, this is a first-order equation. The system of homogeneous equations (42) and (38) has no acceptable solution (see subsection 4.1.2).

(6) If (42) is of second order, we calculate $\mathcal{R}$ and $\mathcal{S}$. If $\mathcal{R} = 0$ and $\mathcal{S} \neq 0$, we have no acceptable solution.
(7) If $\mathcal{R} = 0$ and $\mathcal{S} = 0$, we have to deal only with equation (45) and find its solutions directly (see subsection 4.1.3).
(8) If $\mathcal{R} \neq 0$, we determine H and verify condition (50). If it is not satisfied, we have no solution.
(9) If (50) is true, we verify (54). If it is not satisfied, we have no solution.
(10) If (54) is verified, we obtain w from (55), then from (51) and (52) we determine $F(\alpha, \beta)$.

4.2. Examples

In this section we shall present pertinent examples which cover the general and the special cases too. Let us begin with

Example 1. We consider the two-parametric family of regular orbits

$$f(x, y, z) = xy = c_1, \qquad g(x, y, z) = z/x = c_2;$$

the orbital functions are

$$\alpha = -\frac{y}{x}, \qquad \beta = \frac{z}{x}, \tag{62}$$

and $\alpha_0 = 2y/x^2$, $\beta_0 = 0$. We checked that the four differential conditions are satisfied for this set of regular orbits. From (51) and (52), we have

$$F_\beta = \frac{\alpha^2 \beta}{(1 - \alpha^2 + \beta^2)^2}, \qquad F_\alpha = -\frac{\alpha(1 + \beta^2)}{(1 - \alpha^2 + \beta^2)^2}. \tag{63}$$

The potential is determined uniquely from (63)

$$V = F(\alpha, \beta) = -\frac{\alpha^2}{2(1 - \alpha^2 + \beta^2)}$$

or, equivalently,

$$V(x, y, z) = -\frac{y^2}{2(x^2 - y^2 + z^2)}. \tag{64}$$

We mention that $V = w(r, y) = y^2/2(y^2 - r^2)$, with $r = \sqrt{x^2 + z^2}$, is an axially symmetric potential.

The family (62) is traced isoenergetically by a test particle of unit mass under the influence of the potential (64) (with energy $\mathcal{E} = -1/4$) in the domain of the xyz space defined by $-\sqrt{x^2 + z^2} \leqslant y \leqslant \sqrt{x^2 + z^2}$.

Example 2. For the family

$$f(x, y, z) = y^2/x = c_1, \qquad g(x, y, z) = z/x^2 = c_2,$$

the orbital functions are

$$\alpha = \frac{y}{2x}, \qquad \beta = \frac{2z}{x}, \tag{65}$$

and $\alpha_0 = -y/4x^2$, $\beta_0 = 2z/x^2$. The four differential conditions are satisfied and

$$F_\beta = -\frac{2\beta(3\alpha^2 + 1)}{(4\alpha^2 + \beta^2 + 2)^2}, \qquad F_\alpha = \frac{2\alpha(3\beta^2 + 2)}{(4\alpha^2 + \beta^2 + 2)^2}. \tag{66}$$

The potential is determined uniquely from (66) $V = F(\alpha, \beta) = (3\alpha^2 + 1)/(4\alpha^2 + \beta^2 + 2)$ or, equivalently,

$$V(x, y, z) = \frac{4x^2 + 3y^2}{4(2x^2 + y^2 + 4z^2)}. \tag{67}$$

The family (65) is traced isoenergetically by a test particle of unit mass under the influence of the potential (67) (with energy $\mathcal{E} = 1$). Inequality (9) is $2x^2/(2x^2 + y^2 + 4z^2) \geqslant 0$.

The above potentials (64) and (67) are homogeneous of zero-degree, since they depend on the orbital functions α, β which are in this case homogeneous of zero-degree.

Example 3. For the family

$$f = (e_1 x + y + e_1)\,\mathrm{e}^{-x} = c_1, \qquad g = (e_2 x + y + e_2)\,\mathrm{e}^{-x} = c_2,$$

the orbital functions are

$$\alpha = e_1 x + y, \qquad \beta = e_2 x + z, \qquad e_1, e_2 = \text{const}, \tag{68}$$

with $\alpha_0 = \alpha + e_1$, $\beta_0 = \beta + e_2$. The three conditions (36), (41), (50) are satisfied for any values of e_1, e_2 but the fourth one, namely condition (54), is satisfied only in the case $1 + e_1^2 + e_2^2 = 0$. If we admit complex families, we shall obtain a complex potential (see Contopoulos and Bozis (2000)).

Counter-examples

- For $\alpha = (x + z^2)/(1 - y)$, $\beta = -2z$ functionally independent we have $\mathcal{P} = 0$ and $\mathcal{Q} = -2\alpha_0$. For $\alpha = -2xy/(y^2 - z)$, $\beta = x/(y^2 - z)$ we obtain $\mathcal{P} = -2x\beta_0/(y^2 - z)$ and $\mathcal{Q} = 0$ (case 3 in synthesis of the problem, subsection 4.1).
- For the family of regular orbits $f(x, y, z) = z/x = c_1$, $g(x, y, z) = x^2 + y^2 = c_2$, the orbital functions are

$$\alpha = -\frac{x}{y}, \qquad \beta = \frac{z}{x}, \tag{69}$$

and $\alpha_0 = -(x^2 + y^2)/y^3$, $\beta_0 = 0$. Conditions (36), (41), (50) are fulfilled, but not the last condition (54).

5. The case of straight lines

As was shown by Bozis and Kotoulas (2004), the family (1) consists of straight lines if and only if the orbital functions satisfy

$$\alpha_x + \alpha\alpha_y + \beta\alpha_z = 0, \qquad \beta_x + \alpha\beta_y + \beta\beta_z = 0. \tag{70}$$

The equations for the potential V are (Bozis and Kotoulas 2004, Anisiu 2004b)

$$\alpha V_x - V_y = 0, \qquad \beta V_x - V_z = 0, \tag{71}$$

as it follows easily from (8) in view of (70). We substitute $V = v(\alpha)$ into (71), and recall that $v'(\alpha)$ is not zero on open sets. It follows that $\alpha\alpha_x - \alpha_y = 0$ and $\beta\alpha_x - \alpha_z = 0$; we insert $\alpha_y = \alpha\alpha_x$ and $\alpha_z = \beta\alpha_x$ in the first equation from (70) and get $\alpha_x = 0$, hence also $\alpha_y = \alpha_z = 0$. This is not possible for a nonconstant function α, hence no potential $V = v(\alpha)$ can be compatible with a family of straight lines (70).

Now we look for a solution $V = F(\alpha, \beta)$ of (71), with α, β functionally independent. We get the system

$$\begin{aligned} (\alpha\alpha_x - \alpha_y)F_\alpha + (\alpha\beta_x - \beta_y)F_\beta &= 0 \\ (\beta\alpha_x - \alpha_z)F_\alpha + (\beta\beta_x - \beta_z)F_\beta &= 0, \end{aligned} \tag{72}$$

whose discriminant, in view of (70), is $\Delta = (1+\alpha^2+\beta^2)(\alpha_y\beta_z - \alpha_z\beta_y)$. If $\Delta \neq 0$, it follows that $V = \text{const}$, which is excluded. If $\Delta = 0$, when we eliminate the terms containing α from equations (70), we obtain $\alpha_x\beta_y - \alpha_y\beta_x = 0$. This condition, together with $\alpha_y\beta_z - \alpha_z\beta_y = 0$, shows that α and β must be functionally dependent, which contradicts our hypothesis. It follows that the families of straight lines are not compatible with potentials $V = F(\alpha, \beta)$.

6. Conclusions

In the present study we have examined a solvable version of the three-dimensional inverse problem of dynamics. Given a two-parametric family of orbits (1), the potential function V, which is compatible with it, must satisfy two free-of-energy PDEs (10) and (12). So, our aim is to find special common solutions of these two PDEs for the potential function $V = V(x, y, z)$.

In fact, instead of the two-parametric family (1), we deal with the pair (α, β) and try to find solutions of the form $V = v(\alpha)$ or $V = F(\alpha, \beta)$. We have to impose differential conditions on α and β in order that the problem has solutions of one of these forms. The conditions are not difficult to check once the analytical form of the orbital functions is known. When the conditions are fulfilled, we obtain by quadrature the potential as a function of α, or of α and β correspondingly. Afterwards, we can determine the function of energy dependence from (8) and the allowed area of motion of a test particle of unit mass from (9) (examples 1 and 2 in sections 3.2 and 4.2).

The theoretical results can be applied to a variety of real-world problems. For example, elliptic orbits have been observed in galaxy models (Miller and Smith (1979), figure 5, p 789). The two-parametric family of elliptic orbits

$$f(x, y, z) = x^2 + 2y^2 + z^2 = c_1, \qquad g(x, y, z) = z - x = c_2 \tag{73}$$

leads to the pair

$$\alpha = -\frac{x+z}{2y}, \qquad \beta = 1. \tag{74}$$

Adopting the methodology for potentials of the form $V = v(\alpha)$ which was presented in section 3, we find that the potential (homogeneous of zero-degree) $V(\alpha) = -\alpha^2(2+\alpha^2)/4$, or equivalently,

$$V(x, y, z) = -\frac{(x+z)^2(x^2+8y^2+z^2+2xz)}{64y^4} \tag{75}$$

produces this family of orbits.

Our hypothesis on the expression of the potential has allowed us to obtain pairs of families of curves and potentials under whose action they are traced in specified regions of the space. Thus we offer a method which can generate various examples, which were not numerous until now due to the complexity of calculations, substantiating the 3D inverse problem of dynamics. It can be apllied to various fields of physics where one disposes of information on particle motions, and the form of the potential is needed.

Acknowledgments

Thomas Kotoulas would like to thank the 'Tiberiu Popoviciu' Institute of Numerical Analysis, Cluj, Romania for the hospitality from 7 to 12 November 2005. His work was financially supported by the scientific program 'EPEAEK II, PYTHAGORAS', no 21878, of the Greek Ministry of Education and EU, and the work of Mira-Cristiana Anisiu by 2CEEX0611-96 of the Romanian Ministry of Education and Research. The authors express their gratitude to the Board Member and to the referees for helping to improve the quality of the paper.

References

Anisiu M-C 2004a An alternative point of view on the equations of the inverse problem of dynamics *Inverse Problems* **20** 1865–72

Anisiu M-C 2004b Two- and three-dimensional inverse problem of dynamics *Studia Univ. Babeş-Bolyai, Math.* XLIX(4) 13–26

Anisiu M-C 2005 The energy-free equations of the 3D inverse problem of dynamics *Inverse Probl. Sci. Eng.* **13** 545–58

Binney J and Tremaine S 1987 *Galactic Dynamics* (Princeton, NJ: Princeton University Press)

Boccaletti D and Puccaco G 1996 *Theory of Orbits 1: Integrable Systems and Non-perturbative Methods* (Berlin: Springer)

Bozis G 1983 Determination of autonomous three-dimensional force fields from a two-parametric family *Celest. Mech.* **31** 43–51

Bozis G 1984 Szebehely inverse problem for finite symmetrical material concentrations *Astron. Astrophys.* **134** 360–4

Bozis G 1995 The inverse problem of dynamics: basic facts *Inverse Problems* **11** 687–708

Bozis G and Anisiu M-C 2005 A solvable version of the inverse problem of dynamics *Inverse Problems* **21** 487–97

Bozis G and Caranicolas N D 1997 Potentials creating subfamilies of Keplerian ellipses *The Earth and the Universe Thessaloniki Ziti Editions,* ed G Asteriades, A Bantelas, M E Contadakis, K Katsambalos, A Papolimitriou and I N Tziavos (Thessaloniki: University of Thessaloniki) pp 245–52

Bozis G and Ichtiaroglou S 1994 Boundary curves for families of planar orbits *Celest. Mech. Dyn. Astron.* **58** 371–85

Bozis G and Kotoulas T 2004 Three-dimensional potentials producing families of straight lines (FSL) *Rend. Semin. Facolta Sci. Univ. Cagliari* **74** fasc. 1-2 83-99

Bozis G and Kotoulas T 2005 Homogeneous two-parametric families of orbits in three-dimensional homogeneous potentials *Inverse Problems* **21** 343–56

Bozis G and Nakhla A 1986 Solution of the three-dimensional inverse problem *Celest. Mech.* **38** 357–75

Caranicolas N D 1998 Potentials for the central parts of barred galaxies *Astron. Astrophys.* **332** 88–92

Caranicolas N D 2004 Orbits in global and local potentials *Astron. Astrophys. Trans.* **23** 241–52

Contopoulos G 1960 A third integral of motion in a galaxy *Z. Astrophys.* **49** 273–91

Contopoulos G and Bozis G 2000 Complex force fields and complex orbits *J. Inverse Ill-Posed Probl.* **8** 147–60

Érdi B 1982 A generalization of Szebehely's equation for three dimensions *Celest. Mech.* **28** 209–18

Favard J 1963 *Cours d' Analyse de l' Ecole Polytechnique* Tome 3^2 (Paris: Gauthier-Villars) p 39

Heiss W D Nazmitdinov R G and Radu S 1994 Chaos in axially symmetric potentials with Octupole deformation *Phys. Rev. Lett.* **72** 2351–4

Miller R H and Smith B F 1979 Dynamics of a stellar bar *Astrophys. J.* **227** 785–97

Olaya-Castro A and Quiroga L 2000 Bose–Einstein condensation in an axially symmetric mesoscopic system *Phys. Status Solidi* b **220** 761–4

Puel F 1984 Formulation intrinseque de l'equation de Szebehely *Celest. Mech.* **32** 209–16

Shorokhov S G 1988 Solution of an inverse problem of the dynamics of a particle *Celest. Mech.* **44** 193–206

Smirnov V I 1964 *A Course of Higher Mathematics* vol IV (Oxford: Pergamon) p 359

Szebehely V 1974 On the determination of the potential by satellite observations *Proc. Int. Meeting on Earth's Rotation by Satellite Observation* ed G Proverbio (Bologna: The University of Cagliari) pp 31–5

Váradi F and Érdi B 1983 Existence of the solution of Szebehely's equation in three dimensions using a two-parametric family of orbits *Celest. Mech.* **32** 395–405