Families of orbits in planar anisotropic fields

M.-C. ANISIU and G. BOZIS

[1] T. Popoviciu Institute of Numerical Analysis, Romanian Academy, PO Box 68, 400110 Cluj-Napoca, Romania
[2] Department of Physics, University of Thessaloniki, 54006 Thessaloniki, Greece

Received 13 May 2004; accepted 22 June 2004; published online 30 December 2004

Abstract. The aim of the planar inverse problem of dynamics is: given a monoparametric family of curves $f(x, y) = c$, find the potential $V(x, y)$ under whose action a material point of unit mass can describe the curves of the family. In this study we look for V in the class of the anisotropic potentials $V(x, y) = v(a^2x^2 + y^2)$, ($a$=constant). These potentials have been used lately in the search of connections between classical, quantum, and relativistic mechanics. We establish a general condition which must be satisfied by all the families produced by an anisotropic potential. We treat special cases regarding the families (e. g. families traced isoenergetically) and we present certain pertinent examples of compatible pairs of families of curves and anisotropic potentials.

Key words: celestial mechanics – stellar dynamics

1. Introduction

The planar inverse problem of dynamics consists in finding potentials $V(x, y)$ which can produce as orbits a preassigned monoparametric family of curves traced in the xy plane by a material point of unit mass. The partial differential equations satisfied by the potential, which will be described below, do not provide uniqueness for V. That is why it is desirable to look for solutions in specific classes of potentials.

In this paper we shall focus on anisotropic potentials, which appear in various mathematical models arising in Astronomy and Physics. We mention e.g. some of the first results associated to the anisotropic two-body problem for the Newtonian potential (Gutzwiller 1971; Will 1971; Vinti 1972), for the Manev potential (Craig et al. 1999) and for the Schwarzschild one (Mioc, Pérez-Chavela & Stavinschi 2003). The importance of these anisotropic models comes also from the fact that they were used in the search of connections between classical, quantum, and relativistic mechanics.

The potential $V(x, y)$ which can generate the family

$$f(x, y) = c \tag{1}$$

traced with energy $E = E(f)$ is given by a partial differential equation due to Szebehely (1974). In terms of the 'slope function' γ, introduced by Bozis (1983) and given by

$$\gamma = \frac{f_y}{f_x}, \tag{2}$$

this equation reads

$$V_x + \gamma V_y + \frac{2\Gamma}{1+\gamma^2}\left(E(f) - V\right) = 0. \tag{3}$$

The subscripts denote partial derivatives.

Remark 1. The linearity in V of Szebehely's equation implies that if V_i are solutions of (3) with the energy $E_i(f)$, $i = 1, 2$, then $V = V_1 + V_2$ is a solution of (3) with the energy $E(f) = E_1(f) + E_2(f)$; if V_1 is a solution of (3) with the energy $E_1(f)$, then c_1V_1 is also a solution with the energy $c_1E_1(f)$, and $c_1V_1 + c_2$ is a solution with the energy $c_1E_1(f) + c_2$.

We remark that relation (2) provides a one-to-one correspondence between γ and the family (1).

The function Γ is proportional to the curvature of the family (1) and its value is

$$\Gamma = \gamma\gamma_x - \gamma_y. \tag{4}$$

The families of straight lines are characterized by $\Gamma = 0$. The potentials producing such a family must satisfy the equation (Bozis & Anisiu 2001)

$$V_xV_y\left(V_{xx} - V_{yy}\right) = V_{xy}\left(V_x^2 - V_y^2\right). \tag{5}$$

For the families (1) with $\Gamma \neq 0$, Bozis (1984) obtained a free of energy second order partial differential equation

$$-V_{xx} + \kappa V_{xy} + V_{yy} = \lambda V_x + \mu V_y, \tag{6}$$

where

$$\kappa = \frac{1}{\gamma} - \gamma,\ \lambda = \frac{1}{\gamma\Gamma}\left(\Gamma_y - \gamma\Gamma_x\right),\ \mu = \lambda\gamma + \frac{3\Gamma}{\gamma}. \tag{7}$$

Correspondence to: mira@math.ubbcluj.ro

Real motion on the curves of the family (1) is possible only in the region (Bozis & Ichtiaroglou 1994) defined by the inequality

$$\frac{V_x + \gamma V_y}{\Gamma} \leq 0. \tag{8}$$

Basic facts on the inverse problem of dynamics are to be found in Bozis (1995) and Anisiu (2003).

2. The inverse problem equations for anisotropic potentials

An anisotropic potential V is given by

$$V(x, y) = v\left(a^2x^2 + y^2\right), \tag{9}$$

where a is a real number, $a \notin \{-1, 0, 1\}$. For $a^2 = 1$, the potential becomes isotropic. We shall exclude the trivial case of constant potentials. As it was emphasized in Remark 1, if V is a solution of the inverse problem, so is $c_1V_1 + c_2$; the constant c_2 will be omitted, but c_1 can be chosen adequately to obtain simpler expressions, or, in view of (8), a suitable region of the plane for the orbits to lie.

Having no a priori information on the energy $E(f)$, we intend to rely on eq. (6). So we shall settle at first the case $\Gamma = 0$, corresponding to families (1) of straight lines. Substituting V from (9) into (5) we obtain

$$\left(a^2 - 1\right) v' = 0, \tag{10}$$

where the prime denotes the derivative of v with respect to its unique argument $w = a^2x^2 + y^2$. It follows that no anisotropic potential allows for families of straight lines.

In what follows we shall consider $\Gamma \neq 0$. We substitute V from (9) in eq. (6) and obtain the equation in v

$$\begin{aligned} &2\left(-a^4x^2 + a^2xy\kappa + y^2\right) v'' = \\ &\left(a^2x\lambda + y\mu + a^2 - 1\right) v', \end{aligned} \tag{11}$$

where κ, λ, μ are given by (7) in terms of γ and its derivatives up to the second order.

Remark 2. For the two families of orbits $\gamma_1 = y/(a^2x)$ and $\gamma_2 = -a^2x/y$ the coefficient of v'' in (11) becomes identically zero. For each of them, eq. (11) reduces to $\left(a^2 - 1\right) v' = 0$, hence no nontrivial anisotropic potential (9) gives rise to such families.

Remark 3. If γ is a solution of

$$a^2x\lambda + y\mu + a^2 - 1 = 0, \tag{12}$$

eq. (11) becomes $v'' = 0$ and from (9) we obtain the potential $V(x, y) = c_1\left(a^2x^2 + y^2\right)$, which produces the family of curves corresponding to γ.

From now on we shall exclude from our study the family of conics $f_1(x, y) = a^2x^2 + y^2$ and the family $f_2(x, y) = x^{-1/a^2}y$, corresponding to γ_1, γ_2 in Remark 2.

Equation (11) can be written as

$$\frac{v''}{v'} = U(x, y; a) \tag{13}$$

where

$$U(x, y; a) = \frac{a^2x\lambda + y\mu + a^2 - 1}{2\left(-a^4x^2 + a^2xy\kappa + y^2\right)}. \tag{14}$$

The condition for (13) to admit of a solution of the form (9) is

$$yU_x - a^2xU_y = 0. \tag{15}$$

It follows that the families of curves which can be generated under the action of an anisotropic potential are those determined by the solutions γ of the differential relation (15). For such a γ we have $U(x, y; a) = u\left(a^2x^2 + y^2\right)$, and from (13) we get

$$v = c_1 \int \exp\left(\int u\right) + c_2. \tag{16}$$

Working on condition (15), we obtain

$$x^4\lambda_y a^8 + s_6a^6 + s_4a^4 + s_2a^2 + y^4\mu_x = 0 \tag{17}$$

where

$$\begin{aligned} &s_6 = x\left(x^2y\left(\lambda\kappa_y - \kappa\lambda_y + \mu_y - \lambda_x\right) + x^2\left(\kappa\lambda + \mu\right)\right. \\ &\left. +xy\left(\lambda + \kappa_y\right) + x\kappa + 2y\right) \\ &s_4 = x^2y^2\left(\kappa\lambda_x - \lambda\kappa_x + \mu\kappa_y - \kappa\mu_y - \lambda_y - \mu_x\right) \\ &+x^2y\left(2\lambda - \kappa_y\right) + xy^2\left(2\mu - \kappa_x\right) - \left(x^2 + y^2\right)\kappa \\ &s_2 = y\left(xy^2\left(\kappa\mu_x - \mu\kappa_x + \lambda_x - \mu_y\right) + y^2\left(\lambda - \kappa\mu\right)\right. \\ &\left. +xy\left(\mu + \kappa_x\right) - 2x + y\kappa.\right) \end{aligned} \tag{18}$$

The coefficients in (18) are expressed in terms of the functions κ, λ, μ from (7) and of their first order partial derivatives. Equation (17) represents the necessary condition to be fulfilled by a family (1) in order to be produced by an anisotropic potential.

Remark 4. For $a = 1$ the condition (17) reduces to

$$\begin{aligned} &xy\left(x\lambda + y\mu\right)\left(x\kappa_y - y\kappa_x\right) \\ &+\left(x^2 - xy\kappa - y^2\right)\left(x\left(x\lambda_y - y\lambda_x\right) + y\left(x\mu_y - y\mu_x\right)\right) \\ &+\left(\kappa x^3 + 3x^2y + y^3\right)\lambda + \left(x^3 + 3xy^2 - \kappa y^3\right)\mu = 0. \end{aligned} \tag{19}$$

Equation (19) gives the totality of families (1) produced by central potentials $V = v(r)$, $r = \left(x^2 + y^2\right)^{1/2}$ and is in agreement with pertinent findings by Borghero, Bozis & Melis (1999).

3. The two-dimensional anisotropic harmonic oscillator

The potential $V(x, y) = \left(a^2x^2 + y^2\right)/2$, analyzed in detail by Iro (2002), is one of the simplest anisotropic potentials. In this case the equations of motion

$$\begin{aligned} &\ddot{x} + a^2x = 0 \\ &\ddot{y} + y = 0 \end{aligned} \tag{20}$$

are not coupled, and the solutions for the initial values $x_0 = b_1$, $y_0 = b_3$, $\dot{x}_0 = ab_2$, $\dot{y}_0 = b_4$ are

$$\begin{aligned} &x(t) = b_1 \cos at + b_2 \sin at \\ &y(t) = b_3 \cos t + b_4 \sin t. \end{aligned} \tag{21}$$

It is known that the motion in the configuration plane consists of Lissajous' figures. If a is rational, the orbit is closed; for a irrational, the orbit fills entirely a region of the plane.

We try to obtain families of orbits by eliminating t between the two equations (21), hence we shall consider $a = q/s$, q, s natural numbers.

From the first equation in (21) and its derivative with respect to t we get $\cos at = (ab_1 x + b_2 \dot{x}) / \left(a\left(b_1^2 + b_2^2\right)\right)$ and $\sin at = (ab_2 x - b_1 \dot{x}) / \left(a\left(b_1^2 + b_2^2\right)\right)$, hence

$$t = \frac{1}{a} \arctan \frac{ab_2 x - b_1 \dot{x}}{ab_1 x + b_2 \dot{x}}. \tag{22}$$

Similarly, from the second equation in (21) and its derivative we get

$$t = \arctan \frac{b_4 y - b_3 \dot{y}}{b_3 y + b_4 \dot{y}}, \tag{23}$$

and, eliminating t between the two equations (22) and (23) we obtain

$$a \arctan \frac{b_4 y - b_3 \dot{y}}{b_3 y + b_4 \dot{y}} - \arctan \frac{ab_2 x - b_1 \dot{x}}{ab_1 x + b_2 \dot{x}} = 0. \tag{24}$$

The energy is conserved in each direction, i. e.

$$\dot{x}^2 + a^2 x^2 = a^2 \left(b_1^2 + b_2^2\right), \; \dot{y}^2 + y^2 = b_3^2 + b_4^2. \tag{25}$$

The relations (25) allow us to eliminate $\dot{x}$ and $\dot{y}$ from (24).

In so doing, we obtain a family of orbits which, except for a, includes (not independently, of course) the four constants b_1, b_2, b_3, b_4. Indeed, from (2) we find

$$\gamma = -\frac{a\sqrt{c_1^2 - x^2}}{\sqrt{c_2^2 - y^2}}, \tag{26}$$

where

$$c_1^2 = b_1^2 + b_2^2, \quad c_2^2 = b_3^2 + b_4^2, \tag{27}$$

and from (3)

$$E = \frac{1}{2}\left(a^2 c_1^2 + c_2^2\right). \tag{28}$$

For each a, (26) offers a two-parametric set of slope functions γ (i.e. of monoparametric families) compatible with $V = \left(a^2 x^2 + y^2\right)/2$ and, as expected in view of the Remark 3, with eq. (12) as well. For specific values of a, c_1, c_2 the monoparametric (in c) family corresponding to (26) is

$$f(x, y) = q \arcsin \frac{y}{c_2} - s \arcsin \frac{x}{c_1} = c. \tag{29}$$

In fact (29) is three-parametric and includes *all* orbits produced by the anisotropic harmonic oscillator. Of course, out of the set (29), one may extract e.g. infinitely many two-parametric subsets by imposing *any* relation $\varphi(c, c_1, c_2) = 0$ between the three parameters c, c_1, c_2. However, solving for either of the remaining two parameters is neither an easy nor always an accomplishable task. For this reason we may search *directly* for particular solutions of (12). In this manner we found (with b constant) e.g. for $a = \pm 2$,

$$\begin{array}{l} \gamma = \frac{b-4x}{2y},\ f = \frac{y^2}{4x-b},\ E = -bf + \frac{b^2}{8},\ \frac{4x+b}{4x-b} \le 0, \text{ and} \\ \gamma = by,\ f = by^2 + 2x,\ E = \frac{f^2}{2} + \frac{f}{b},\ \frac{4x+by^2}{b} \ge 0; \end{array} \tag{30}$$

for: $a = \pm\frac{1}{2}$,

$$\gamma = \frac{b - x^2}{2xy},\ f = \frac{y}{x^2 - b},\ E = \frac{b^2}{2} f^2 + \frac{b}{4},\ x^2 - 2b \le 0. \tag{31}$$

Beside the family, we gave the energy dependence E and the region (8) where real motion is allowed.

The potential of the harmonic oscillator being homogeneous of order two, it is natural to look for compatible homogeneous families (for which γ is homogeneous of order zero),

$$\gamma = g(z) \text{ with } z = y/x. \tag{32}$$

In this case, eq. (12) reads

$$\begin{array}{l} \left(zg + a^2\right)(zg + 1)\ddot{g} + \left(2zg + 3a^2 - 1\right) g\dot{g} \\ -z\left(2zg - a^2 + 3\right)\dot{g}^2 = 0, \end{array} \tag{33}$$

where the dot denotes the derivative with respect to z.

From the solutions of the form $g(z) = bz^m$ the only ones which verify (33) (with $a^2 \neq 1$) are obtained for $m = -1$ and are given by $g = \pm a/z$; they correspond to the families

$$f(x, y) = x^{\mp 1/a} y, \tag{34}$$

traced with zero energy. The inequality (8) becomes $y^2 \le 0$, which means that real motion is not allowed on the curves of the family (34) under the action of the potential of an anisotropic harmonic oscillator; these curves can be described under a potential with opposite sign, namely $V = -\left(a^2 x^2 + y^2\right)/2$.

Specifying a, we can obtain further solutions of (33), which give rise to families f traced with the specified energy, under the action of the potential of the harmonic oscillator, in certain regions of the plane (again b denotes a constant):

$$\begin{array}{l} a = \pm 2: \\ \gamma = -\frac{2}{z} + bz,\ f = \frac{by^2 - x^2}{y^4},\ E = \frac{b+b^2}{2f},\ \frac{1}{x^2 - by^2} \le 0; \\ a = \pm 3: \\ \gamma = b - \frac{3}{z},\ f = \frac{2x - by}{y^3},\ E = -\frac{4b+b^3}{6f},\ \frac{6x+by}{2x-by} \le 0; \\ a = \pm\frac{1}{3}: \\ \gamma = -\frac{b}{3(bz-1)},\ f = \frac{2by-3x}{x^3},\ E = -\frac{9+4b^2}{18b^2 f},\ \frac{x+2by}{3x-2by} \ge 0. \end{array} \tag{35}$$

4. Isoenergetic families

The isoenergetic families have the total energy $E(f) = e$, e being a constant which can be considered zero. For the anisotropic potential V given by (9), Szebehely's equation (3) can be written as

$$\frac{v'}{v} = \frac{\Gamma}{(y\gamma + a^2 x)(\gamma^2 + 1)}. \tag{36}$$

The condition that the right hand side is a function of $a^2x^2 + y^2$ reads

$$\begin{array}{l} y\left(\frac{\left(y\gamma + a^2 x\right)\left(\gamma^2 + 1\right)}{\Gamma}\right)_x \\ -a^2 x\left(\frac{\left(y\gamma + a^2 x\right)\left(\gamma^2 + 1\right)}{\Gamma}\right)_y = 0, \end{array}$$

or

$$\begin{array}{l} \left(y\gamma + a^2 x\right)\left(\gamma^2 + 1\right)\left(a^2 x \Gamma_y - y\Gamma_x\right) + \Gamma\left(-2a^4 x^2 \gamma\gamma_y\right. \\ +a^2\left(\left(\gamma^2 + 1\right)(y - x\gamma) + xy\left(2\gamma\gamma_x - \left(3\gamma^2 + 1\right)\gamma_y\right)\right) \\ \left.+y^2\left(3\gamma^2 + 1\right)\gamma_x\right) = 0. \end{array} \tag{37}$$

If γ satisfies the differential condition (37), the anisotropic potential will be given by $v = c_1 \exp\left(\int F\right)$, where $F(a^2x^2 + y^2) = \Gamma / \left(\left(y\gamma + a^2 x\right)\left(\gamma^2 + 1\right)\right)$.

It can be checked that $\gamma = y/x$, representing a family of homocentric circles

$$x^2 + y^2 = c, \tag{38}$$

is a solution of eq. (37). Equation (36) becomes $v'/v = -1/w$, with $w = a^2x^2 + y^2$ and has the solution $v = c_1/w$, corresponding to

$$V(x, y) = \frac{c_1}{a^2x^2 + y^2}. \tag{39}$$

The circles are traced isoenergetically all over the plane for $c_1 < 0$. The anisotropic potential V given by (39) is a member of the totality of homogeneous potentials, found by Borghero & Bozis (2002), which produce isoenergetically the family (38).

Remark 5. It is known that the Newtonian potential is compatible with the family of circles (38); it follows by Remark 1 that the perturbed Newtonian potential

$$V = -\frac{1}{\sqrt{x^2 + y^2}} + \frac{c_1}{a^2x^2 + y^2},\ c_1 < 0$$

can give rise to family (38) all over the plane with

$$E = -1/(2\sqrt{x^2 + y^2}).$$

5. Concluding remarks

We studied real anisotropic potentials $V(x, y) = v\left(a^2x^2 + y^2\right)$ in the light of the planar inverse problem of dynamics and mainly from the viewpoint of the monoparametric families (1) which they can produce. We established the general differential condition (17) which must be satisfied by all the families $\gamma = \gamma(x, y)$ compatible with such potentials. Written explicitly the condition (17) would include the slope function $\gamma(x, y)$ and partial derivatives of it up to the third order. As such, it is a highly nonlinear PDE in the unknown function $\gamma(x, y)$.

The (superintegrable case of the) two-dimensional anisotropic harmonic oscillator $V = \left(a^2x^2 + y^2\right)/2$ was reviewed from the same viewpoint. The pertinent three-parametric family of orbits was given by the eq. (29). There exist, of course, infinitely many ways of extracting out of (29) families with one or two parameters but this task is not always possible, depending on the value of the constant a at hand. For this reason, and for specific values of a, we found certain examples by *direct* reference to the PDE (6).

To aid the algebra, we assumed either homogeneity of the orbits (i. e. families of the form $\gamma = g(y/x)$) or isoenergeticity of the families (i. e. all the members of each family are traced with the same total energy, say $E = 0$). Sporadic findings are given by (30), (31) and (35), respectively.

References

Anisiu, M.-C.: 2003, in: V. Barbu et al. (eds.), *Analysis and Optimization of Differential Systems*, Kluwer Acad. Publ., Boston (MA), p. 13

Borghero, F., Bozis, G.: 2002, Mecc 37, 545

Borghero, F., Bozis, G., Melis, A.: 1999, Rendiconti Mat. Appl. 19, 303

Bozis, G.: 1983, CeMec 31, 129

Bozis, G.: 1984, A&A 134, 360

Bozis, G.: 1995, Inverse Problems 11, 687

Bozis, G., Anisiu, M.-C.: 2001, RoAJ 11, 27

Bozis, G., Ichtiaroglou, S.: 1994, CeMDA 58, 371

Craig, S., Diacu, F.N., Lacomba, E.A., Pérez, E.: 1999, JMP 40, 1359

Gutzwiller, M.C.: 1971, JMP 12, 343

Iro, H.: 2002, *A Modern Approach to Classical Mechanics*, World Scientific

Mioc, V., Pérez-Chavela, E., Stavinschi, M.: 2003, CeMDA 86, 81

Szebehely, V.: 1974, in: E. Proverbio (ed.), Proc. of the Internat. Meeting on Earth's Rotations by Satellite Observations, The University of Cagliari, Bologna, p. 31

Vinti, J.P.: 1972, CeMec 6, 198

Will, C.M.: 1971, ApJ 169, 141