INSTITUTE OF PHYSICS PUBLISHING INVERSE PROBLEMS

Inverse Problems **20** (2004) 1865–1872 PII: S0266-5611(04)74589-X

An alternative point of view on the equations of the inverse problem of dynamics

Mira-Cristiana Anisiu

'T Popoviciu' Institute of Numerical Analysis, Romanian Academy, PO Box 68, 400110 Cluj-Napoca, Romania

E-mail: mira@math.ubbcluj.ro

Received 12 January 2004, in final form 3 September 2004
Published 11 October 2004
Online at stacks.iop.org/IP/20/1865
doi:10.1088/0266-5611/20/6/011

Abstract
The version of the inverse problem of dynamics considered here is: given a family of planar curves $f(x, y) = c$, find the potentials $V(x, y)$ which give rise to this family. Its solution is based on two linear partial differential equations satisfied by V: one of first order, containing the total energy function $E(f)$, given by Szebehely in 1974, and the other one of second order, derived by Bozis in 1984 by eliminating the energy from Szebehely's equation. In this paper, Bozis' partial differential equation is obtained directly by eliminating the time derivatives of $x(t)$ and $y(t)$ up to the third order between seven differential relations based on the equations of motion and on the given family. Szebehely's equation is then derived as a consequence. This shows the importance of Bozis' equation, which is traditionally considered as following from Szebehely's one. The connection with the nonconservative case is emphasized.

1. Introduction

We consider the following version of the inverse problem for one material point of unit mass, moving in the xy inertial Cartesian plane. Given a family of curves

$$f(x, y) = c \tag{1}$$

with f of C^3-class (continuous and with continuous derivatives up to third order on a domain of the plane), find the potentials $V(x, y)$ under whose action, for appropriate initial conditions, the particle will describe the curves of that family. The equations of the motion are

$$\ddot{x} = -V_x, \qquad \ddot{y} = -V_y, \tag{2}$$

where the dots denote derivatives with respect to the time t, and the subscripts partial derivatives.

We emphasize that in this version of the inverse problem a *family* of curves (1) is given, which is in fact determined by the ratio f_y/f_x. Up to now, in the research connected to the inverse problem of dynamics, the families of curves were selected on the grounds of theoretical reasons: families of conic sections, of homogeneous functions or of other special analytic forms. It would be important to consider the inverse problem from the numerical viewpoint. An orbit will be obtained as a result of a curve-fitting process applied to some observed data. As Bozis and Blaga (2004) have shown, this single orbit can be classified in different monoparametric families of curves (1). A practical application would be to find the Newtonian potential of the nonspherical Earth from observed satellite orbits.

Therefore a family of curves (1) can be obtained either from theory or from measured data.

By making use of the energy integral, Szebehely (1974) proved that the desired potentials satisfy the first-order partial differential equation

$$f_x V_x + f_y V_y + \frac{2(V - E(f))}{f_x^2 + f_y^2}\left(f_{xx} f_y^2 - 2 f_{xy} f_x f_y + f_{yy} f_x^2\right) = 0, \tag{3}$$

where $E(f)$ denotes the total energy, which is constant on each curve of the family (1). Using the functions

$$\gamma = \frac{f_y}{f_x} \qquad \text{and} \qquad \Gamma = \gamma\gamma_x - \gamma_y, \tag{4}$$

Bozis (1983) wrote Szebehely's equation in the simpler form

$$V_x + \gamma V_y + \frac{2\Gamma\,(E(f) - V)}{1 + \gamma^2} = 0. \tag{5}$$

Remark 1. The function γ is related to the slope of the curves in family (1); more precisely, it represents the slope at each point of a family $f^*(x, y) = c^*$ which is orthogonal to family (1). The function Γ has also a geometrical meaning, the curvature K of the members of family (1) being given by $K = |\Gamma|/(1 + \gamma^2)^{3/2}$.

Under the action of a potential that satisfies equation (5), the curves (1) are traced by a material point only in the allowed region, defined by the inequality (Bozis and Ichtiaroglou 1994)

$$\frac{V_x + \gamma V_y}{\Gamma} \leqslant 0. \tag{6}$$

By eliminating the energy from (5) (using the fact that $E_y/E_x = f_y/f_x$), Bozis (1984) obtained the equation of second order which is energy free

$$-V_{xx} + \kappa V_{xy} + V_{yy} = \lambda V_x + \mu V_y, \tag{7}$$

where

$$\kappa = \frac{1}{\gamma} - \gamma, \qquad \lambda = \frac{\Gamma_y - \gamma\Gamma_x}{\gamma\Gamma}, \qquad \mu = \lambda\gamma + \frac{3\Gamma}{\gamma}. \tag{8}$$

The basic equations (5) and (7) of the inverse problem of dynamics present the connection between geometry (described by γ and Γ) and dynamics (the planar potential V). Their derivation and other related results are described by Bozis (1995) and by Anisiu (2003a, 2003b).

When we are facing an inverse problem related to the family of curves (1), we have to calculate the functions γ and Γ from (4) and afterwards plug them into equation (5); from (8) we get κ, λ and μ and insert them into (7). Therefore we have at our disposal two partial differential equations in the unknown function V. If we can get some information on the energy (e.g. if we are interested in isoenergetic families, with $E(f) = e =$ const, the case considered by Borghero and Bozis (2002)), we can use the first-order equation (5). Otherwise we are bound to work with the energy-free equation (7) in order to find the potentials (or at least some particular ones) which can give rise to the family of curves (1). The fact that equations (5) and (7) do not have a unique solution can be used to look for the potential in various classes of functions with physical significance, such as homogeneous or quasihomogeneous ones.

We remark that Szebehely (1974) obtained the first-order equation (3) intending to use it for the determination of the potential of the Earth by means of satellite observations, while Bozis (1984) used equation (7) to check if a given family of orbits may be generated in the plane of symmetry outside a material concentration.

In what follows we derive in a unified manner the two basic equations (5) and (7), as well as inequality (6). The special case of families of straight lines will also be treated.

2. Main results

Let us consider a particle whose motion is described by equations (2), where V is of C^2-class on a domain of the xy plane. We shall use a procedure similar to that followed by Kasner (1906) while he obtained the differential equation of the trajectories corresponding to a general (not necessarily conservative) force field. By differentiating (1) with respect to t we get $f_x\dot{x} + f_y\dot{y} = 0$, or, using the notation (4)

$$\gamma = -\frac{\dot{x}}{\dot{y}}. \tag{9}$$

By differentiating (9) we get $\gamma_x\dot{x} + \gamma_y\dot{y} = (\dot{x}\ddot{y} - \dot{y}\ddot{x})/\dot{y}^2$, or, using (4) again,

$$-\Gamma = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{y}^3}. \tag{10}$$

Inserting in (10) $\ddot{x}$ and $\ddot{y}$ from (2), and $\dot{x}$ from (9) we obtain

$$\Gamma\dot{y}^2 = -(V_x + \gamma V_y).$$

If $\Gamma = 0$ (which corresponds to a family (1) of straight lines, studied by Bozis and Anisiu (2001)) we have by necessity

$$V_x + \gamma V_y = 0, \tag{11}$$

which represents Szebehely's equation for this special case. The straight lines are traced with arbitrary energy.

Remark 2. The case of a family of straight lines appeared here as a special case in the mathematical reasoning. Another problem, namely that of Darboux integrability, revealed the importance of families of parallel or concurrent lines (Grigoriadou 1999). Isolated straight lines were found for the Hénon–Heiles model by Antonov and Timoshkova (1993) or van der Merwe (1991). Contopoulos and Zikides (1980), as well as Caranicolas and Innanen (1992), identified straight lines in galactic models.

Example 1. The central potential $V(x, y) = v(r)$, where $r = (x^2 + y^2)^{1/2}$, is compatible with the family of straight lines $\gamma = -x/y$ which can be described equivalently by $f(x, y) = y/x = c$ (Bozis and Anisiu 2001).

Let us consider now a general family (1) with $\Gamma \neq 0$. In this case we have

$$\dot{y}^2 = -\frac{V_x + \gamma V_y}{\Gamma}. \tag{12}$$

We differentiate (10), divide both members by $\dot{y}$ and get

$$\gamma\Gamma_x - \Gamma_y = \frac{\dot{y}(\dot{x}\dddot{y} - \dot{y}\dddot{x}) - 3\ddot{y}(\dot{x}\ddot{y} - \dot{y}\ddot{x})}{\dot{y}^5}. \tag{13}$$

As explained in remark 1, the functions γ and Γ represent the geometry of the family of curves (1). The formulae (9), (10) and (13) relate these geometrical entities to the kinematics derivatives, namely to the velocity and acceleration of the particle describing the curves of the family.

Two additional equations are obtained by differentiating equations (2) with respect to t, namely

$$\dddot{x} = -(V_{xx}\dot{x} + V_{xy}\dot{y}), \qquad \dddot{y} = -(V_{xy}\dot{x} + V_{yy}\dot{y}). \tag{14}$$

Now we eliminate the derivatives $\dot{x}, \dot{y}, \ddot{x}, \ddot{y}, \dddot{x}, \dddot{y}$ between the seven relations in (2), (9), (12), (13) and (14), and get

$$\Gamma(-\gamma V_{xx} + V_{xy} - \gamma^2 V_{xy} + \gamma V_{yy}) = -(V_x + \gamma V_y)(\gamma\Gamma_x - \Gamma_y) + 3V_y\Gamma^2. \tag{15}$$

This is a differential equation which must be satisfied by all the potentials which admit as trajectories the curves of the family (1). After dividing both members by $\gamma\Gamma$ we get Bozis' equation (7), with λ and μ given in (8).

A straightforward calculation shows that equation (7) can be written as

$$\gamma W_x - W_y = 0, \tag{16}$$

where

$$W = V - \frac{1 + \gamma^2}{2\Gamma}(V_x + \gamma V_y). \tag{17}$$

But $\gamma = f_y/f_x$ implies $f_y W_x - f_x W_y = 0$. This equation has the general solution $W = E(f)$, where E denotes an arbitrary function. It follows that

$$V - \frac{1 + \gamma^2}{2\Gamma}(V_x + \gamma V_y) = E(f). \tag{18}$$

In view of relations (2), (9) and (10) we obtain

$$V + \frac{\dot{x}^2 + \dot{y}^2}{2} = E(f), \tag{19}$$

which means that $E(f)$ represents the total energy, constant on each curve of the family (1). Therefore equation (18), obtained this time from Bozis' equation, is in fact Szebehely's equation. From (19) we obtain $E(f) - V \geqslant 0$, and from (18) it follows that only the curves of the family (1) or parts of them which are situated in the plane region (6) can be described by the unit mass particle.

Example 2. For the family of homocentric circles

$$f(x, y) = x^2 + y^2 = c \tag{20}$$

and arbitrary energy $E(r)\big(r = \sqrt{x^2 + y^2}\big)$, Broucke and Lass (1977) have found the general solution, in polar coordinates r, θ, of Szebehely's equation (5) as

$$V(r, \theta) = g(r) + \frac{1}{r^2}h(\theta), \tag{21}$$

with g and h arbitrary functions of their arguments. The energy in this case is $E = g(r) + rg'(r)/2$, and inequality (6) becomes $g'(r) \geqslant 2h(\theta)/r^3$. A special case of (21) is the Newtonian potential $V = -1/r$ (with $g(r) = -1/r$ and $h(\theta) = 0$), under whose action the circles (20) are traced all over the plane. Another central potential compatible with the family (20) on the entire plane is Maneff's (1924) potential $V = -1/r - \alpha/r^2$ (with $g(r) = -1/r$ and $h(\theta) = -\alpha, \alpha > 0$).

The special form of the family in example 2 allowed the integration of Szebehely's equation; when we do not have at our disposal information on the energy, Bozis' equation is more suitable. It was used, e.g., by Anisiu and Pal (1999) to find out of the Hénon–Heiles type potentials

$$V(x, y) = x^2 + a_1y^2 + a_2x^2y + a_3y^3, \qquad a_1, a_2, a_3 \in \mathbb{R}, \quad a_1 > 0 \tag{22}$$

those which are compatible with a family of polytropic curves. This kind of potential was introduced by Hénon and Heiles (1964) as a model for the motion of a star in a galaxy; it can be used to represent the gravitational field of the Earth, other planets and their satellites (Agekian 2003).

Example 3. The curves of the family

$$f(x, y) = x^{-4}y = c \tag{23}$$

can be traced by a unit mass particle under the action of the potential

$$V(x, y) = x^2 + 16y^2 + a_2x^2y + (16/3)a_2y^3$$

with the energy $E(f) = -a_2/(24f)$, in the region described by the inequality $(a_2(x^2 + 8y^2) + 24y)y \leqslant 0$. This result can be obtained by inserting $\gamma = -x/(4y)$, $\Gamma = -3x/(16y^2)$ and V from (22) in equation (7), and selecting adequately the coefficients in V. Afterwards the energy is determined from Szebehely's equation (5) and the allowed region from (6).

Remark 3. As expected, the general solution of the second-order equation (7) will depend on two arbitrary functions; the same situation occurs for equation (5), one arbitrary function being the energy. So, even if the general solution cannot be found, sometimes it is useful to look for the potential in certain classes of functions (e.g. homogeneous (Borghero and Bozis 2002), or quasihomogeneous, as in example 3). Several pairs (f, V) can be found in the papers of Bozis (1995), Anisiu (2003a) and in the references therein.

3. The case of a general force field

Bertrand (1877) raised the problem of finding the force, not necessarily conservative, depending merely on the position (x, y) of the planets moving on conic sections under the action of that force. Dainelli (1880) solved the problem of Bernard for arbitrary families of curves (1) and obtained, using different notation, formulae similar to (32) and (33). In what follows we derive a partial differential equation satisfied by the force components, and find the region where real motion is possible; finally we provide the formulae for the components of the most general force which is compatible with the family of curves (1). These formulae can be useful whenever the force field is not supposed *a priori* to be conservative. The advantage of working with general force fields is that we do not have to integrate partial differential equations, because we dispose of formulae (32) and (33).

We apply the procedure in section 2 for the system

$$\ddot{x} = X, \qquad \ddot{y} = Y, \tag{24}$$

the force components X and Y being of C^1-class on a domain of the plane xy. If the family (1) consists of straight lines ($\Gamma = 0$), instead of (11) we obtain

$$X + \gamma Y = 0, \tag{25}$$

this being the relation satisfied by the components of the force field in this special case.

Let us consider now a general family (1) with $\Gamma \neq 0$. Instead of (12) we have this time

$$\dot{y}^2 = \frac{X + \gamma Y}{\Gamma}. \tag{26}$$

The differentiation of equations (24) with respect to t gives

$$\ddot{x} = X_x \dot{x} + X_y \dot{y}, \qquad \ddot{y} = Y_x \dot{x} + Y_y \dot{y}. \tag{27}$$

The elimination of the derivatives of x and y between the seven relations in (24), (9), (26), (13) and (27) leads to

$$\Gamma(\gamma X_x - X_y + \gamma^2 Y_x - \gamma Y_y) = (X + \gamma Y)(\gamma \Gamma_x - \Gamma_y) - 3Y\Gamma^2, \tag{28}$$

a differential relation satisfied by the force field in order to admit as trajectories the curves of the family (1). After dividing both members by $\gamma\Gamma$ we get

$$-X_x + \frac{1}{\gamma} X_y - \gamma Y_x + Y_y = \lambda X + \mu Y, \tag{29}$$

where λ and μ are given in (8). This equation was obtained by Bozis (1983), using a different method. From (26) it follows that the motion of the particle is possible only in the plane region (Bozis 1994) described by the inequality

$$\frac{X + \gamma Y}{\Gamma} \geqslant 0. \tag{30}$$

It is obvious that Bozis' equation (7) and the inequality (6) found by Bozis and Ichtiaroglou (1994) follow from (29), respectively from (30), after replacing $X = -V_x$ and $Y = -V_y$.

We remark that, if we denote by

$$\xi = \frac{X + \gamma Y}{\Gamma}, \tag{31}$$

equation (29) can be written as $\gamma \xi_x - \xi_y = -2Y$, or

$$Y = -\tfrac{1}{2}\gamma \xi_x + \tfrac{1}{2}\xi_y. \tag{32}$$

From relation (31) we get then

$$X = \tfrac{1}{2}\gamma^2 \xi_x - \tfrac{1}{2}\gamma \xi_y + \Gamma \xi. \tag{33}$$

Therefore for an arbitrary positive function ξ we obtain the components of the force given by (33) and (32), which were found by a different method by Bozis (1983).

Example 4. For the monoparametric family

$$f(x, y) = x - \frac{x^2 - y^2}{\sqrt{x^2 + y^2}} = c \tag{34}$$

we obtain from (4)

$$\gamma = \frac{y(3x^2 + y^2)}{\sqrt{(x^2 + y^2)^3} - x(x^2 + 3y^2)} \qquad \text{and} \qquad \Gamma = \gamma \gamma_x - \gamma_y.$$

For an arbitrary function ξ we get from formulae (33) and (32) the components X, Y of the force compatible with the family (34).

Specifying the following value of the arbitrary function

$$\xi = \frac{\left(x\sqrt{(x^2+y^2)^3} - x^4 + y^4\right)\left(x^3 + 3xy^2 - \sqrt{(x^2+y^2)^3}\right)^2}{\sqrt{(x^2+y^2)^7}\left\{2x^4 + (y^2 - x^2)\left(2x\sqrt{x^2+y^2} + y^2\right)\right\}},$$

we obtain the obviously nonconservative force with components

$$X = \frac{x(y^2 - x^2)}{\sqrt{(x^2+y^2)^5}} \qquad \text{and} \qquad Y = \frac{y(y^2 - x^2)}{\sqrt{(x^2+y^2)^5}}. \tag{35}$$

The force (35) was considered by Borghero *et al* (1999) in view of the direct problem; they proved its compatibility with the family (34).

4. Conclusions

We assert that Szebehely's and Bozis' equations are of equal importance for the inverse problem attached to a family (1) and a system (2); when we have no *a priori* information on the energy, it is useful (and fully justified) to start working with equation (7) and then to obtain the energy from equation (18).

We have derived the basic equations of the inverse problem in a simple and natural way, by a process of elimination of the time derivatives of x and y. Doing so, the case of families of straight lines presented its particularities and the allowed region emerged.

This unifying consideration of conservative and general force field systems explains also the connection (already mentioned by Bozis (1995)) between Bozis' equation and the differential relation (29).

Acknowledgments

I express my thanks to an unknown referee for carefully reading the paper and for making several useful suggestions.

References

Agekian T A 2003 A basic system of equations in the field of rotationally symmetric potential *Astron. Lett.* **29** 348–51

Anisiu M-C 2003a *The Equations of the Inverse Problem of Dynamics* (Cluj-Napoca: House of the Book of Science) (in Romanian)

Anisiu M-C 2003b PDEs in the inverse problem of dynamics *Analysis and Optimization of Differential Systems* ed V Barbu *et al* (Boston, MA: Kluwer Academic) pp 13–20

Anisiu M-C and Pal A 1999 Special families of orbits for the Hénon–Heiles type potential *Rom. Astron. J.* **9** 179–85

Antonov V A and Timoshkova E I 1993 Simple trajectories in a rotationally symmetric gravitational field *Astron. Rep.* **37** 138–44

Bertrand J 1877 Sur la possibilité de déduire d'une seule des lois de Kepler le principe de l'attraction *C. R. Acad. Sci., Paris* **84** 671–4

Borghero F and Bozis G 2002 Isoenergetic families of planar orbits generated by homogeneous potentials *Meccanica* **37** 545–54

Borghero F, Bozis G and Melis A 1999 Inverse and direct problem of the dynamics of central motions *Rend. Mat.* VII **19** 303–16

Bozis G 1983 Inverse problem with two-parametric families of planar orbits *Celest. Mech.* **31** 129–43

Bozis G 1984 Szebehely inverse problem for finite symmetrical material concentrations *Astron. Astrophys.* **134** 360–4

Bozis G 1994 Family boundary curves for autonomous dynamical systems *Celest. Mech. Dyn. Astron.* **60** 161–72

Bozis G 1995 The inverse problem of dynamics: basic facts *Inverse Problems* **11** 687–708

Bozis G and Anisiu M-C 2001 Families of straight lines in planar potentials *Rom. Astron. J.* **11** 27–43

Bozis G and Blaga C 2004 The role of single orbits in dynamics *Inverse Problems* **20** 61–73

Bozis G and Ichtiaroglou S 1994 Boundary curves for families of planar orbits *Celest. Mech. Dyn. Astron.* **58** 371–85

Broucke R and Lass H 1977 On Szebehely's equation for the potential of a prescribed family of orbits *Celest. Mech.* **16** 215–25

Caranicolas N D and Innanen K A 1992 Periodic motion in perturbed elliptic oscillators *Astron. J.* **103** 1308–12

Contopoulos G and Zikides M 1980 Periodic orbits and ergodic components of a resonant dynamical system *Astron. Astrophys.* **90** 198–203

Dainelli U 1880 Sul movimento per una linea qualunque *G. Mat.* **18** 271–300

Grigoriadou S 1999 The inverse problem of dynamics and Darboux's integrability criterion *Inverse Problems* **15** 1621–37

Hénon M and Heiles C 1964 The applicability of the third integral of motion: some numerical experiments *Astron. J.* **69** 73–9

Kasner E 1906 The trajectories of dynamics *Trans. Am. Math. Soc.* **7** 401–24

Maneff G 1924 La gravitation et le principe de l'égalité de l'action et de la réaction *C. R. Acad. Sci., Paris* **178** 2159–61

van der Merwe P du T 1991 Solvable forms of a generalized Hénon–Heiles system *Phys. Lett.* A **156** 216–20

Szebehely V 1974 On the determination of the potential by satellite observations *Proc. Int. Meeting on Earth's Rotation by Satellite Observation (The University of Cagliari Bologna, Italy)* ed G Proverbio pp 31–5