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Fixed-Point Results and the Ekeland Variational Principle in Vector B-Metric Spaces

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Abstract: In this paper, we extend the concept of b-metric spaces to the vectorial case, where the distance is vector-valued, and the constant in the triangle inequality axiom is replaced by a matrix. For such spaces, we establish results analogous to those in the b-metric setting: fixed-point theorems, stability results, and a variant of Ekeland's variational principle. As a consequence, we also derive a variant of Caristi's fixed-point theorem.

Keywords: triangle inequality axiom; b-metric space; variational principle; fixed point

MSC: 47H10; 54H25

1. Introduction

The concept of a b-metric space arises as a natural generalization of a metric space, where the triangle inequality axiom is relaxed by introducing a constant $b \geq 1$ on its right-hand side. Early ideas in this direction can be traced back to the notion of "quasimetric" spaces, as discussed in [1]. However, the formal definition and terminology of b-metric spaces are widely attributed to Bakhtin [2] and Czerwik [3]. Notably, one of the earliest works to introduce a mapping satisfying the properties of a b-metric dates back to 1970 in [4], where such a mapping was referred to as a "distance". A concept related to that of a b-metric is the notion of a quasi-norm, which can be traced back to Hyers [5] and Bourgin [6], who originally used the term "quasi-norm". For an overview on b-metric spaces, we refer the reader to [7,8].

Various results from the classical theory of metric spaces have been extended to b-metric spaces, including fixed-point theorems (see, e.g., [9–14]), estimations (see, e.g., [15,16]), stability results (see, e.g, [17,18]), and variational principles (see, e.g., [19,20]). In [21], the metric was allowed to take vector values, and results analogous to those for b-metric spaces were established, with matrices converging to zero replacing the contraction constants but not the constant b from the triangle inequality axiom.

In this paper, we introduce the concept of a vector B-metric space, where the scalar constant b in the triangle inequality is replaced by a matrix B. This generalization introduces new challenges in establishing results analogous to those for classical b-metric spaces. To the best of our knowledge, this concept, along with the corresponding results presented here, is novel. Notably, some of the results appear to be new even in the scalar particular case where the matrix B is reduced to a constant.

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Throughout this paper, we consider $\mathbb{R}^n$-valued vector metrics ($n \geq 1$) on a set X, i.e., mappings $d : X \times X \to \mathbb{R}^n_+$. In the scalar case ($n = 1$), we use the special notation ρ to denote a standard metric or a b-metric.

The classical definition of a b-metric reads as follows:

Definition 1. *Let X be a set and let $b \geq 1$ be a given real number. A mapping $\rho : X \times X \to \mathbb{R}_+$ is said to be a b-metric if for all $x, y, z \in X$, the following conditions are satisfied: $\rho(x, y) \geq 0$, $\rho(x, y) = 0$ if and only if $x = y$, $\rho(x, y) = \rho(y, x)$ and $\rho(x, z) \leq b(\rho(x, y) + \rho(y, z))$. The pair (X, ρ) is called a b-metric space.*

In case the mapping ρ is allowed to be vector-valued, if we replace the constant b with a matrix B and we define the ordering in $\mathbb{R}^n$ by the components, then we obtain our definition of a vector B-metric space.

Definition 2. *Let X be a set, $n \geq 1$, and let $B \in \mathcal{M}_{n\times n}(\mathbb{R})$ be an arbitrary matrix. A mapping $d = (d_1, d_2, \ldots, d_n) \colon X \times X \to \mathbb{R}^n_+$ is called a vector B-metric if for all $u, v, w \in X$, one has*

(positivity): $d(u, v) \geq 0$ and $d(u, v) = 0$ if and only if $u = v$;
(symmetry): $d(u, v) = d(v, u)$;
(triangle inequality): $d(u, w) \leq B(d(u, v) + d(v, w))$.

The pair (X, d) is called a vector B-metric space.

2. Preliminaries

In this paper, the vectors in $\mathbb{R}^n$ are regarded as column matrices, and the ordering between them, and more generally between matrices of the same size, is understood by the components. Likewise, the convergence of a sequence of vectors or matrices is understood componentwise.

The spaces of square matrices of size n with real number entries and nonnegative entries are denoted by $\mathcal{M}_{n\times n}(\mathbb{R})$ and $\mathcal{M}_{n\times n}(\mathbb{R}_+)$, respectively. An element of $\mathcal{M}_{n\times n}(\mathbb{R}_+)$ is referred to as a *positive matrix*, while a matrix $M \in \mathcal{M}_{n\times n}(\mathbb{R})$ is called *inverse-positive* if it is invertible and its inverse M^{-1} is positive.

A positive matrix M is said to be *convergent to zero* if its power M^k tends to the zero matrix 0_n as $k \to \infty$.

We have the following characterizations of matrices which are convergent to zero (see, e.g., [22,23]).

Proposition 1. *Let $M \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ and let I be the identity matrix of size n. The following statements are equivalent:*

(a) *M is convergent to zero;*

(b) *The spectral radius $r(M)$ of matrix M is less than 1, i.e., $r(M) < 1$;*

(c) *$I - M$ is invertible and $(I - M)^{-1} = I + M + M^2 + \ \ldots;$*

(d) *$I - M$ is inverse-positive.*

The following proposition collects the various properties equivalent to the notion of an inverse-positive matrix (see, e.g., [23,24]).

Proposition 2. *Let $M \in \mathcal{M}_{n\times n}(\mathbb{R})$. The following statements are equivalent:*

(a) *M is inverse-positive;*

(b) *M is monotone, i.e., $Mx \geq 0$ $(x \in \mathbb{R}^n)$ implies $x \geq 0$;*

(c) *A positive matrix $\overline{M}$ and a real number $s > r(\overline{M})$ exist such that the following representation holds: $M = sI - \overline{M}$.*

Clearly, if M is inverse-positive, from the representation $M = sI - \overline{M}$, we immediately see that all its entries except those from the diagonal are ≤ 0; also, the matrix $\frac{1}{s}\overline{M}$ is convergent to zero. If a matrix M is both positive and inverse-positive, using the representation $M = sI - \overline{M}$, we deduce that M must be a diagonal matrix with strictly positive diagonal entries.

A mapping $N : X \to X$ defined on a vector B-metric space (X, d) is said to be a *Perov contraction mapping* if a matrix A convergent to zero exists such that

$$d(N(x), N(y)) \leq Ad(x, y) \tag{1}$$

for all $x, y \in X$.

The next proposition is about the relationship between vector B-metrics and both vector and scalar b-metrics.

Proposition 3. *(1^0) Any vector-valued b-metric d can be identified with a vector B_b-metric, where B_b is the diagonal matrix whose diagonal entries are all equal to b.*

(2^0) If d is a vector B-metric with an inverse-positive matrix B, then d is also a vector $\underline{B}$-metric with respect to the diagonal matrix $\underline{B}$ that preserves the diagonal of B, as well as a vector-valued $\tilde{b}$-metric with $\tilde{b} = \max\{b_{ii} : 1 \leq i \leq n\}$. Here, $B = (b_{ij})_{1 \leq i,j \leq n}$.

(3^0) If d is a vector B-metric with a positive matrix B, then with each norm in $\mathbb{R}^n$, one can associate a scalar b-metric; for example,

$$\begin{aligned}
\rho_1(x, y) &:= \sum_{i=1}^{n} d_i(x, y), && \text{is a } b_1\text{-metric}, \quad b_1 := \sum_{i=1}^{n} \max_{1 \leq j \leq n} b_{ij}, \\
\rho_\infty(x, y) &:= \max_{1 \leq i \leq n} d_i(x, y), && \text{is a } b_\infty\text{-metric}, \quad b_\infty := \max_{1 \leq i \leq n} \sum_{j=1}^{n} b_{ij}, \\
\rho_2(x, y) &:= \left(\sum_{i=1}^{n} d_i(x, y)^2 \right)^{\frac{1}{2}}, && \text{is a } b_2\text{-metric}, \quad b_2 := \left(\sum_{i,j=1}^{n} b_{ij}^2 \right)^{\frac{1}{2}}.
\end{aligned}$$

Thus, with any vector B-metric, one can associate different (scalar) b-metrics, depending on the chosen metric on $\mathbb{R}^n$. However, as shown in [22], working in a vector setting with matrices instead of numbers is more accurate, especially when a connection with other matrices is necessary. It will also be the case in this work that some conditions or conclusions will connect the matrix B with the matrix A involved in (1).

If Y is a nonempty subset of a vector B-metric space (X, d), we define the *diameter* of the set Y by

$$\operatorname{diam}_d(Y) := \sup\{\rho_1(x, y) : x, y \in Y\} = \sup\left\{ \sum_{i=1}^{n} d_i(x, y) : x, y \in Y \right\}.$$

From this definition, it follows immediately that if $\operatorname{diam}_d(Y) = a$, then $d(x, y) \leq ae$ for all $x, y \in Y$, where $e = (1, 1, \ldots, 1) \in \mathbb{R}^n$. Conversely, if $d(x, y) \leq ae$ for all $x, y \in Y$, then $\operatorname{diam}_d(Y) \leq na$.

Although a b-metric does not generate a topology (see, e.g., [25]), several topological properties can still be defined in terms of sequences (e.g., closed sets, continuous operators, or lower semicontinuous functionals).

We conclude this section with two examples of vector B-metrics.

Example 1. *Let* $d\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2_+$ *be given by*

$$d(x,y) = \begin{pmatrix} |x_1 - y_1|^2 + |x_2 - y_2| \\ |x_2 - y_2|, \end{pmatrix},$$

for $x = (x_1, x_2),\ y = (y_1, y_2) \in \mathbb{R}^2$. *Then,* $(\mathbb{R}^2, d)$ *is a vector B-metric space, where*

$$B = \begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix}.$$

Here, the matrix B is inverse-positive but not positive.

Example 2. *We present an example of a vector-valued mapping d which is a vector B-metric with respect to a positive matrix but for which no inverse-positive matrix exists such that d remains a vector B-metric. Let*

$$S = \{(t,t) \ :\ t \in \mathbb{R}\} \subset \mathbb{R}^2,$$

and let $d\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2_+$ *be given by*

$$d(x,y) = \begin{cases} (0,0) & \text{if } x = y, \\ (|x-y|^2, |x-y|) & \text{if } x, y \in S, \\ (|x-y|, |x-y|^2) & \text{otherwise,} \end{cases}$$

where $|z| = |(z_1, z_2)| = |z_1| + |z_2|$ *is a norm on* $\mathbb{R}^2$. *Note that d is a vector* B_0*-metric, where*

$$B_0 = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}.$$

Let us show that B_0 *is the smallest matrix for which the triangle inequality holds for d. To this aim, let* $B = (b_{ij})_{1 \le i,j \le n}$ *be any matrix for which the triangle inequality is satisfied. Then, for* $x, y \in S$ *and* $z \notin S$, *we have*

$$\begin{pmatrix} |x-y|^2 \\ |x-y| \end{pmatrix} \le \begin{pmatrix} b_{11}(|x-z| + |z-y|) + b_{12}(|x-z|^2 + |z-y|^2) \\ b_{21}(|x-z| + |z-y|) + b_{22}(|x-z|^2 + |z-y|^2) \end{pmatrix}. \tag{2}$$

Let $t, \alpha \in \mathbb{R} \setminus \{0\}$, *and set* $x = (t,t) \in S$, $y = (0,0) \in S$ *and* $z = (\alpha, 0) \notin S$. *The first inequality in* (2) *yields*

$$4t^2 \le b_{11}(|t-\alpha| + |t| + |\alpha|) + b_{12}\Big((|t-\alpha| + |t|)^2 + \alpha^2\Big).$$

Clearly, taking $\alpha = t$ *and the limit as* $t \to \infty$, *this inequality holds only if* $b_{12} \ge 2$. *Similarly, from the second inequality, we obtain*

$$2|t| \le b_{21}(|t-\alpha| + |t| + |\alpha|) + b_{22}\Big((|t-\alpha| + |t|)^2 + \alpha^2\Big).$$

Setting $\alpha = \frac{t}{2}$, *we find that*

$$2|t| \le 2b_{21}|t| + 5b_{22}\frac{t^2}{2}, \quad \text{or equivalently,} \quad 5b_{22}\frac{t^2}{2} + 2|t|(b_{21} - 1) \ge 0.$$

Clearly, this inequality required for all t implies thats $b_{21} \geq 1$. To determine the values of b_{11} and b_{22}, we apply the triangle inequality with $x, y, z \in S$ $(x \neq y \neq z \neq x)$, which gives

$$\begin{pmatrix} |x-y|^2 \\ |x-y| \end{pmatrix} \leq \begin{pmatrix} b_{11}\left(|x-z|^2+|z-y|^2\right)+b_{12}(|x-z|+|z-y|) \\ b_{21}\left(|x-z|^2+|z-y|^2\right)+b_{22}(|x-z|+|z-y|) \end{pmatrix}.$$

Similar arguments to those above imply that $b_{11} \geq 2$ and $b_{22} \geq 1$. Thus, $B \geq B_0$ as claimed.

3. Fixed-Point Theorems in Vector B-Metric Spaces

In this section, we establish some fixed-point results in vector B-metric spaces analogous to the well-known classical results.

3.1. The Perov-Type Fixed-Point Theorem

Our first result is a version of Perov's fixed-point theorem (see, [26,27]) for such spaces.

Theorem 1. *Let (X, d) be a complete vector B-metric space, where B is either a positive or an inverse-positive matrix, and let $N\colon X \to X$ be an operator. Assume that a matrix $A \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ convergent to zero exists such that*

$$d(N(x), N(y)) \leq Ad(x,y), \quad \text{for all } x, y \in X, \tag{3}$$

i.e., N is a Perov contraction mapping. Then, N has a unique fixed point.

Proof. Let $x_0 \in X$, and recursively define

$$x_k = N(x_{k-1}), \quad \text{for } k \geq 1.$$

Since the matrix A is convergent to zero, for each $\alpha > 0$, $k_0 = k_0(\alpha)$ such that

$$A^{k_0} \leq \Lambda,$$

where Λ is the square matrix of size n whose entries are all equal to α. Let $k, p \geq 0$ and k_0 be such that $A^{k_0} \leq \Lambda$ for some $\alpha > 0$ to be specified later.

Case (a): B is inverse-positive. The triangle inequality yields

$$\begin{aligned} B^{-2}d(x_k, x_p) &\leq B^{-1}d(x_k, x_{k+k_0}) + B^{-1}d(x_p, x_{k+k_0}) \\ &\leq B^{-1}A^k d(x_0, x_{k_0}) + d(x_p, x_{p+k_0}) + d(x_{p+k_0}, x_{k+k_0}) \\ &\leq B^{-1}A^k d(x_0, x_{k_0}) + A^p d(x_0, x_{k_0}) + A^{k_0} d(x_k, x_p) \\ &\leq B^{-1}A^k d(x_0, x_{k_0}) + A^p d(x_0, x_{k_0}) + \Lambda d(x_k, x_p), \end{aligned}$$

which gives

$$(B^{-2} - \Lambda) d(x_k, x_p) \leq B^{-1}A^k d(x_0, x_{k_0}) + A^p d(x_0, x_{k_0}). \tag{4}$$

Given that the right-hand side of (4) is a vector that converges to zero as $k, p \to \infty$, our goal is to show that a linear combination of the components of the vector $d(x_k, x_p)$ is bounded above by the corresponding components of the right-hand side of (4). With this aim, we use the following notations:

$$\begin{aligned} &B^{-2} = (\gamma_{ij})_{1\leq i,j\leq n}, \\ &B^{-1}A^k d(x_0, x_{k_0}) + A^p d(x_0, x_{k_0}) = \varphi_{k,p} = (\varphi^i_{k,p})_{1\leq i\leq n}. \end{aligned}$$

Hence,

$$\sum_{i=1}^{n} \varphi_{k,p}^{i} \to 0 \quad \text{as} \quad k,p \to \infty. \tag{5}$$

Under these notations, relation (4) gives

$$\sum_{j=1}^{n} (\gamma_{ij} - \alpha) d_j(x_k, x_p) \leq \varphi_{k,p}^{i}, \; i = 1,2,\ldots,n. \tag{6}$$

Summing in (6) over all $i \in \{1,2,\ldots,n\}$, we obtain

$$\sum_{i,j=1}^{n} (\gamma_{ij} - \alpha) d_j(x_k, x_p) \leq \sum_{i=1}^{n} \varphi_{k,p}^{i}. \tag{7}$$

Sinc e B^{-2} is invertible and positive, the sum of its elements in each column must be positive, i.e.,

$$\sum_{i=1}^{n} \gamma_{ij} > 0, \quad j = 1,2,\ldots,n.$$

If we denote

$$\gamma = \min\left\{ \sum_{i=1}^{n} \gamma_{ij} \; : \; j = 1,2,\ldots,n \right\},$$

relation (7) implies that

$$\begin{aligned} \sum_{i=1}^{n} \varphi_{k,p}^{i} &\geq \sum_{i,j=1}^{n} \gamma_{ij} d_j(x_k, x_p) - n\alpha \sum_{j=1}^{n} d_j(x_k, x_p) \\ &= \sum_{j=1}^{n} \left(\sum_{i=1}^{n} \gamma_{ij} \right) d_j(x_k, x_p) - n\alpha \sum_{j=1}^{n} d_j(x_k, x_p) \\ &\geq (\gamma - n\alpha) \sum_{j=1}^{n} d_j(x_k, x_p). \end{aligned}$$

Choosing $\alpha < \gamma/n$, one has

$$\sum_{j=1}^{n} d_j(x_k, x_p) \leq \frac{1}{\gamma - n\alpha} \sum_{i=1}^{n} \varphi_{k,p}^{i}. \tag{8}$$

In (8), we observe that the factor $\frac{1}{\gamma - n\alpha}$ depends only on n and B, whence (5) yields

$$\sum_{j=1}^{n} d_j(x_k, x_p) \to 0 \quad \text{as} \quad k,p \to \infty,$$

so the sequence (x_k) is Cauchy.

Case (b): B is positive. One has

$$\begin{aligned} d(x_k, x_p) &\leq Bd(x_k, x_{k+k_0}) + Bd(x_p, x_{k+k_0}) \\ &\leq BA^k d(x_0, x_{k_0}) + B^2 d(x_p, x_{p+k_0}) + B^2 d(x_{p+k_0}, x_{k+k_0}) \\ &\leq BA^k d(x_0, x_{k_0}) + B^2 A^p d(x_0, x_{k_0}) + B^2 A^{k_0} d(x_k, x_p) \\ &\leq BA^k d(x_0, x_{k_0}) + B^2 A^p d(x_0, x_{k_0}) + B^2 \Lambda d(x_k, x_p), \end{aligned}$$

which gives

$$(I - B^2\Lambda) d(x_k, x_p) \leq BA^k d(x_0, x_{k_0}) + B^2 A^p d(x_0, x_{k_0}). \tag{9}$$

Note that since $\Lambda^k = (n\alpha)^{k-1}\Lambda$, if α is chosen to be smaller than one divided by the greatest element of B^2 multiplied with n, the matrix $B^2\Lambda$ is convergent to zero. Consequently, $I - B^2\Lambda$ is invertible and $\left(I - B^2\Lambda\right)^{-1} \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$. Hence, (9) is equivalent to

$$d(x_k, x_p) \le \left(I - B^2\Lambda\right)^{-1}\left(BA^k d(x_0, x_{k_0}) + B^2 A^p d(x_0, x_{k_0})\right). \tag{10}$$

As the right-hand side of (10) converges to zero when $k, p \to \infty$, we conclude that (x_k) is Cauchy.

Therefore, in both cases, the sequence (x_k) is Cauchy, and since X is complete, it has a limit x^*; that is, $d(x_k, x^*) \to 0$ as $k \to \infty$. Then, from

$$d(N(x_k), N(x^*)) \le Ad(x_k, x^*),$$

it follows that $N(x_k) \to N(x^*)$ as $k \to \infty$, while from $x_{k+1} = N(x_k)$, passing to the limit, one obtains $x^* = N(x^*)$. Hence, N has a fixed point. To prove the uniqueness, suppose that another fixed point x^{**} exists. Then, from

$$d(x^*, x^{**}) = d(N(x^*), N(x^{**})) \le Ad(x^*, x^{**}),$$

recursively, we obtain

$$d(x^*, x^{**}) \le A^k d(x^*, x^{**}),$$

for all $k \ge 1$. Since $A^k \to 0_n$ as $k \to \infty$, we deduce that $d(x^*, x^{**}) = 0$, i.e., $x^{**} = x^*$. □

If we are not interested in the uniqueness of the fixed point for N, the condition (3) can be relaxed and replaced with a weaker assumption on the graph of N.

Theorem 2. *Let (X, d) be a complete vector B-metric space, where B is either positive or inverse-positive, and let $N\colon X \to X$ be an operator. Assume a matrix $A \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ convergent to zero exists such that*

$$d\left(N(x), N^2(x)\right) \le Ad(x, N(x)), \quad \text{for all } x \in X. \tag{11}$$

Then, N has at least one fixed point.

Proof. Following the proof of Theorem 1, from any initial point x_0, the sequence $x_k = N^k(x_0)$ is convergent to a fixed point x^* of N, which clearly depends on the starting point x_0, but condition (11) is insufficient to guarantee the uniqueness. □

The next result is a version for vector B-metric spaces of Maia's fixed-point theorem [28]. The contraction condition on the operator is considered with respect to a vector B_1-metric d_1, not necessarily complete, while the convergence of the sequence of successive approximations is guaranteed in a complete vector B_2-metric d_2 in a subordinate relationship to d_1.

Theorem 3. *Let X be a set equipped with two $\mathbb{R}^n$-vector metrics, a B_1-metric d_1 and a B_2-metric d_2, where B_2 is either positive or inverse-positive, and let $N\colon X \to X$ be an operator. Assume that the following conditions hold:*

(i) (X, d_1) *is a complete vector B_1-metric space;*
(ii) $d_1(x, y) \le Cd_2(x, y)$ *for all $x, y \in X$ and some matrix $C \in \mathcal{M}_{n\times n}(\mathbb{R})$;*
(iii) *A matrix A convergent to zero exists such that*

$$d_2(N(x), N(y)) \le Ad_2(x, y), \quad \text{for all } x, y \in X; \tag{12}$$

(iv) *The operator N is continuous in (X, d_1).*

Then, the operator N has a unique fixed point.

Proof. Let $x_0 \in X$ be fixed, and consider the iterative sequence $x_{k+1} = N(x_k)$ for $k \geq 0$. For any $k, k_0, p \geq 0$, applying the triangle inequality twice and using condition (iii), we derive either

$$(B_2^{-2} - A^{k_0}) d_2(x_k, x_p) \leq B_2^{-1} A^k d_2(x_0, x_{k_0}) + A^p d_2(x_0, x_{k_0}),$$

in the case that B_2 is inverse-positive, or

$$(I - B_2^2 A^{k_0}) d_2(x_k, x_p) \leq B_2 A^k d_2(x_0, x_{k_0}) + B_2^2 A^p d_2(x_0, x_{k_0}),$$

if B_2 is positive. With a similar argument to that for the proof of Theorem 1, we deduce that (x_k) is a Cauchy sequence in (X, d_2). From (ii), it follows immediately that (x_k) is also a Cauchy sequence in (X, d_1); hence, (x_k) is convergent with respect to metric d_1 to some x^*; that is,

$$d_1(N(x_k), x^*) = d_1(x_{k+1}, x^*) \to 0, \quad \text{as } k \to \infty,$$

while the continuity of N yields $d_1(N(x^*), x^*) = 0$, i.e., $N(x^*) = x^*$. To establish the uniqueness, suppose that x^{**} is another fixed point of N, i.e., $N(x^{**}) = x^{**}$. Then, according to (12), one has

$$(I - A) d_2(x^*, x^{**}) \leq 0.$$

Since A is convergent to zero, according to Proposition 1 (d), we can multiply by $(I - A)^{-1}$ without changing the sense of the inequality and obtain $d_2(x^*, x^{**}) = 0$, i.e., $x^* = x^{**}$. □

3.2. Error Estimates

The classical Banach and Perov fixed-point theorems are accompanied by certain error estimates in terms of the contraction constant and the matrix, respectively. These estimates allow us to obtain stopping criteria for the iterative approximation process. It is the aim of this subsection to obtain such stopping criteria when working in vector B-metric spaces.

Theorem 4. *Assume that all of the conditions of Theorem 1 hold, and let (x_k) be a sequence of successive approximations of the fixed point x^*.*

(1^0) *If B is inverse-positive, then*

$$\left(B^{-1} - A\right) d(x_k, x^*) \leq A^k d(x_0, x_1) \quad (k \geq 0). \tag{13}$$

If in addition the matrix $B^{-1} - A$ is inverse-positive, then

$$d(x_k, x^*) \leq \left(B^{-1} - A\right)^{-1} A^k d(x_0, x_1) \quad (k \geq 0). \tag{14}$$

(2^0) *If B is positive, then*

$$(I - BA) d(x_k, x^*) \leq BA^k d(x_0, x_1) \quad (k \geq 0). \tag{15}$$

If in addition $I - BA$ is inverse-positive, then

$$d(x_k, x^*) \leq (I - BA)^{-1} BA^k d(x_0, x_1) \quad (k \geq 0). \tag{16}$$

Proof. (1^0): We have

$$\begin{aligned} B^{-1}d(x_k, x^*) &\leq d(x_k, x_{k+1}) + d(x_{k+1}, x^*) \\ &\leq A^k d(x_0, x_1) + Ad(x_k, x^*), \end{aligned}$$

whence we deduce (13). The second part is obvious.

(2^0): We have

$$\begin{aligned} d(x_k, x^*) &\leq Bd(x_k, x_{k+1}) + Bd(x_{k+1}, x^*) \\ &\leq BA^k d(x_0, x_1) + BAd(x_k, x^*), \end{aligned}$$

that is, (15). The additional conclusion is obvious. □

Remark 1. *Clearly, since A^k tends to the zero matrix as $k \to \infty$, Formulas (14) and (16) provide stopping criteria for the iterative fixed-point approximation algorithm starting from x_0, when an admissible error is given. It should be emphasized that these estimates are in terms of matrices A and B. In contrast, if we make the transition to (scalar) b-metric spaces, as discussed in Section 2, the resulting estimates will depend on the chosen norm in $\mathbb{R}^n$ and may vary across different norms. So, from this point of view, the vector approach not only unifies the results that can be obtained using the scalar method but also provides the best estimates.*

3.3. Stability Results

We now present two stability properties of the Perov contraction mappings in vector B-metric spaces.

The first property is the Reich–Zaslavski property and generalizes that obtained in [18] for b-metric spaces.

Theorem 5. *Let (X, d) be a complete vector B-metric space, and let $N\colon X \to X$ be an operator such that (3) holds with a matrix A convergent to zero. In addition, assume that either*

(a) *B and $B^{-1} - A$ are inverse-positive*

or

(b) *B is positive and $I - BA$ is inverse-positive.*

Then, N is stable in the sense of the Reich–Zaslavski property; i.e., N has a unique fixed point x^*, and for every sequence $(x_k) \subset X$ satisfying

$$d(x_k, N(x_k)) \to 0 \quad \text{as} \quad k \to \infty, \tag{17}$$

one has

$$x_k \to x^* \quad \text{as} \quad k \to \infty.$$

Proof. According to Theorem 1, the operator N has a unique fixed point x^*. In addition, for any sequence (x_k) satisfying (17), in case (a), we have

$$\begin{aligned} B^{-1}d(x_k, x^*) &\leq d(x_k, N(x_k)) + d(N(x_k), x^*) \\ &= d(x_k, N(x_k)) + d(N(x_k), N(x^*)) \\ &\leq d(x_k, N(x_k)) + Ad(x_k, x^*), \end{aligned}$$

that is,

$$d(x_k, x^*) \leq (B^{-1} - A)^{-1} d(x_k, N(x_k)),$$

while in case (b),

$$d(x_k, x^*) \leq (I - BA)^{-1} B d(x_k, N(x_k)).$$

These estimates immediately yield the conclusion. □

The second stability result is related to Ostrowski's theorem and extends to vector B-metric spaces a similar property established in [18] for b-metric spaces.

Theorem 6. *Let (X, d) be a complete vector B-metric space, and let $N\colon X \to X$ be an operator. Assume N satisfies (3) with a matrix A convergent to zero. In addition, assume that either*

(a) *B and $I - \tilde{b}A$ are inverse-positive, where $\tilde{b} = \max\{b_{ii} : i = 1, 2, \ldots, n\}$*

or

(b) *B is positive and $I - BA$ is inverse-positive.*

Then, N has the Ostrowski property, i.e., N has a unique fixed point x^*, and for every sequence $(x_k) \subset X$ satisfying

$$d(x_{k+1}, N(x_k)) \to 0 \ \text{ as } \ k \to \infty,$$

one has

$$x_k \to x^* \quad \text{as} \ \ k \to \infty.$$

Proof. As previously established, the operator N has a unique fixed point x^*. In case (a), we have

$$\begin{aligned} d(x_{k+1}, x^*) &\leq \tilde{b}\, d(x_{k+1}, N(x_k)) + \tilde{b}\, d(N(x_k), N(x^*)) \\ &\leq \tilde{b}\, d(x_{k+1}, N(x_k)) + \tilde{b}A\, d(x_k, x^*) \\ &\leq \ \ldots \\ &\leq \tilde{b} \sum_{p=0}^{k} (\tilde{b}A)^p d(x_{k+1-p}, N(x_{k-p})) + (\tilde{b}A)^{k+1} d(x_0, x^*), \end{aligned}$$

while in case (b), a similar estimation gives

$$d(x_{k+1}, x^*) \leq \sum_{p=0}^{k} (BA)^p B d(x_{k+1-p}, N(x_{k-p})) + (BA)^k B d(x_0, x^*).$$

Since $I - \tilde{b}A$ is inverse-positive and $\tilde{b}A$ is positive in the first case, and $I - BA$ is inverse-positive and BA is positive in the second case, the series $\sum_{p=0}^{k} (\tilde{b}A)^p$ and $\sum_{p=0}^{k} (BA)^p$ are convergent. Moreover, $(\tilde{b}A)^k$ and $(BA)^k$ converge to the zero matrix as $k \to \infty$. Therefore, using the Cauchy–Toeplitz lemma (see [29]), it follows that $d(x_{k+1}, x^*) \to 0$ as $k \to \infty$. □

3.4. The Avramescu-Type Fixed-Point Theorem

Our next result is a variant of Avramescu's fixed-point theorem (see [30]) in vector B-metric spaces.

Theorem 7 (Avramescu's theorem in vector B-metric spaces)**.** *Let (X, d) be a complete vector B-metric space, D a nonempty closed convex subset of a normed space Y, and $N_1 : X \times D \to X$ and $N_2 : X \times D \to D$ be two mappings. Assume that the following conditions are satisfied:*

(i) *$N_1(x, .)$ is continuous for every $x \in X$ and there is a matrix A convergent to zero such that*

$$d(N_1(x, y), N_1(\overline{x}, y)) \leq A\, d(x, \overline{x}),$$

for all $x, \overline{x} \in X$ and $y \in D$;

(ii) *Either*

(a) *B and $B^{-1} - A$ is inverse-positive*

or

(b) *B is positive and $I - BA$ is inverse-positive.*

(iii) *N_2 is continuous and $N_2(X \times D)$ is a relatively compact subset of Y .*

Then, $(x^*, y^*) \in X \times D$ such that

$$N_1(x^*, y^*) = x^*, \quad N_2(x^*, y^*) = y^*.$$

Proof. For each $y \in D$, Theorem 1 applies to the operator $N_1(., y)$ and gives a unique $S(y) \in X$ such that

$$N_1(S(y), y) = S(y). \tag{18}$$

We claim that the mapping $S : D \to X$ is continuous. To prove this, let $y, \overline{y} \in D$. In case (a), we have

$$\begin{aligned} B^{-1}d(S(y), S(\overline{y})) &= B^{-1}d(N_1(S(y), y), N_1(S(\overline{y}), \overline{y})) \\ &\leq d(N_1(S(y), y), N_1(S(\overline{y}), y)) + d(N_1(S(\overline{y}), y), N_1(S(\overline{y}), \overline{y})) \\ &\leq Ad(S(y), S(\overline{y})) + d(N_1(S(\overline{y}), y), N_1(S(\overline{y}), \overline{y})), \end{aligned}$$

which implies

$$\left(B^{-1} - A\right)d(S(y), S(\overline{y})) \leq d(N_1(S(\overline{y}), y), N_1(S(\overline{y}), \overline{y})),$$

while in case (b), one has

$$(I - BA)d(S(y), S(\overline{y})) \leq Bd(N_1(S(\overline{y}), y), N_1(S(\overline{y}), \overline{y})).$$

Since $B^{-1} - A$ and $I - BA$ are inverse-positive, respectively, in case (a), we deduce that

$$d(S(y), S(\overline{y})) \leq \left(B^{-1} - A\right)^{-1} d(N_1(S(\overline{y}), y), N_1(S(\overline{y}), \overline{y})), \tag{19}$$

and in case (b),

$$d(S(y), S(\overline{y})) \leq (I - BA)^{-1}Bd(N_1(S(\overline{y}), y), N_1(S(\overline{y}), \overline{y})). \tag{20}$$

Then, for any convergent sequence $(y_k) \subset D$, $y_k \to y^*$ as $k \to \infty$, the continuity of $N_1(S(y^*), .)$ together with relations (19) and (20) imply that $d(S(y_k), S(y^*)) \to 0$ as $k \to \infty$. Thus, S is continuous, and since N_2 is continuous, the composed mapping

$$N_2(S(.), .) : D \to D$$

is continuous too. Since its range is relatively compact according to condition (iii), Schauder's fixed-point theorem applies and guarantees the existence of a point $y^* \in D$ such that

$$N_2(S(y^*), y^*) = y^*. \tag{21}$$

Finally, denoting $x^* := S(y^*)$, from (18) and (21), we have the conclusion. □

Remark 2. *Without the invariance condition $N_2(X \times D) \subset D$, a similar result holds if D is a closed ball B_R centered at the origin and of the radius R in the space $(Y, \|.\|)$, provided that Schaefer's fixed-point theorem is used instead of Schauder's theorem. In this case, in addition to conditions (i) and (ii), we need the Leray–Schauder condition*

$$y \neq \lambda N_2(x, y),$$

for all $x \in X, y \in Y$ with $\|y\| = R$, and $\lambda \in (0, 1)$.

In particular, for scalar b-metric spaces, conditions (a) and (b) from hypothesis (ii) of Theorem 7 are the same and reduce to the unique requirement that the product of b and the Lipschitz constant a of N are less than one. More precisely, Theorem 7 reads as follows.

Theorem 8 (Avramescu's theorem in b-metric spaces)**.** *Let (X, ρ) be a complete b-metric space $(b \geq 1)$, D a nonempty closed convex subset of a normed space Y, and $N_1 : X \times D \to X$ and $N_2 : X \times D \to D$ be two mappings. Assume that the following conditions are satisfied:*

(i) *$N_1(x, .)$ is continuous for every $x \in X$ and there is a constant $a \geq 0$ such that*

$$\rho(N_1(x, y), N_2(\overline{x}, y)) \leq a\rho(x, \overline{x}),$$

for all $x, \overline{x} \in X$ and $y \in D$;

(ii) $ab < 1$;

(iii) *N_2 is continuous and $N_2(X \times D)$ is a relatively compact subset of Y .*

Then, $(x^*, y^*) \in X \times D$ such that $N_1(x^*, y^*) = x^*$ and $N_2(x^*, y^*) = y^*$.

4. Ekeland's Principle and Caristi's Fixed-Point Theorem in Vector *B*-Metric Spaces

4.1. Classical Results

We first recall for comparison the classical results in metric spaces (see, [31–34]).

Theorem 9 (Weak Ekeland's variational principle)**.** *Let (X, ρ) be a complete metric space, and let $f : X \to \mathbb{R}$ be a lower semicontinuous function bounded from below. Then, for given values of $\varepsilon > 0$ and $x_0 \in X$, a point $x^* \in X$ exists such that*

$$f(x^*) \leq f(x_0) - \varepsilon\rho(x^*, x_0)$$

and

$$f(x^*) < f(x) + \varepsilon\rho(x^*, x) \quad \text{for all } x \in X,\ x \neq x^*.$$

Theorem 10 (Strong Ekeland's variational principle)**.** *Let (X, ρ) be a complete metric space, and let $f : X \to \mathbb{R}$ be a lower semicontinuous function that is bounded from below. For given values $\varepsilon > 0$, $\delta > 0$, and $x_0 \in X$ satisfying*

$$f(x_0) \leq \inf_{x \in X} f(x) + \varepsilon,$$

a point $x^* \in X$ exists such that the following hold:

$$\begin{aligned}
&f(x^*) \leq f(x_0), \\
&\rho(x^*, x_0) \leq \delta, \\
&f(x^*) < f(x) + \frac{\varepsilon}{\delta}\rho(x^*, x) \quad \text{for all } x \in X,\ x \neq x^*.
\end{aligned}$$

Below, we have a version of Ekeland's variational principle for scalar b-metric spaces (see, [19]).

Theorem 11 ([19])**.** *Let (X, ρ) be a complete b-metric space with $b > 1$, where the b-metric ρ is continuous. Let $f : X \to \mathbb{R}$ be a lower semicontinuous function bounded from below. For given values of $\varepsilon > 0$ and $x_0 \in X$ satisfying*

$$f(x_0) \leq \inf_{x \in X} f(x) + \varepsilon,$$

a sequence $(x_k) \subset X$ and a point $x^* \in X$ exist such that

$$\begin{aligned} &x_k \to x^* \quad \text{as } k \to \infty, \\ &\rho(x^*, x_k) \leq \frac{\varepsilon}{2^k}, \quad k \in \mathbb{N}, \\ &f(x^*) \leq f(x_0) - \sum_{k=0}^{\infty} \frac{1}{b^k} \rho(x^*, x_k), \\ &f(x^*) + \sum_{k=0}^{\infty} \frac{1}{b^k} \rho(x^*, x_k) < f(x) + \sum_{k=0}^{\infty} \frac{1}{b^k} \rho(x, x_k), \quad \text{for } x \neq x^*. \end{aligned}$$

The proof of Theorem 11 in [19] is based on the version for scalar b-metric spaces of Cantor's intersection lemma.

Lemma 1 ([19])**.** *Let (X, ρ) be a complete b-metric space. For every descending sequence $(F_k)_{k \geq 1}$ of nonempty closed subsets of X with $\operatorname{diam}_\rho(F_k) \to 0$ as $k \to \infty$, the intersection $\bigcap_{k=1}^{\infty} F_k$ contains one and only one element.*

Let us first note that a version of Cantor's intersection lemma remains true in complete vector B-metric spaces.

Lemma 2. *Let (X, d) be a complete vector B-metric space, and let $(F_k)_{k \geq 1}$ be a descending sequence of nonempty closed subsets of X. Assume that for every $\varepsilon > 0$, $k_0 \geq 1$ such that*

$$d(x, y) \leq \varepsilon e \quad \text{for all } x, y \in F_k \text{ and } k \geq k_0, \tag{22}$$

where $e = (1, 1, \ldots, 1)$. Then, the intersection $\bigcap_{k=1}^{\infty} F_k$ contains exactly one element.

Proof. As stated in the preliminaries, condition (22) implies that the diameter of F_k with respect to the scalar b-metric ρ_1 tends to zero. Since (X, d) is complete, it follows that (X, ρ_1) is also complete. From Cantor's lemma in scalar b-metric spaces (Lemma 1), we conclude that the intersection $\bigcap_{k=1}^{\infty} F_k$ has exactly one element. □

4.2. Ekeland's Variational Principle in Vector B-Metric Spaces

First, we state and prove a version of the weak form of Ekeland's variational principle in vector B-metric spaces.

Theorem 12 (Weak Ekeland's variational principle in vector B-metric spaces)**.** *Let (X, d) be a complete vector B-metric space such that the B-metric d is continuous, and let $f : X \to \mathbb{R}^n$ be a lower semicontinuous function bounded from below. Assume that f satisfies the following condition:*

(H) *For every nonempty closed subset $F \subset X$ and every $\varepsilon > 0$, a point $x_{\varepsilon,F} \in F$ exists such that*

$$f(x_{\varepsilon,F}) \leq f(x) + \varepsilon e, \quad \text{for all } x \in F, \tag{23}$$

where $e = (1, 1, \ldots, 1) \in \mathbb{R}^n$.

Then, for a given $x_0 \in X$, a sequence $\{x_k\} \subset X$ and a point $x^* \in X$ exist such that $x_k \to x^*$ as $k \to \infty$,

$$f(x^*) \leq f(x_0) - d(x^*, x_0), \tag{24}$$

and

$$f(x^*) + d(x^*, x_k) \geq f(x) + d(x, x_k) \quad \text{for all } k \geq 0 \quad \text{implies } x = x^*. \tag{25}$$

Moreover,

$$f(x^*) \geq f(x) + Bd(x^*, x) + (B - I)d(x^*, x_k) \quad \text{for all } k \geq 0 \quad \text{implies } x = x^*. \tag{26}$$

Proof. Let us fix a sequence (ε_k) of positive numbers satisfying $\varepsilon_k \to 0$ as $k \to \infty$. We now proceed to construct the sequence (x_k). Let

$$F(x_0) := \{x \in X : \ f(x) + d(x, x_0) \leq f(x_0)\}.$$

Clearly, $x_0 \in F(x_0)$ and $F(x_0)$ is closed because d is continuous and f is lower semi-continuous. Then, according to assumption (23), a point $x_1 \in F(x_0)$ exists with

$$f(x_1) \leq f(x) + \varepsilon_1 e \quad \text{for all } x \in F(x_0).$$

Define

$$F(x_1) := \{x \in F(x_0) : \ f(x) + d(x, x_1) \leq f(x_1)\},$$

and recursively, having $x_k \in F(x_{k-1})$ with

$$f(x_k) \leq f(x) + \varepsilon_k e \quad \text{for all } x \in F(x_{k-1}),$$

we define

$$F(x_k) := \{x \in F(x_{k-1}) : \ f(x) + d(x, x_k) \leq f(x_k)\}.$$

The sets $F(x_k)$ are nonempty and closed and by their definition form a descending sequence. To apply Cantor's intersection lemma, we verify that their diameters tend to zero as $k \to \infty$. Indeed, for any $y \in F(x_k) \subset F(x_{k-1})$, one has

$$f(y) + d(y, x_k) \leq f(x_k).$$

Also, from the definition of x_k,

$$f(x_k) \leq f(y) + \varepsilon_k e.$$

Consequently, using the definition of $F(x_k)$, we deduce

$$d(y, x_k) \leq f(x_k) - f(y) \leq \varepsilon_k e,$$

whence, for every $y_1, y_2 \in F(x_k)$, we have

$$d(y_1, y_2) \leq B(d(y_1, x_k) + d(y_2, x_k)).$$

As a result, $\mathrm{diam}_d(F(x_k)) \to 0$ as $k \to \infty$. Thus, according to Cantor's lemma,

$$\bigcap_{k=0}^{\infty} F(x_k) = \{x^*\}.$$

From $x^* \in F(x_0)$, one has (24).

Next, we prove (25). To this end, we show the equivalent statement: if $x \neq x^*$, then $k = k(x) \geq 0$ such that

$$f(x^*) + d(x^*, x_k) \not\geq f(x) + d(x, x_k),$$

that is,

$$f_i(x^*) + d_i(x^*, x_k) < f_i(x) + d_i(x, x_k)$$

for at least one index $i \in \{1, 2, \ldots, n\}$.

Let $x \in X,\ x \neq x^*$ be arbitrary. Then, $x \notin \bigcap_{k=0}^{\infty} F(x_k)$. We distinguish two cases:

(a) $x \notin F(x_0)$; (b) $x \in F(x_{k-1})$ and $x \notin F(x_k)$ for some $k = k(x) \geq 1$.

In case (a), we have $f(x) + d(x, x_0) \not\leq f(x_0)$. In case (b), we have $f(x) + d(x, x_k) \not\leq f(x_k)$. Thus, in both cases, $k = k(x) \geq 0$ such that $f(x) + d(x, x_k) \not\leq f(x_k)$. This implies that $i \in \{1, 2, \ldots, n\}$ with

$$f_i(x) + d_i(x, x_k) > f_i(x_k).$$

On the other hand, since $x^* \in F(x_k)$, one has $f(x^*) + d(x^*, x_k) \leq f(x_k)$. In particular, for the index i identified above, it holds that

$$f_i(x_k) \geq f_i(x^*) + d_i(x^*, x_k).$$

Then, from these two inequalities, we obtain

$$f_i(x^*) + d_i(x^*, x_k) < f_i(x) + d_i(x, x_k), \tag{27}$$

which equivalently proves (25).

In order to establish (26), we apply the triangle inequality for d on the right-hand side of (27), which gives

$$f_i(x^*) + d_i(x^*, x_k) < f_i(x) + d_i(x, x_k) \leq f_i(x) + (Bd(x^*, x_k))_i + (Bd(x^*, x))_i.$$

Hence,

$$f_i(x^*) + d_i(x^*, x_k) < f_i(x) + (Bd(x^*, x_k))_i + (Bd(x^*, x))_i,$$

that is,

$$f(x^*) \not\geq f(x) + Bd(x^*, x) + (B - I)d(x^*, x_k).$$

Thus, (26) holds. □

A version of the strong form of Ekeland's variational principle in vector B-metric spaces is as follows.

Theorem 13 (Strong Ekeland's variational principle in vector B-metric spaces)**.** *Let (X, d) be a complete B-metric space such that the B-metric d is continuous, and let $f : X \to \mathbb{R}^n$ be a lower semicontinuous function bounded from below and satisfying condition (H). Then, for given values of $\varepsilon, \delta > 0$ and $x_0 \in X$ with*

$$f(x_0) \leq f(x) + \varepsilon e \quad \text{for all } x \in X, \tag{28}$$

a sequence $(x_k) \subset X$ *exists and* $x^* \in X$ *such that* $x_k \to x^*$ *as* $k \to \infty$,

$$f(x^*) \leq f(x_0), \tag{29}$$

$$d(x^*, x_0) \leq \delta e, \tag{30}$$

$$f(x^*) + \frac{\varepsilon}{\delta} d(x^*, x_k) \geq f(x) + \frac{\varepsilon}{\delta} d(x, x_k) \text{ for all } k \geq 0 \text{ implies } x = x^*.$$

Moreover,

$$f(x^*) \geq f(x) + \frac{\varepsilon}{\delta} B d(x^*, x) + \frac{\varepsilon}{\delta}(B - I) d(x^*, x_k) \text{ for all } k \geq 0 \text{ implies that } x = x^*.$$

Proof. We apply the weak form of Ekeland's variational principle, Theorem 12, to the vector B-metric $\frac{\varepsilon}{\delta} d$. From (24), we immediately obtain (29), while from $x^* \in F(x_0)$ and (28), we deduce

$$\frac{\varepsilon}{\delta} d(x^*, x_0) \leq f(x_0) - f(x^*) \leq \varepsilon e,$$

whence (30). The remaining conclusions follow directly. □

A consequence of the weak form of Ekeland's variational principle is the following version of Caristi's fixed-point theorem (see [35]) in vector B-metric spaces.

Theorem 14. *Let* (X, d) *be a complete vector B-metric space such that the B-metric d is continuous, and let* $f : X \to \mathbb{R}^n$ *be a lower semicontinuous function bounded from below and satisfying condition (H). Assume that for an operator* $N : X \to X$*, the following conditions are satisfied:*

$$d(N(x), y) \leq d(x, y) + B d(N(x), x), \quad x, y \in X \tag{31}$$

and

$$B d(N(x), x) \leq f(x) - f(N(x)), \quad x \in X. \tag{32}$$

Then, N has at least one fixed point.

Proof. Assume that N has no fixed points. Then, applying Ekeland's variational principle to f (Theorem 12), from (25), one has

$$f(x^*) + d(x^*, x_k) \ngeq f(N(x^*)) + d(N(x^*), x_k)$$

for some k. Therefore, there is an index i with

$$f_i(x^*) + d_i(x^*, x_k) < f_i(N(x^*)) + d_i(N(x^*), x_k).$$

Using (32) gives

$$(B d(N(x^*), x^*))_i \leq f_i(x^*) - f_i(N(x^*)) < d_i(N(x^*), x_k) - d_i(x^*, x_k),$$

that is,

$$d_i(x^*, x_k) + (B d(N(x^*), x^*))_i < d_i(N(x^*), x_k),$$

which contradicts (31). Consequently, N has a fixed point. □

4.3. New Versions of Ekeland's Variational Principle in b-Metric Spaces

We emphasize that in the scalar case, that is, when $n = 1$, $B = b \geq 1$, and $d = \rho$ is a b-metric, our theorems from the previous subsection offer more natural versions in b-metric spaces of the classical results as follows.

Theorem 15 (Weak Ekeland's variational principle in b-metric spaces)**.** *Let (X, ρ) be a complete b-metric space ($b \geq 1$) such that the b-metric ρ is continuous, and let $f : X \to \mathbb{R}$ be a lower semicontinuous function bounded from below. Then, for a given value of $x_0 \in X$, a sequence $(x_k) \subset X$ exists and $x^* \in X$ such that $x_k \to x^*$ as $k \to \infty$,*

$$f(x^*) \leq f(x_0) - \rho(x^*, x_0),$$

and for each $x \in X$, $x \neq x^*$, an index $k = k(x)$ exists with

$$f(x^*) + \rho(x^*, x_k) < f(x) + \rho(x, x_k).$$

Moreover, for each $x \in X$, $x \neq x^*$, an index $k = k(x)$ exists with

$$f(x^*) < f(x) + b\rho(x^*, x) + (b-1)\rho(x^*, x_k). \tag{33}$$

Theorem 16 (Strong Ekeland's variational principle in b-metric spaces)**.** *Let (X, ρ) be a complete b-metric space ($b \geq 1$) such that the b-metric ρ is continuous, and let $f : X \to \mathbb{R}$ be a lower semicontinuous function bounded from below. Then, for given values of $\varepsilon, \delta > 0$ and $x_0 \in X$ with*

$$f(x_0) \leq \inf_{x \in X} f(x) + \varepsilon,$$

a sequence $(x_k) \subset X$ exists and $x^* \in X$ such that $x_k \to x^*$ as $k \to \infty$,

$$f(x^*) \leq f(x_0),$$

$$\rho(x^*, x_0) \leq \delta,$$

and for each $x \in X$, $x \neq x^*$, an index $k = k(x)$ exists with

$$f(x^*) + \frac{\varepsilon}{\delta}\rho(x^*, x_k) < f(x) + \frac{\varepsilon}{\delta}\rho(x, x_k).$$

Moreover, for each $x \in X$, $x \neq x^*$, an index $k = k(x)$ exists with

$$f(x^*) < f(x) + b\frac{\varepsilon}{\delta}\rho(x^*, x) + (b-1)\frac{\varepsilon}{\delta}\rho(x^*, x_k). \tag{34}$$

Theorem 17 (Caristi's fixed-point theorem in b-metric spaces)**.** *Let (X, ρ) be a complete b-metric space ($b \geq 1$) such that the b-metric ρ is continuous, and let $f : X \to \mathbb{R}$ be a lower semicontinuous function bounded from below. If for an operator $N : X \to X$, one has*

$$\rho(N(x), y) \leq \rho(x, y) + b\rho(N(x), x), \quad x, y \in X \tag{35}$$

and

$$b\rho(N(x), x) \leq f(x) - f(N(x)), \quad x \in X, \tag{36}$$

then N has at least one fixed point.

The last three results reduce to the classical ones in ordinary metric spaces, i.e., if $b = 1$. Thus, (33) reduces to

$$f(x^*) < f(x) + \rho(x^*, x), \quad x \neq x^*;$$

(34) reduces to

$$f(x^*) < f(x) + \frac{\varepsilon}{\delta}\rho(x^*, x), \quad x \neq x^*;$$

assumption (35) trivially holds, while (36) becomes the classical Caristi's inequality

$$\rho(N(x), x) \leq f(x) - f(N(x)), \quad x \in X.$$

5. Conclusions and Further Research

In this paper, we introduced the concept of a vector B-metric space. Several fixed-point theorems analogous to those in scalar b-metric spaces, as well as their classical counterparts, were presented. Additionally, we discussed some stability results. Finally, we provided a variant of Ekeland's variational principle alongside a version of Caristi's theorem. It remains an open question whether the assumption that $B^{-1} - A$ or $I - BA$ is inverse-positive can be omitted in Theorems 7, 5, and 6. Additionally, one may explore a variant of Ekeland's variational principle where Caristi's theorem holds without requiring the additional assumption (31). Lastly, it would be interesting to study a case where the matrix B is neither positive nor inverse-positive, for instance, when it has positive diagonal elements but contains both positive and negative entries elsewhere.

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