CRITICAL POINT LOCALIZATION AND MULTIPLICITY RESULTS IN BANACH SPACES VIA NEHARI MANIFOLD TECHNIQUE

RADU PRECUP AND ANDREI STAN

Abstract. In the paper, results on the existence of critical points in annular subsets of a cone are obtained with the additional goal of obtaining multiplicity results. Compared to other approaches in the literature based on the use of Krasnoselskii's compression-extension theorem or topological index methods, our approach uses the Nehari manifold technique in a surprising combination with the cone version of Birkhoff-Kellogg's invariant-direction theorem. This yields a simpler alternative to traditional methods involving deformation arguments or Ekeland's variational principle. The new method is illustrated on a boundary value problem for p-Laplacian equations and we believe that it will be useful for proving the existence, localization and multiplicity of solutions for other classes of problems with variational structure.

Keywords: Critical point, Nehari manifold, Birkoff-Kellogg invariant-direction, cone, p-Laplace operator, positive solution, multiple solutions
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1. Introduction

Finding localized solutions of equations (often equivalent to finding critical points of a given functional) in predefined domains is of interest in mathematical models, as it provides a certain degree of control over the solutions of the modeled system – for example, one may seek a solution whose energy remains within specific bounds. However, this approach introduces additional challenges compared to critical point theory in the entire space, primarily due to the behavior of the functional at the boundaries. For instance, if a functional attains its minimum at a boundary point, that point is not necessarily a critical point in the usual sense, as the directional derivatives may not vanish in every direction of the space.

Some of the earliest attempts to localize critical points date back to Schechter [19, 20] (see also [21]). Using pseudogradients and deformation arguments and imposing a boundary condition on the sphere, Schechter established the existence of critical points within a ball under a compactness Palais-Smale condition on the functional. It can be said that Schechter's theorems (for minimizers and points of mountain pass type) are critical point versions of Schaefer's fixed point theorem in a ball [10, p. 139], a particular case of the general Leray-Schauder fixed point theorem [10, Theorem 6.5.4]. Since 2008 [13], the first author has been interested in locating critical points in annular subsets of a cone, with the adjacent goal of obtaining multiple solutions in such disjoint sets. Critical point results in annular conical sets can be seen as extensions of Krasnoselskii's fixed point theorem for cones. Similar to Schechter's approach, the methods in [13] relied on deformation arguments within Hilbert spaces, exploiting their rich geometric structure. Later, in papers [12] and [15], analogous results were obtained in Banach spaces with some geometric properties. The alternative method of obtaining critical points, based on Ekeland's variational principle, has also been used for localization in bounded conical sets [14], [15].

An interesting idea is to search for critical points on specific subsets where they are likely to lie. A classical example is the Nehari manifold, which has been extensively studied in the literature. A particularly insightful and comprehensive reference on this topic is the paper by Szulkin and Weth [18]; see also [17], [25], [7], [3], [1]. For a real Banach space X and a C^1 functional E, the corresponding Nehari manifold is defined as

$$\mathcal{N} := \{u \in X \setminus \{0\} : \langle E'(u), u\rangle = 0\},$$

where $\langle \cdot, \cdot \rangle$ denotes the dual pairing between X^* and X. Obviously, any nonzero critical point of E, i.e., solution of the equation $E'(u) = 0$, belongs to $\mathcal{N}$. It may happen that the converse statement is also valid for certain points of $\mathcal{N}$ with appropriate properties. For example, we can look for points that minimize the functional E on $\mathcal{N}$, even if E is unbounded from below on the entire domain.

Given the parallelism that can be highlighted between the fixed point and critical point theory, it is natural to assume that a deeper interaction of the two theories would be possible. This is exactly the purpose of this work, which for the first time combines the Nehari manifold technique with the topological invariant-direction theorem of Birkoff-Kellogg, thus obtaining results for locating solutions in annular conical sets without using Ekeland's principle. The idea is to use a cone version of the Birkoff-Kellogg theorem for a given operator defined on a domain whose boundary coincides with the Nehari manifold, to guarantee the existence of an eigenvalue and an eigenvector. Then, using the definition of the Nehari manifold, it is shown that the eigenvalue must be equal to one, which implies that the corresponding direction is a critical point of the functional. Since the invariant-direction theorem is fundamentally derived using the fixed point index, our approach effectively combines critical point techniques (the Nehari manifold method) with fixed point methods.

In [22], the second author exploits the method of the Nehari manifold, combining it with Ekeland's variational principle to obtain solutions within annular domains. The present work aims not only to demonstrate a natural and somewhat unexpected application of the Birkhoff-Kellogg theorem but also to extend and strengthen the results of [22] in several key directions: first and foremost, we generalize the theory from Hilbert spaces to Banach spaces. Secondly, we relax the regularity assumption on the functional, requiring only C^1 smoothness instead of C^2, and finally, some conditions imposed in [22] are no longer necessary in our framework.

2. Preliminaries

In this section we recall some basic notions and results that are used throughout the paper.

2.1. **The duality mapping.** Let X be a real Banach space, X^* its dual space and let $\langle \cdot, \cdot \rangle$ denote the dual pairing between X^* and X. A function $\varphi \colon \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a normalization function if it is continuous, strictly increasing, $\varphi(0) = 0$ and $\varphi(\tau) \to \infty$ as $\tau \to \infty$. The duality mapping corresponding to the normalization function φ is the set-valued mapping $J : X \to 2^{X^*}$

given by

$$J(x) := \{x^* \in X^* : \langle x^*, x\rangle = \varphi(|x|)|x|,\ |x^*|_{X^*} = \varphi(|x|)\}. \tag{2.1}$$

Several fundamental properties of the duality mapping are summarized in the following proposition. For proofs and additional information, we refer the reader to [4–6].

Proposition 2.1. *The duality mapping* (2.1) *has the following properties:*

(a) For each $x \in X$, the set $J(x)$ is nonempty, bounded, convex and closed;
(b) J is monotone, i.e., for all $x, y \in X$, $x^* \in J(x)$ and $y^* \in J(y)$, we have

$$\langle x^* - y^*, x - y\rangle \geq (\varphi(|x|) - \varphi(|y|))(|x| - |y|) \geq 0. \tag{2.2}$$

(c) If X is strictly convex, then J is strictly monotone, i.e., (2.2) holds with strict inequality for $x \neq y$;
(d) If X^* is strictly convex, then J is single-valued;
(e) If X is reflexive and J is single-valued, then $J(X) = X^*$ and J is demicontinuous, i.e., if $x_n \to x$ strongly, then $J(x_n) \rightharpoonup J(x)$ weakly;
(f) If X is reflexive and locally uniformly convex and J is single-valued, then J is bijective from X to X^* and its inverse J^{-1} is bounded, continuous and monotone.

2.2. **Energetic Harnack inequality for the p-Laplacian.** As an example of duality mapping, we mention the case of the Sobolev space $W_0^{1,p}(0,1)$ for $1 < p < \infty$, endowed with the energetic norm

$$|u|_{1,p} = \left(\int_0^1 \left|u'(s)\right|^p ds\right)^{\frac{1}{p}},$$

corresponding to the p-Laplace operator $J(u) = -\left(|u'|^{p-2}u'\right)'$. This operator is the duality mapping of the space $W_0^{1,p}(0,1)$ corresponding to the normalization function $\varphi(\tau) = \tau^{p-1}$. In virtue of the very good geometry of the space, J is invertible and its inverse

$$J^{-1} : W^{-1,p'}(0,1) \to W_0^{1,p}(0,1), \quad \left(\tfrac{1}{p} + \tfrac{1}{p'} = 1\right),$$

is bounded, continuous, and strictly monotone.

With respect to the p-Laplace operator, we have the following Harnack type inequality in terms of the energetic norm, obtained in [12]. This result proves to be extremely useful for localization in annular sets, as it allows for obtaining certain lower bounds (see also [15]).

Lemma 2.2 (Lemma 3.1 from [12])**.** *For every function $u \in W_0^{1,p}(0,1)$ such that $u(t) \geq 0$ and $u(t) = u(1-t)$ for all $t \in \left(0, \frac{1}{2}\right)$, if $Ju \in C([0,1], \mathbb{R}_+)$ and Ju is nondecreasing on $\left(0, \frac{1}{2}\right)$, then*

$$u(t) \geq t(1-2t)^{1/(p-1)}|u|_{1,p}, \tag{2.3}$$

for all $t \in \left(0, \frac{1}{2}\right)$.

2.3. **Birkhoff-Kellogg type theorem in cones.** One of our tools in this paper is the version in cones due to Krasnoselskii and Ladyzenskii [11] (see also [10, p.139], [23] and [1]), of the classical theorem of Birkhoff and Kellogg invariant-direction theorem [2] (see also [9], [10, Theorem 6.6]) regarding the existence of a 'nonlinear' eigenvalue and eigenvector for compact maps in Banach spaces.

Theorem 2.3 (Krasnoselskii and Ladyzenskii)**.** *Let X be a real Banach space, $U \subset X$ be an open bounded set with $0 \in U$, $K \subset X$ a cone, and $T : K \cap \overline{U} \to K$ a compact operator. If*

$$\inf_{x \in K \cap \partial U} |T(x)| > 0,$$

then, there exist $\lambda_0 > 0$ and $x_0 \in K \cap \partial U$ such that

$$x_0 = \lambda_0 T(x_0).$$

3. Main result

In what follows, X is a reflexive and locally uniformly convex Banach space with a single-valued duality mapping J; K is a nondegenerate cone in X, i.e., $K \subset X$ is closed, convex, $\lambda K \subset K$ for all $\lambda \in \mathbb{R}_+$ and $K \setminus \{0\} \neq \emptyset$; and $E : X \to \mathbb{R}$ is a C^1 Fréchet differentiable functional.

Given $0 < r < R < \infty$, our aim is to determine a critical point of E within the conical annular set

$$K_{r,R} := \{u \in K : r \leq |u| \leq R\}.$$

The Nehari manifold of the functional E restricted to $K_{r,R}$ is the set

$$\mathcal{N}_{r,R} = \{u \in K_{r,R} : \langle E'(u), u\rangle = 0\}.$$

In the subsequent, we consider the operators

$$N\colon X \to X^*, \quad N(u) := J(u) - E'(u),$$

and

$$T : X \to X, \quad T = J^{-1}N.$$

Our first two assumptions regard the operator T, and read as follows:

(H1): The operator T is completely continuous from X to X, and moreover, it is invariant with respect to the cone K, i.e.,

$$T(K) \subset K.$$

(H2): The operator T is bounded away from zero on the set $\mathcal{N}_{r,R}$, that is,

$$\inf_{u \in \mathcal{N}_{r,R}} |T(u)| > 0.$$

To state the third assumption, for each point $u \in K \setminus \{0\}$, define the function

$$\alpha_u\colon \mathbb{R}_+ \to \mathbb{R}, \quad \alpha_u(\sigma) = E(\sigma u). \tag{3.1}$$

(H3): For every $u \in K \setminus \{0\}$, there exists a unique $s(u) \in \left(\frac{r}{|u|}, \frac{R}{|u|}\right)$ such that

$$\alpha_u'(s(u)u) = 0.$$

Moreover, α_u' is positive on $\left[\frac{r}{|u|}, s(u)\right)$ and negative on $\left(s(u), \frac{R}{|u|}\right]$.

From condition (H3), we see that the Nehari manifold $N_{r,R}$ allows for the representation

$$\mathcal{N}_{r,R}=\{s(u)u: \ u\in K\setminus\{0\}\}.$$

Another important consequence of assumption (H3), essential to our analysis, is the continuity of the mapping s, as shown in the following lemma.

Lemma 3.1. *The mapping* $s: K\setminus\{0\}\to\mathbb{R}$ *is continuous.*

Proof. Let $u_k\in K\setminus\{0\}$ with $u_k\to u\in K\setminus\{0\}$. Since

$$\frac{r}{|u_k|}<s(u_k)<\frac{R}{|u_k|},\ \ \frac{r}{|u_k|}\to\frac{r}{|u|},\ \ \frac{R}{|u_k|}\to\frac{R}{|u|},$$

there exists $k_0\in\mathbb{N}$ such that

$$s(u_k)\in\left(\frac{r}{|u|},\frac{R}{|u|}\right)\quad\text{for all } k\geq k_0.$$

The boundedness of $s(u_k)$ ensures the existence of a convergent subsequence, which we still denote by $s(u_k)$. Let $\widetilde{s}$ be its limit. Then, one clearly has

$$\widetilde{s}\in\left[\frac{r}{|u|},\frac{R}{|u|}\right].$$

By definition, we have that $\langle E'(s(u_k)u_k),u_k\rangle=0$, and therefore $\langle E'(\widetilde{s}u),u\rangle=0$. From assumption (H3), it follows that $\widetilde{s}=s(u)$. Consequently, the entire sequence $s(u_k)$ is convergent to $s(u)$, which completes our proof. □

We are now ready to state the main result of this paper whose proof relies on the Krasnoselskii-Ladyzhenskaya theorem.

Theorem 3.2. *Assume that conditions (H1)-(H3) are satisfied. Then, there exists* $u^*\in\mathcal{N}_{r,R}$ *such that* $E'(u^*)=0$.

Proof. The central idea of the proof is to apply the Krasnoselskii-Ladyzenskii theorem to the operator $T=J^{-1}N$ on an open set U chosen such that

$$K\cap\partial U=\mathcal{N}_{r,R}. \tag{3.2}$$

To this end, we define the set

$$\tilde{U}:=\{\sigma u: u\in\mathcal{N}_{r,R},\ \sigma\in[0,1)\}.$$

Clearly, $0\in\tilde{U}$. To verify that $\tilde{U}$ is open in the relative topology of K, it is sufficient to show that $K\setminus\tilde{U}$ is closed. Let $w_k\in K\setminus\tilde{U}$ and $w_k\to w$. Then, there exist $u_k\in\mathcal{N}_{r,R}$ and $\sigma_k\geq 1$ such that

$$w_k=\sigma_k u_k. \tag{3.3}$$

Since the sequence u_k is bounded (recall that $|u_k|\in(r,R)$), it follows that the sequence σ_k is also bounded and therefore admits a convergent subsequence, which we denote again by σ_k. Let σ be the limit of this subsequence. Clearly, $\sigma_k\geq 1$ implies that $\sigma\geq 1$.

From (3.3), we may write $u_k = \frac{1}{\sigma_k} w_k$, hence u_k is convergent to $u := \frac{1}{\sigma} w$. As the Nehari manifold $\mathcal{N}_{r,R}$ is closed, it follows that $u \in \mathcal{N}_{r,R}$. Consequently, the limit point w satisfies

$$w = \sigma u \quad \text{with } \sigma \geq 1 \text{ and } u \in \mathcal{N}_{r,R},$$

which implies that $w \in K \setminus \tilde{U}$.

Let us observe that the relative boundary of $\tilde{U}$ is

$$\partial_K \tilde{U} := \mathrm{cl}(\tilde{U}) \setminus \tilde{U} = s^{-1}(1) = \mathcal{N}_{r,R}.$$

To strictly comply with the conditions of Theorem 2.3, let us indicate the open set U of the space X to which it applies. Recall that, by a theorem due to Dugundji (see [8, Corollary 4.2]), every nonempty closed convex subset of a real Banach space X is a retract of X. In particular, every cone of X is a retract of X. Let $\rho\colon X \to K$ be a retract of K, i.e., a continuous mapping such that $\rho(u) = u$ for all $u \in K$. Then, the set $U := \rho^{-1}(\tilde{U})$ is open in X and $K \cap \partial U = \mathcal{N}_{r,R}$. Indeed, since $\rho(u) = u$ for all $u \in K$, one has

$$K \cap \rho^{-1}(\tilde{U}) = \tilde{U}.$$

Moreover,

$$K \cap \partial U = K \cap \partial(\rho^{-1}(\tilde{U})) = \partial_K \left(K \cap \rho^{-1}(\tilde{U})) \right) = \partial_K \left(\tilde{U} \right),$$

which proves our claim.

Now, based on assumptions (H1) and (H2), Theorem 2.3 guarantees the existence of an element $u^* \in \mathcal{N}_{r,R}$ and a positive scalar $\lambda_0 > 0$ such that

$$J^{-1} N(u^*) = \lambda_0 u^*,$$

or equivalently

$$N(u^*) = J(\lambda_0 u^*).$$

Moreover, since $u^* \in \mathcal{N}_{r,R}$, it follows that

$$0 = \langle J(u^*) - N(u^*), u^* \rangle = \langle J(u^*) - J(\lambda_0 u^*), u^* \rangle. \tag{3.4}$$

From this, we also have

$$0 = \langle J(u^*) - N(u^*), u^* \rangle = \langle J(u^*) - J(\lambda_0 u^*), \lambda_0 u^* \rangle. \tag{3.5}$$

Combining (3.4) and (3.5), we derive

$$\langle J(u^*) - J(\lambda_0 u^*), u^* - \lambda_0 u^* \rangle = 0.$$

Using the strong monotonicity property of the duality mapping J (Proposition 2.1 (c)), we infer that

$$\lambda_0 u^* = u^*,$$

which implies $\lambda_0 = 1$. Consequently, the identity $N(u^*) = J(u^*)$ holds, therefore u^* is a critical point of the functional E. □

Let us note that hypotheses (H2) and (H3) require a certain behavior of the functional E only relative to an interval $[r, R]$. The situation in which this behavior occurs on several such intervals leads us to the multiplicity of critical points, with their location in disjoint annular conical sets. Thus, Theorem 3.2 directly yields the following multiplicity principle.

Theorem 3.3 (Multiplicity)**.** *Let condition (H1) hold.*

(1^0): *If there are finite sequences of numbers* $(r_k)_{1\le k\le m}$ *and* $(R_k)_{1\le k\le m}$ *with*

$$0 < r_1 < R_1 < r_2 < R_2 < \cdots < r_m < R_m$$

such that conditions (H2) and (H3) are satisfied for every pair (r_k, R_k), $k = 1,2,...,m$, *then there exist m points* x_k^* *with*

$$E'\left(x_k^*\right) = 0, \ \ x_k^* \in K, \ \ r_k < \left|x_k^*\right| < R_k \quad (k = 1,2,...,m).$$

(2^0): *If there are increasing sequences of numbers* $(r_k)_{k\ge 1}$ *and* $(R_k)_{k\ge 1}$ *with*

$$0 < r_k < R_k < r_{k+1} \ \ (k \ge 1), \quad r_k \to \infty \ \ as\ k \to \infty$$

such that conditions (H2) and (H3) are satisfied for every pair (r_k, R_k), $k \ge 1$, *then there exists a sequence of points* $\left(x_k^*\right)_{k\ge 1}$ *with*

$$E'\left(x_k^*\right) = 0, \ \ x_k^* \in K, \ \ r_k < \left|x_k^*\right| < R_k; \ \ \left|x_k^*\right| \to \infty \quad as\ k \to \infty.$$

(3^0): *If there are decreasing sequences of numbers* $(r_k)_{k\ge 1}$ *and* $(R_k)_{k\ge 1}$ *with*

$$0 < R_{k+1} < r_k < R_k \ \ (k \ge 1), \quad R_k \to 0 \ \ as\ k \to \infty$$

such that conditions (H2) and (H3) are satisfied for every pair (r_k, R_k), $k \ge 1$, *then there exists a sequence of points* $\left(x_k^*\right)_{k\ge 1}$ *with*

$$E'\left(x_k^*\right) = 0, \ \ x_k^* \in K, \ \ r_k < \left|x_k^*\right| < R_k; \ \ x_k^* \to 0 \quad as\ k \to \infty.$$

Proof. The result follows directly by applying Theorem 3.2 to each pair (r_k, R_k). □

4. Application

To illustrate the theoretical results, we consider the Dirichlet problem for a p-Laplace equation

$$\begin{cases} -\left(|u'|^{p-2}u'\right)'(t) = f(u(t)), & t \in (0,1) \\ u(0) = u(1) = 0, \end{cases} \tag{4.1}$$

where $1 < p < \infty$, and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function that is nonnegative and nondecreasing on $\mathbb{R}_+$.

Consider the Banach space $X := W_0^{1,p}(0,1)$ endowed with the usual norm $|u|_{1,p} := |\nabla u|_{L^p}$. It is well known (see, e.g., [5, Chapter 1.2]) that $W_0^{1,p}(0,1)$ is a uniformly convex and reflexive Banach space with $W_0^{-1,p'}(0,1)$ its dual, where p' is the conjugate of p, i.e., $\frac{1}{p} + \frac{1}{p'} = 1$. If $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $W_0^{-1,p'}(0,1)$ and $W_0^{1,p}(0,1)$, and $v \in L^q(0,1) \subset W_0^{-1,p'}(0,1)$, then

$$\langle v, u \rangle = \int_0^1 v(t)u(t)dt,$$

for all $u \in W_0^{1,p}(0,1)$ (see [16, Proposition 8.14]).

Let λ_p denote the first eigenvalue of the Euler-Lagrange equation

$$J(u) = \lambda |u|_{1,p}^{p-2} u \quad \text{in } (0,1), \quad u(0) = u(1) = 0.$$

It is known (see, e.g., [5]) that

$$\lambda_p = \min_{u \in W_0^{1,p}(0,1)\setminus\{0\}} \frac{|u|_{1,p}^p}{|u|_p^p},$$

that is, $c_p = \lambda_p^{-\frac{1}{p}}$ is the smallest constant such that

$$|u|_p \le c_p |u|_{1,p}, \tag{4.2}$$

for all $u \in W_0^{1,p}(0,1)$. Also, for Nu trebuie "from"?? the continuous embedding $W_0^{1,p}(0,1) \subset C[0,1]$ one has

$$|u(t)| \le |u|_{1,p} \quad (t \in [0,1]),$$

for every $u \in W_0^{1,p}(0,1)$.

The energy functional of the problem (4.1) is

$$E(u) = \frac{1}{p}|u|_{1,p}^p - \int_0^1 F(u(t))dt,$$

where $F(\xi) = \int_0^\xi f(s)ds$. One has

$$E'(u) = J(u) - N_f(u),$$

where N_f is the Nemytski superposition operator $N(u) := f(u)$. Hence the solutions of problem (4.1) are the critical points of functional E.

In $W^{1,p}(0,1)$, we consider the cone

$$K = \Big\{u \in H_0^1(0,1) : u \ge 0,\ u \text{ is nondecreasing on } [0,1/2],$$
$$u(t) = u(1-t) \text{ and } u(t) \ge \phi(t)|u|_{1,p} \text{ for all } t \in [0,1/2]\Big\},$$

where ϕ is the function involved in the energetic Harnack inequality (2.3), namely

$$\phi : \left(0, \frac{1}{2}\right) \to \mathbb{R}, \quad \phi(t) = t(1-2t)^{1/(p-1)}.$$

Let $\beta \in (0,1/4)$, and denote

$$\Phi := \int_\beta^{1/2} \phi(t)dt.$$

For $0 < r < R < \infty$, we assume that the following conditions hold:

(h1): The function f satisfies the inequalities

$$\frac{f(r)}{r^{p-1}} < \frac{1}{c_p} \quad \text{and} \quad \frac{f(R\phi(\beta))}{R^{p-1}} > \frac{1}{2\Phi}. \tag{4.3}$$

(h2): The function
$$g(t) := \frac{f(t)}{t^{p-1}}$$
is strictly increasing on $(0, R]$.

Given the above two conditions, the following existence result holds.

Theorem 4.1. *Under conditions (h1) and (h2), problem* (4.1) *admits a solution* $u \in K$ *such that*
$$r < |u|_{1,p} < R.$$

Proof. We apply Theorem 3.2.

Check of (H1).

(a). Complete continuity of the operator $J^{-1}N_f$*.*The operator $J^{-1}N_f$ is completely continuous from $W_0^{1,p}(0,1)$ to itself. Indeed, since $W_0^{1,p}(0,1)$ compactly embeds into $C[0,1]$ (see, e.g., [16, Theorem 8.8]), and $C[0,1]$ continuously embeds into $W^{-1,p'}(0,1)$, the continuity of f ensures that the Nemytskii operator N_f is completely continuous from $W_0^{1,p}(0,1)$ to $W^{-1,p'}(0,1)$. Finally, since J^{-1} is a continuous bounded operator, it follows that $J^{-1}N_f$ is completely continuous, as claimed.

We now show that the cone K is invariant under the operator $J^{-1}N_f$. Let $u \in K$, and denote $v = J^{-1}N_f(u)$. We show that $v \in K$.

By the comparison principle for the p-Laplace operator (see, e.g., [24, Lemma 1.3]), since $N_f(u) \geq 0$, it follows that $J^{-1}N_f(u) \geq 0$. To prove that v is symmetric, denote $w(t) := v(1-t)$. Since u is symmetric, so is $f(u)$, and hence
$$N_f(u(1-t)) = N_f(u(t)).$$
Moreover, we have
$$J(v)(1-t) = N_f(u(1-t)),$$
and
$$J(v)(1-t) = J(w)(t).$$
Therefore,
$$J(w) = N_f(u),$$
which shows that both v and w solve the same Dirichlet problem for the p-Laplace equation. By uniqueness of the solution to this problem, it follows that $v = w$, that is, v is symmetric. Finally, we observe that $J(v) = J(J^{-1}N_f(u)) = N_f(u)$, which is nonnegative and nondecreasing on $[0,1/2]$. Therefore, Lemma 2.2 applies and guarantees that the inequality (2.3) holds for v. Consequently, $v \in K$ as claimed, hence condition (H1) holds.

Check of (H2). Suppose, by contradiction, that (H2) does not hold. Then, one can find a sequence $\{u_k\} \subset \mathcal{N}_{r,R}$ such that
$$J^{-1}N_f(u_k) \to 0. \tag{4.4}$$
Since J is demicontinuous (Proposition 2.1 (e)), relation (4.4) implies that
$$J(J^{-1}N_f(u_k)) = N_f(u_k) \rightharpoonup 0,$$

weakly. Therefore, for any given $\chi \in K \setminus \{0\}$, we have

$$\langle N_f(u_k), \chi \rangle \to 0.$$

From the Harnack inequality and the monotonicity of the functions u_k and χ on the interval $[0, 1/2]$, for every $t \in (\beta, 1/2)$, we have

$$\begin{aligned} &u_k(t) \geq u_k(\beta) \geq \phi(\beta)|u_k|_{1,p}, \text{ and} \\ &\chi(t) \geq \phi(t)|\chi|_{1,p}. \end{aligned} \tag{4.5}$$

Using the monotonicity of f, the symmetry of u_k and χ, and the bounds in (4.5), we obtain

$$\begin{aligned} \langle N_f(u_k), \chi \rangle &= \int_0^1 f(u_k(t))\chi(t)\,dt \geq 2\int_\beta^{1/2} f(u_k(t))\chi(t)\,dt \\ &\geq 2|\chi|_{1,p} f(\phi(\beta)|u_k|_{1,p}) \int_\beta^{1/2} \phi(t)\,dt = 2|\chi|_{1,p} f(\phi(\beta)|u_k|_{1,p})\Phi. \end{aligned}$$

Since $|u_k|_{1,p} \geq r$ for all k, we conclude that

$$0 \leq 2|\chi|_{1,p} f(\phi(\beta)r)\Phi \leq \langle N_f(u_k), \chi \rangle \to 0,$$

whence $f(\phi(\beta)r) = 0$, which contradicts the strict positivity of f on $(0, R]$ implied by (h2). Hence, (H2) holds.

Check of (H3). Let $u \in K \setminus \{0\}$ and denote $w := \frac{u}{|u|_{1,p}}$. We immediately see that the derivative of the mapping α_u defined in (3.1) is

$$\alpha_u'(\sigma) = \sigma^{p-1}|u|_{1,p}^p - \int_0^1 f(\sigma u(t))u(t)\,dt.$$

We claim that

$$\alpha_u'\left(\frac{r}{|u|_{1,p}}\right) > 0 \quad \text{and} \quad \alpha_u'\left(\frac{R}{|u|_{1,p}}\right) < 0. \tag{4.6}$$

Since $w(t) \leq 1$ for all $t \in [0, 1]$, it follows that $f(rw(t)) \leq f(r)$ for all $t \in [0, 1]$. Moreover, by Hölder's inequality and (4), we have

$$\int_0^1 w(t)\,dt \leq \left(\int_0^1 w(t)^p\,dt\right)^{1/p} \leq c_p|w|_{1,p} = c_p,$$

since $|w|_{1,p} = 1$ by definition. Thus, using the first inequality in (4.3), we obtain

$$\begin{aligned}
\alpha_u'\left(\frac{r}{|u|_{1,p}}\right) &= r^{p-1}|u|_{1,p} - \int_0^1 f(rw(t))u(t)dt \\
&= |u|_{1,p}\left(r^{p-1} - \int_0^1 f(rw(t))w(t)dt\right) \\
&\geq |u|_{1,p}\left(r^{p-1} - f(r)\int_0^1 w(t)dt\right) \\
&\geq |u|_{1,p}\left(r^{p-1} - c_p f(r)\right) \\
&> 0,
\end{aligned}$$

that is, the first inequality in (4.6). To prove the second claim, note that the monotonicity of w on $[0,1/2]$, together with the Harnack inequality, yields that

$$w(t) \geq \phi(\beta) \quad \text{for all } t \in [\beta, 1/2]. \tag{4.7}$$

Using the symmetry of w and the second inequality in (4.3), we find that

$$\begin{aligned}
\alpha_u'\left(\frac{R}{|u|_{1,p}}\right) &= R^{p-1}|u|_{1,p} - \int_0^1 f(Rw(t))u(t)dt \\
&= |u|_{1,p}\left(R^{p-1} - 2\int_0^{1/2} f(Rw(t))w(t)dt\right) \\
&\leq |u|_{1,p}\left(R^{p-1} - 2\int_\beta^{1/2} f(R\phi(t))\phi(t)dt\right) \\
&\leq |u|_{1,p}\left(R^{p-1} - 2\Phi f(R\phi(\beta))\right) \\
&< 0.
\end{aligned}$$

Consequently, the second inequality in (4.6) also holds.

To continue with the verification of $(H3)$, let us denote $w := \frac{u}{|u|_{1,p}}$ and $\gamma := \sigma|u|_{1,p}$. Then,

$$\begin{aligned}
\alpha_u'(\sigma) &= \sigma^{p-1}|u|_{1,p}^p - \int_0^1 f(\sigma u(t))u(t)dt \\
&= \sigma^{p-1}|u|_{1,p}^p\left(1 - \int_0^1 \frac{f(\gamma w(t))}{\gamma^{p-1}}w(t)dt\right) \\
&= \sigma^{p-1}|u|_{1,p}^p h(\gamma),
\end{aligned}$$

where

$$h(\gamma) := 1 - \int_0^1 \frac{f(\gamma w(t))}{\gamma^{p-1}}w(t)dt \quad (\gamma \in [r,R]).$$

We now show that the function h is strictly decreasing on $[r,R]$. For this, let $r \le \gamma_1 < \gamma_2 \le R$. One has

$$h(\gamma_1) - h(\gamma_2) = 2\int_0^{\frac{1}{2}} \left(\frac{f(\gamma_2 w(t))}{\gamma_2^{p-1}} - \frac{f(\gamma_1 w(t))}{\gamma_1^{p-1}} \right) w(t)\, dt.$$

Since $0 < w(t) \le 1$ for all $t \in (0,1/2]$, it follows that $0 < \gamma_1 w(t) < \gamma_2 w(t) \le R$. Then, using assumption (h2), we have

$$g(\gamma_2 w(t)) > g(\gamma_1 w(t)),$$

which implies

$$\begin{aligned} \left(\frac{f(\gamma_2 w(t))}{\gamma_2^{p-1}} - \frac{f(\gamma_1 w(t))}{\gamma_1^{p-1}} \right) w(t) &= \left(\frac{f(\gamma_2 w(t))}{\gamma_2^{p-1} w(t)^{p-1}} - \frac{f(\gamma_1 w(t))}{\gamma_1^{p-1} w(t)^{p-1}} \right) w(t)^p \\ &= (g(\gamma_2 w(t)) - g(\gamma_1 w(t)))\, w(t)^p \\ &> 0, \end{aligned}$$

for all $t \in (0,1/2]$. Therefore $h(\gamma_1) > h(\gamma_2)$ and thus h is strictly decreasing on $[r,R]$. Moreover, since $h(r) > 0$ and $h(R) < 0$, it follows that h has exactly one zero γ_0 in (r,R), is positive on (r,γ_0) and negative on (γ_0,R). Correspondingly, α_u' has the unique zero at

$$s(u) := \frac{\gamma_0}{|u|_{1,p}},$$

is positive on $\left(\frac{r}{|u|_{H_0^1}}, s(u)\right)$ and negative on $\left(s(u), \frac{R}{|u|_{H_0^1}}\right)$. So condition (H3) is verified.

Since all the conditions (H1)-(H3) are satisfied, Theorem 3.2 applies and gives the conclusion. □

Instead of condition (h2), we may consider an alternative assumption formulated in relation to the annular conical set $K_{r,R}$. More exactly,

(h2') The function f is of class C^1 on $\mathbb{R}$ and

$$\min_{t \in [r\phi(\beta),R]} f'(t) > \frac{(p-1)R^{p-2}}{2\Psi}\ , \tag{4.8}$$

where

$$\Psi = \int_\beta^{1/2} \phi(t)^2 dt.$$

Theorem 4.2. *Under conditions (h1) and (h2'), the problem (4.1) has a solution $u \in K$ satisfying*

$$r < |u|_{1,p} < R.$$

Proof. Similar to the proof of Theorem 4.1, assumption (H1) from Theorem 3.2 is satisfied. In addition, it is easy to see that the strict positivity of $f'(r\phi(\beta))$ implies $f(r\phi(\beta)) > 0$, which as above guarantees (H2). Moreover, by (h1), relation (4.6) also holds. To verify condition (H2), it remains to prove that α_u' has a unique zero $s(u)$ within the interval $\left(\frac{r}{|u|_{1,p}}, \frac{R}{|u|_{1,p}}\right)$, is positive on $\left(\frac{r}{|u|_{1,p}}, s(u)\right)$ and negative on $\left(s(u), \frac{R}{|u|_{1,p}}\right)$. Under the notations from the proof of the previous theorem, consider the function

$$\tilde{h}(\gamma) := \gamma^{p-1} - \int_0^1 f(\gamma w(t))w(t)dt = \gamma^{p-1}h(\gamma) \quad (\gamma \in [r,R]).$$

Then, we have

$$\alpha_u'(\sigma) = |u|_{1,p}\tilde{h}(\sigma|u|_{1,p}) = |u|_{1,p}\tilde{h}(\gamma). \tag{4.9}$$

We now show that $\tilde{h}$ is strictly decreasing on $[r,R]$. Since f is of class C^1, it suffices to show that

$$\tilde{h}'(\gamma) < 0, \quad \text{for all } \gamma \in [r,R].$$

Differentiating, and using that f is nondecreasing, we obtain

$$\begin{aligned}\tilde{h}'(\gamma) &= (p-1)\gamma^{p-2} - 2\int_0^{1/2} f'(\gamma w(t))w(t)^2\,dt \\ &\le (p-1)\gamma^{p-2} - 2\int_\beta^{1/2} f'(\gamma w(t))w(t)^2\,dt.\end{aligned}$$

For $t \in [\beta, 1/2]$, one has $r\phi(\beta) \le \gamma w(t) \le R$, whence

$$f'(\gamma w(t)) \ge \min_{s \in [r\phi(\beta), R]} f'(s).$$

Then, also using (4.8), we obtain

$$\tilde{h}'(\gamma) \le (p-1)R^{p-2} - 2\Psi \min_{s \in [r\phi(\beta), R]} f'(s) < 0,$$

as we desired. Finally, by (4.6) and (4.9), we conclude that α_u' has a unique zero within the interval $\left(\frac{r}{|u|_{1,p}}, \frac{R}{|u|_{1,p}}\right)$. Moreover, α_u' is positive on $\left(\frac{r}{|u|_{1,p}}, s(u)\right)$ and α_u' is negative on $\left(s(u), \frac{R}{|u|_{1,p}}\right)$, so condition (H2) is verified.

Therefore, Theorem 3.2 applies and gives the conclusion. □

Remark 4.3. Condition (h2) given on the whole interval $(0,R]$ does not lead to multiplicity. To see why, suppose there are two pairs (r_i, R_i), $i = 1,2$, with $0 < r_1 < R_1 < r_2 < R_2$ for which conditions (h1) and (h2) are both satisfied. By assumption (h1), we have

$$\frac{f(r_2)}{r_2^{p-1}} < \frac{1}{c_p}.$$

On the other hand, (h2) yields

$$\frac{f(R_1)}{R_1^{p-1}} \le \frac{f(r_2)}{r_2^{p-1}} < \frac{1}{c_p}.$$

This implies

$$\begin{aligned}\alpha_u'\left(\frac{R_1}{|u|_{1,p}}\right) &= R_1^{p-1}|u|_{1,p} - \int_0^1 f(R_1 w(t))u(t)dt \\ &= |u|_{1,p}\left(R_1^{p-1} - \int_0^1 f(R_1 w(t))w(t)dt\right) \\ &\geq |u|_{1,p}\left(R_1^{p-1} - f(R_1)\right) \\ &\geq |u|_{1,p}\left(R_1^{p-1} - c_p f(R_1)\right) \\ &> 0,\end{aligned}$$

which contradicts (4.6). However, condition (h2') can be applied separately to each of several disjoint annular sets, which leads to multiple solutions, as shown in the following result illustrating the general multiplicity principle given by Theorem 3.3

Theorem 4.4. **(1^0):** *If there are finite sequences of numbers $(r_k)_{1\leq k\leq m}$ and $(R_k)_{1\leq k\leq m}$ with*

$$0 < r_1 < R_1 < r_2 < R_2 < \cdots < r_m < R_m$$

such that conditions (h1) and (h2') are satisfied for every pair (r_k, R_k), $k = 1,2,...,m$, then there exist m solutions u_k^* of problem (4.1) with

$$u_k^* \in K,\ \ r_k < \left|u_k^*\right|_{1,p} < R_k \quad (k = 1,2,...,m).$$

(2^0): *If there are increasing sequences of numbers $(r_k)_{k\geq 1}$ and $(R_k)_{k\geq 1}$ with*

$$0 < r_k < R_k < r_{k+1} \ \ (k \geq 1), \quad r_k \to \infty \ \ as\ k \to \infty$$

such that conditions (h1) and (h2') are satisfied for every pair (r_k, R_k), $k \geq 1$, then there exists a sequence of solutions $\left(u_k^*\right)_{k\geq 1}$ of problem (4.1) with

$$u_k^* \in K,\ \ r_k < \left|u_k^*\right|_{1,p} < R_k;\ \ \left|u_k^*\right|_{1,p} \to \infty \quad as\ k \to \infty.$$

(3^0): *If there are decreasing sequences of numbers $(r_k)_{k\geq 1}$ and $(R_k)_{k\geq 1}$ with*

$$0 < R_{k+1} < r_k < R_k \ \ (k \geq 1), \quad R_k \to 0 \ \ as\ k \to \infty$$

such that conditions (h1) and (h2') are satisfied for every pair (r_k, R_k), $k \geq 1$, then there exists a sequence of solutions $\left(u_k^*\right)_{k\geq 1}$ of problem (4.1) with

$$u_k^* \in K,\ \ r_k < \left|x_k^*\right|_{1,p} < R_k;\ \ \left|u_k^*\right|_{1,p} \to 0 \quad as\ k \to \infty.$$

Remark 4.5. It is worth mentioning that requirement (h1) can be satisfied for a sequence of pairs (r_k, R_k) as in (2^0), if for example

$$\liminf_{\tau\to\infty} \frac{f(\tau)}{\tau^{p-1}} < \frac{1}{c_p} \quad \text{and} \quad \limsup_{\tau\to\infty} \frac{f(\tau)}{\tau^{p-1}} > \frac{1}{2\Phi\phi(\beta)^{p-1}}.$$

Similarly, (h1) can be satisfied for a sequence of pairs (r_k, R_k) as in (3^0), if

$$\liminf_{\tau\to 0+}\frac{f(\tau)}{\tau^{p-1}}<\frac{1}{c_p}\quad\text{and}\quad\limsup_{\tau\to 0+}\frac{f(\tau)}{\tau^{p-1}}>\frac{1}{2\Phi\phi(\beta)^{p-1}}.$$

Both situations mean a very strong oscillation towards infinity and zero, respectively, from below $\frac{1}{c_p}$ to above $\frac{1}{2\Phi\phi(\beta)^{p-1}}$.

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Email address: r.precup@ictp.acad.ro
Email address: andrei.stan@ubbcluj.ro

R. Precup, Faculty of Mathematics and Computer Science and Institute of Advanced Studies in Science and Technology, Babeş-Bolyai University, 400084 Cluj-Napoca, Romania & Tiberiu Popoviciu Institute of Numerical Analysis, Romanian Academy, P.O. Box 68-1, 400110 Cluj-Napoca, Romania

A. Stan, Department of Mathematics, Babeş-Bolyai University, 400084 Cluj-Napoca, Romania & Tiberiu Popoviciu Institute of Numerical Analysis, Romanian Academy, P.O. Box 68-1, 400110 Cluj-Napoca, Romania