

Some variants of Ćirić's multi-valued contraction principle

Cristian Daniel Alecsa and Adrian Petrușel

Dedicated to the memory of Professor Ștefan Mărușter

Abstract. In this article, a study of the fixed point problem for Ćirić type multi-valued operators is presented. More precisely, some variants of Ćirić's contraction principle for multi-valued operators, as well as a strict fixed point principle for Ćirić type multi-valued will be given.

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1 Introduction

The aim of this paper is to present a study on Ćirić type multi-valued operators. Following the approach given in [16], where the author considered some variants of the multivalued contraction principle given by Nadler [14], respectively a so-called strict multi-valued contraction principle, we will consider here the case of Ćirić type multi-valued operators, see [3].

We also notice that in [24] Reich developed some fixed point theorems for multi-valued generalized contractions. A fully comprehensive study on Reich operators was made in [12] by T. Lazăr et al. Also, qualitative properties, namely data dependence, Ulam-Hyers stability and so on, were studied

for the case of multi-valued φ -contractions by V.L. Lazăr in [13]. Moreover, C. Chifu and G. Petruşel in [5] studied qualitative properties concerning Hardy-Rogers multi-valued operators (see [7] for the single-valued case) in the framework of b-metric spaces, while T. Lazăr, D. O'Regan et al. [11] studied the case of multi-valued operators of Ćirić type defined on a set endowed with two metrics. Finally, we point out that in [2], M. Boriceanu studied existence and uniqueness of the fixed point and data dependence for multi-valued Ćirić type operators in the context of b-metric spaces. At the same time, Ćirić type multi-valued operators were studied in [17] and [19].

Regarding terminology and basic concepts for fixed point problems related to multi-valued operators, we will follow the works [1],[9], [18] and [23]. Furthermore, for the approximation of strict fixed points (also called end-points) of multi-valued mappings, we refer to [6], [8] and [22]. Finally, regarding data dependence, multi-valued fractal operators, selections and qualitative properties for the fixed point inclusion and for multi-valued fractals, we will refer to [4], [10] and [20].

Led (X, d) be a metric space. Denote by $P(X)$ the family of all nonempty subsets of X . Also, $P_b(X)$ stands for the family of nonempty, bounded subsets of X and $P_{cl}(X)$ the family of nonempty, closed subsets of X . In a similar manner, by $P_{cp}(X)$ we refer to the family of nonempty, compact subsets of X . From now on, $\overline{B}(x_0, r)$ means the closure in (X, d) of the ball $B(x_0, r)$, where $B(x_0, r) := \{x \in X \mid d(x_0, x) < r\}$ is the open ball with radius $r > 0$ and the center $x_0 \in X$. By $\widetilde{B}(x_0; r) := \{x \in X \mid d(x_0, x) \leq r\}$ we denote the closed ball centered in x_0 with radius r . We recall now some important functionals which will be used through the paper:

- the gap functional $D : P(X) \times P(X) \rightarrow \mathbb{R}_+$, $D(A, B) := \inf_{a \in A, b \in B} \{d(a, b)\}$.
- the generalized Pompeiu-Hausdorff functional $H : P(X) \times P(X) \rightarrow \mathbb{R}_+ \cup \{+\infty\}$, where $H(A, B) = \max\{\sup_{a \in A} D(a, B), \sup_{b \in B} D(b, A)\}$.

Furthermore, if $T : X \rightarrow P(X)$ is a multi-valued operator, then an element $x \in X$ is a fixed point for T if and only if $x \in T(x)$. We denote by F_T the set of all fixed points of the operator T and by $(SF)_T$ the set of all strict fixed points of T , where $x \in X$ is a strict fixed point of T (or an endpoint, or a stationary point) if and only if $\{x\} = Tx$.

For a multi-valued operator $T : X \rightarrow P(Y)$ we can also define the following useful notions. The graph of the operator T , defined by $Graph(T) := \{(x, y) \in X \times Y \mid y \in T(x)\}$, and the image of the set $Y \in P(X)$ will be denoted by $T(Y) := \bigcup_{x \in Y} T(x)$. A single-valued mapping $t : X \rightarrow Y$ is called

a selection of T if for each $x \in X$, we have that $t(x) \in T(x)$.

We present now an important concept, which appears naturally by Nadler's contraction principle. By [21], we recall here the notion of multi-valued weakly Picard operator.

Definition 1.1. Let (X, d) be a metric space.

Consider $T : X \rightarrow P(X)$ be a multi-valued operator. By definition, T is a multi-valued weakly Picard operator (briefly MWP operator) if for each $x \in X$ and for each $y \in T(x)$, there exists a sequence $(x_n)_{n \in \mathbb{N}}$, satisfying the following

- (i) $x_0 = x$ and $x_1 = y$,
- (ii) $x_{n+1} \in T(x_n)$, for each $n \in \mathbb{N}$,
- (iii) the sequence $(x_n)_{n \in \mathbb{N}}$ is convergent to a fixed point of T .

Remark 1.1. A sequence $(x_n)_{n \in \mathbb{N}}$ satisfying conditions (i) and (ii) is called a sequence of successive approximations of T starting from $(x, y) \in \text{Graph}(T)$. If $T : X \rightarrow P(X)$ is a MWP operator, then we define the operator $T^\infty : \text{Graph}(T) \rightarrow P(F_T)$, by $T^\infty(x, y) := \{z \in F_T \mid \text{there exists a sequence of successive approximations of } T \text{ starting from } (x, y) \text{ that converges to } z\}$.

Furthermore, if (X, d) is a metric space and $T : X \rightarrow P(X)$ a multi-valued operator, then T is said to be closed if $\text{Graph}(T)$ is a closed set in $X \times X$. By $T^1(x) := T(x), \dots, T^n(x) := T(T^{n-1}(x))$ we denote the iterates of the multi-valued mapping T , while the set $V^0(Y; \varepsilon) := \{x \in X \mid D(x, Y) < \varepsilon\}$ is called the (open) ε -neighborhood of $Y \in P(X)$.

From [14], we shall recall some important lemmas that are used throughout the article.

Lemma 1.1. Let A and B from $P(X)$ and $q > 1$. Then, for each $a \in A$, there exists $b \in B$, such that $d(a, b) \leq qH(A, B)$.

Lemma 1.2. Let A and B from $P(X)$. Also, consider $\eta > 0$, such that

- (i) for each $a \in A$, there exists $b \in B$, with $d(a, b) \leq \eta$,
 - (ii) for each $b \in B$, there exists $a \in A$, with $d(a, b) \leq \eta$.
- Then $H(A, B) \leq \eta$.

Now, we recall the basic concepts for the qualitative properties of the fixed point inclusion and of the fixed point iteration. The first two definitions are related to well-posedness of the fixed point problem. For the concept of well-posedness, we let the reader follow [12] and [19].

Definition 1.2. Let (X, d) be a metric space and $T : Y \rightarrow P_d(X)$ be a multi-valued operator. Then the fixed point problem is well-posed for T with

respect to the gap functional D if and only if:

- (i) $F_T = \{x^*\}$;
- (ii) if $(x_n) \in X$ has the property that $D(x_n, T(x_n)) \rightarrow 0$, then $x_n \rightarrow x^*$.

Definition 1.3. Let (X, d) be a metric space, $Y \in P(X)$ and $T : Y \rightarrow P_d(X)$ be a multi-valued operator. Then the fixed point problem is well-posed for T with respect to the Pompeiu-Hausdorff functional H if and only if:

- (i) $(SF)_T = \{x^*\}$;
- (ii) if $(x_n) \in X$ is a sequence such that $H(x_n, Tx_n) \rightarrow 0$, then $x_n \rightarrow x^*$.

Now, the second important concept related to the fixed point problem is limit shadowing or Ostrowski property, which can be found in [12] and [13].

Definition 1.4. Let (X, d) be a metric space and $T : X \rightarrow P(X)$ be a multi-valued operator. By definition, the multi-valued operator T has the Ostrowski property, if $F_T = \{x^*\}$ and for any sequence $(y_n)_{n \in \mathbb{N}} \subset X$, such that $D(y_{n+1}, Ty_n) \rightarrow 0$, we have $(y_n)_{n \in \mathbb{N}} \rightarrow x^*$, as $n \rightarrow \infty$.

We introduce now the notions of ψ -MWP operator and of generalized Ulam-Hyers stabilités. For the study of generalized Ulam-Hyers stability we refer to [15].

Definition 1.5. Let (X, d) be a metric space and $T : X \rightarrow P(X)$ be a MWP operator. Let $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be continuous in 0, increasing, such that $\psi(0) = 0$. By definition, T is ψ -MWP operator, if there exists a selection $t^\infty : \text{Graph}(T) \rightarrow F_T$ of T^∞ , such that $d(x, t^\infty(x, y)) \leq \psi(d(x, y))$, for each $(x, y) \in \text{Graph}(T)$.

Definition 1.6. Let (X, d) be a metric space and $T : X \rightarrow P(X)$. By definition, the fixed point inclusion

$$x \in T(x) \tag{1.1}$$

is called generalized Ulam-Hyers stable if and only if there exists an increasing, continuous in 0 function $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, with $\psi(0) = 0$, such that for every $\varepsilon > 0$ and for each $y^* \in X$ for which $D(y, T(y)) \leq \varepsilon$, there exists a solution x^* a solution of the fixed point inclusion (1.1), such that $d(y^*, x^*) \leq \psi(\varepsilon)$.

Definition 1.7. Let (X, d) be a metric space and $T : Y \rightarrow P(X)$. By definition, the strict fixed point inclusion

$$\{x\} = T(x) \tag{1.2}$$

is called generalized Ulam-Hyers stable if and only if there exists an increasing, continuous in 0 function $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, with $\psi(0) = 0$, such that for

every $\varepsilon > 0$ and for each $y^* \in X$ for which $H(y, T(y)) \leq \varepsilon$, there exists a solution x^* a solution of the strict fixed point inclusion (1.2), such that $d(y^*, x^*) \leq \psi(\varepsilon)$.

Finally, following [6], [8] and [22], we recall the last important concepts.

Definition 1.8. Let $X \neq \emptyset$ and $T : X \rightarrow P(X)$ be a multi-valued operator. Then, T has the approximate endpoint property if $\inf_{x \in X} \sup_{y \in Tx} d(x, y) = 0$.

2 Main results

The aim of this paper is to extend to the case of Ćirić type multi-valued generalized contractions, the results given in [16], where the author studied extended properties for the fixed point problem related to Nadler's multi-valued contractions through relevant metrical and topological properties. In the present section some variants of the multi-valued Ćirić principle are given. We shall enhance the classical result of Ćirić [3] with additional metrical and topological conclusions with respect to the fixed point problem.

Theorem 2.1 (An extended version of the Ćirić's multi-valued contraction principle). Let (X, d) be a complete metric space and $T : X \rightarrow P_d(X)$ be a multi-valued α -Ćirić type operator, i.e., there exists $\alpha \in (0, 1)$, such that

$$H(T(x), T(y)) \leq \alpha \cdot M(x, y), \text{ for each } x, y \in X,$$

where

$$M(x, y) := \{d(x, y), D(x, T(x)), D(y, T(y)), \frac{1}{2} [D(x, T(y)) + D(y, T(x))]\}.$$

Then, the following conclusions hold:

- (a) there exists $x^* \in F_T$;
- (b) for each $(x, y) \in \text{Graph}(T)$, there exists a sequence $(x_n)_{n \in \mathbb{N}}$ of successive approximations for T starting from (x, y) , convergent to a fixed point of T ;
- (c) there exists a selection $t^\infty : \text{Graph}(T) \rightarrow F_T$ of T^∞ , such that

$$d(x, t^\infty(x, y)) \leq \frac{1}{1 - \alpha} d(x, y), \forall (x, y) \in \text{Graph}(T);$$

- (d) F_T is closed in (X, d) ;
- (e) if $(x_n)_{n \in \mathbb{N}}$ is a sequence of successive approximations for T , starting from a pair $(x, y) \in \text{Graph}(T)$, which converges to a fixed point $x^*(x, y)$ of T , then

$$d(x_n, x^*) \leq \frac{\alpha^n}{1 - \alpha} d(x, y), \forall n \in \mathbb{N};$$

- (f) if $G : X \rightarrow P_{cl}(X)$ is a Ćirić-type multi-valued operator with coefficient β , and there exists $\eta > 0$, such that $H(T(x), G(x)) \leq \eta$, for all $x \in X$, then $H(F_T, F_G) \leq \eta \cdot \max \left\{ \frac{1}{1-\alpha}, \frac{1}{1-\beta} \right\}$;
- (g) if $T_n : X \rightarrow P_{cl}(X)$ is a sequence of multi-valued α -Ćirić-type operators, with $T_n(x) \xrightarrow{H} T(x)$ as $n \rightarrow \infty$, uniformly with respect to $x \in X$, then

$$\lim_{n \rightarrow \infty} H(F_{T_n}, F_T) = 0;$$

- (h) if there exists $x_0 \in X$ and $r > 0$, such that $D(x_0, T(x_0)) < (1-\alpha)r$, then there exists $x^* \in F_T \cap B(x_0, r)$;
- (i) if there exists $x_0 \in X$ and $r > 0$ such that $\delta(x_0, T(x_0)) < (1-\alpha)r$, then $T : \tilde{B}(x_0, r) \rightarrow P \left(\tilde{B} \left(x_0, \frac{1}{1-\alpha}r \right) \right)$ and there exists $x^* \in F_T \cap B(x_0, r)$;
- (j) if X is a Banach space, U an open subset of X and $T : U \rightarrow P_{cl}(X)$ is a Ćirić multi-valued operator, then the associated multi-valued field $G : U \rightarrow P(X)$, $G(x) := x - T(x)$ is open;
- (k) there exists a Caristi selection of T ;
- (m) if, additionally, $T : X \rightarrow P_{cp}(X)$, then the fixed point inclusion $x \in T(x)$ is generalized Ulam-Hyers stable;
- (n) the multi-valued operator T has the approximate fixed point property;
- (o) if the multi-valued operator T is lower semicontinuous, then it has the approximate endpoint property if and only if it has a unique strict fixed point;
- (p) if $\alpha < \frac{1}{2}$, then the fixed point set F_T is compact.
- (q) if $T : X \rightarrow P_{b,cl}(X)$, then for each $p > 0$, one has $H(F_p^*, F_T) \leq \frac{p}{1-\alpha}$, where $F_p^* := \{x \in X \mid D(x, T(x)) < p\}$.

Proof. (a), (b), (c) and (e) (In fact (a) and (b) means that T is a MWP operator, while (a), (b) and (c) can be concise represented by saying that T is a ψ -MWP operator, with $\psi(t) = \frac{1}{1-\alpha}t$).

Let $x_0 \in X$ and $x_1 \in T(x_0)$ be arbitrary elements. Then $H(T(x_0), T(x_1)) \leq \alpha M(x_0, x_1)$. Furthermore, consider $q \in \left(1, \frac{1}{\alpha}\right)$.

Now, for x_1 , there exists $x_2 \in T(x_1)$, such that $d(x_1, x_2) \leq qH(T(x_0), T(x_1))$, so $d(x_1, x_2) \leq q\alpha M(x_0, x_1)$. We consider the following cases :

If $M(x_0, x_1) = d(x_0, x_1)$, then $d(x_1, x_2) \leq (q\alpha)d(x_0, x_1)$.

If $M(x_0, x_1) = D(x_0, T(x_0)) \leq d(x_0, x_1)$, then $d(x_1, x_2) \leq (q\alpha)d(x_0, x_1)$.

If $M(x_0, x_1) = D(x_1, T(x_1)) \leq d(x_1, x_2)$, then $d(x_1, x_2) \leq (q\alpha)d(x_1, x_2)$, which is a contradiction, So $M(x_0, x_1)$ can not be $d(x_1, x_2)$.

Finally, if $M(x_0, x_1) = \frac{1}{2} [D(x_1, T(x_0)) + D(x_0, T(x_1))]$, then by using the fact that $D(x_1, T(x_0)) \leq d(x_1, x_1) = 0$ and the fact that $D(x_0, T(x_1)) \leq d(x_0, x_2) \leq d(x_0, x_1) + d(x_1, x_2)$, it follows that $M(x_0, x_1) \leq \frac{1}{2}d(x_0, x_1) + \frac{1}{2}d(x_1, x_2)$. So $d(x_1, x_2) \leq \frac{q\alpha}{2}d(x_0, x_1) + \frac{q\alpha}{2}d(x_1, x_2)$. Then $d(x_1, x_2) \leq \frac{q\alpha}{2 - q\alpha}d(x_0, x_1)$.

Since $q \in \left(1, \frac{1}{\alpha}\right)$, we get that $d(x_1, x_2) \leq (q\alpha)d(x_0, x_1)$.

Let us denote by $\lambda := q\alpha$. Then, by all the cases $d(x_1, x_2) \leq \lambda d(x_0, x_1)$.

Also, denote by $\rho_n := d(x_n, x_{n+1})$, for each $n \in \mathbb{N}$. By induction, we can construct a sequence $(x_n)_{n \in \mathbb{N}}$, such that for $x_n \in T(x_{n-1})$, there exists $x_{n+1} \in T(x_n)$, for which $\rho_n \leq \lambda \rho_{n-1}$, for each $n \in \mathbb{N}$. Then $\rho_n \leq \lambda^n \rho_0$, so by triangle inequality $d(x_n, x_{n+p}) \leq \lambda^n \frac{1 - \lambda^p}{1 - \lambda} \rho_0$. Taking $n \rightarrow \infty$, it follows up that the sequence $(x_n)_{n \in \mathbb{N}}$ is Cauchy, so there exists $x^* \in X$, such that $x_n \rightarrow x^*$. Furthermore, in the estimate $d(x_n, x_{n+p}) \leq \lambda^n \frac{1 - \lambda^p}{1 - \lambda} \rho_0$. taking $p \rightarrow \infty$, it

follows that $d(x_n, x^*) \leq \frac{\lambda^n}{1 - \lambda} d(x_0, x_1)$. Taking $n = 0$ and making $q \searrow 1$, it follows the estimate $d(x, x^*) = d(x, t^\infty(x, y)) \leq \frac{1}{1 - \alpha} d(x, y)$, with $y \in T(x)$. Here, we denoted by $x := x_0$ and $y := x_1 \in T(x_0)$.

The final step is to show that $x^* \in F_T$, i.e., to prove that $D(x^*, T(x^*)) = 0$. We have the following estimation:

$$D(x^*, T(x^*)) \leq d(x^*, x_{n+1}) + H(T(x_n), T(x^*)) \leq d(x^*, x_{n+1}) + \alpha M(x_n, x^*).$$

Moreover, since

$$M(x_n, x^*) \leq \max\{d(x_n, x^*), d(x_n, x_{n+1}), D(x^*, T(x^*))\},$$

$$\frac{1}{2} [d(x^*, x_n) + d(x^*, x_{n+1}) + D(x^*, T(x^*))],$$

by letting $n \rightarrow \infty$, we obtain that $\lim_{n \rightarrow \infty} M(x_n, x^*) \leq D(x^*, T(x^*))$.

This means that

$$D(x^*, T(x^*)) \leq \alpha \max\{D(x^*, T(x^*)), \frac{1}{2}D(x^*, T(x^*))\} < D(x^*, T(x^*)),$$

and the conclusion follows.

(d) We know that $F_T \in P(X)$. We shall show that F_T is closed in (X, d) . For this, let $x_n \in F_T$, such that $x_n \rightarrow x^*$. So, for each $n \in \mathbb{N}$, $D(x_n, T(x_n)) = 0$.

We shall show that $x^* \in F_T$, i.e. $x^* \in T(x^*)$. Also, since the operator T has closed values, then it is enough to show that $D(x^*, T(x^*)) = 0$. We have the following inequalities :

$$\begin{aligned} D(x^*, T(x^*)) &\leq d(x^*, x_n) + D(x_n, T(x^*)) \leq d(x^*, x_n) + H(T(x_n), T(x^*)) \\ &\leq d(x^*, x_n) + \alpha M(x^*, x_n) \end{aligned}$$

We have the following cases:

If $M(x^*, x_n) = d(x^*, x_n)$, then $D(x^*, T(x^*)) \leq (1 + \alpha)d(x_n, x^*) \rightarrow 0$. Furthermore, if $M(x^*, x_n) = D(x_n, T(x_n)) = 0$, then $D(x^*, T(x^*)) \leq d(x^*, x_n) \rightarrow 0$. Moreover, if $M(x^*, x_n) = D(x^*, T(x^*))$, then we obtain that

$$D(x^*, T(x^*)) \leq \frac{1}{1 - \alpha} d(x^*, x_n) \rightarrow 0.$$

Finally, if $M(x^*, x_n) = \frac{1}{2} [D(x_n, T(x^*)) + D(x^*, T(x_n))]$ $\leq \frac{1}{2} D(x_n, T(x^*)) + \frac{1}{2} d(x^*, x_n)$.

Also, $D(x_n, T(x^*)) \leq H(T(x_n), T(x^*)) \leq \alpha M(x_n, x^*) \leq \frac{\alpha}{2} D(x_n, T(x^*)) + \frac{\alpha}{2} d(x^*, x_n)$, so $D(x_n, T(x^*)) \leq \frac{\alpha}{2 - \alpha} d(x_n, x^*)$. This implies that

$$D(x^*, T(x^*)) \leq d(x^*, x_n) + \frac{\alpha}{2 - \alpha} d(x_n, x^*) = \frac{2}{2 - \alpha} d(x_n, x^*) \rightarrow 0, n \rightarrow \infty.$$

Thus, by all cases $x^* \in F_T$, so F_T is closed.

(f) By (a),(b),(c) and (e), we have that $d(x, x^*) \leq \frac{1}{1 - \alpha} d(x, y)$, where x is an arbitrary element of X and $y \in T(x)$, where $x^* \in F_T$. Taking $x = y^* \in F_G$, then we obtain that $d(x^*, y^*) \leq \frac{1}{1 - \alpha} d(y, y^*)$, where $y \in T(y^*)$. Furthermore, since $y \in T(y^*)$ is arbitrary, we can make the following assertion: for $y^* \in F_G$, there exists $y \in T(y^*)$, such that $d(y, y^*) \leq H(G(y^*), T(y^*)) \leq \eta$, so $d(x^*, y^*) \leq \frac{\eta}{1 - \alpha}$.

Now, also from the global principle of the existence of the fixed point of G , we get that $d(x, x^*) \leq \frac{1}{1 - \beta} d(x, y)$, with $x^* \in F_G$, x is an arbitrary element of X and $y \in Gx$.

Taking $x = y^* \in F_T$, then we obtain that $d(x^*, y^*) \leq \frac{1}{1 - \beta} d(y, y^*)$, where $y \in G(y^*)$.

As in the first case, since $y \in G(y^*)$ is arbitrary, then for $y^* \in F_T$, there exists

$y \in G(y^*)$, such that $d(y^*, y) \leq H(T(y^*), G(y^*)) \leq \eta$. So $d(x^*, y^*) \leq \frac{\eta}{1-\beta}$. From the first case we get that for $y^* \in F_G$, there exists $x^* \in F_T$, such that

$$d(x^*, y^*) \leq \eta \cdot \max \left\{ \frac{1}{1-\alpha}, \frac{1}{1-\beta} \right\},$$

while from the second case we infer that for $y^* \in F_T$, there exists $x^* \in F_G$, such that

$$d(x^*, y^*) \leq \eta \cdot \max \left\{ \frac{1}{1-\alpha}, \frac{1}{1-\beta} \right\}.$$

By Lemma 1.2, we get the conclusion $H(F_T, F_G) \leq \eta \cdot \max \left\{ \frac{1}{1-\alpha}, \frac{1}{1-\beta} \right\}$.

(g) Let $\varepsilon > 0$ be an arbitrary fixed element. Since $T_n(x) \xrightarrow{H} T(x)$ as $n \rightarrow \infty$, uniformly for each $x \in X$, then for all $x \in X$, we have that $\lim_{n \rightarrow \infty} H(T_n x, T x) = 0$.

This means that for $\varepsilon > 0$, there exists $N(\varepsilon) \in \mathbb{N}$, such that for each $n \geq N(\varepsilon)$, we have that $\sup_{x \in X} H(T_n(x), T(x)) < \varepsilon$. From the conclusion (f) of data dependence, we have that for $\varepsilon > 0$, there exists $N(\varepsilon) \in \mathbb{N}$, such that for all $n \geq N(\varepsilon)$, one has $H(F_{T_n}, F_T) < \frac{1}{1-\alpha} \cdot \varepsilon$. So, the conclusion is valid.

(h) Let $s \in (0, r)$, such that $\tilde{B}(x_0, s) \subset B(x_0, r)$, with $D(x_0, T(x_0)) < (1-\alpha)s < (1-\alpha)r$. Since $D(x_0, T(x_0)) < (1-\alpha)s$, then there exists $x_1 \in T(x_0)$, such that $d(x_0, x_1) < (1-\alpha)s < s$, so $x_1 \in B(x_0, s) \subset \tilde{B}(x_0, s)$. From the hypothesis, we have that $H(T(x_0), T(x_1)) \leq \alpha M(x_0, x_1)$, where:

$$\begin{aligned} M(x_0, x_1) &= \\ &= \max\{d(x_0, x_1), D(x_0, T(x_0)), D(x_1, T(x_1)), \frac{1}{2} [D(x_0, T(x_1)) + D(x_1, T(x_0))]\} \\ &\leq \max\{d(x_0, x_1), D(x_0, T(x_0)), D(x_1, T(x_1)), \frac{1}{2} [d(x_0, x_1) + D(x_1, T(x_1))]\} \\ &= \max\{d(x_0, x_1), D(x_0, T(x_0)), D(x_1, T(x_1))\} \\ &\leq \max\{d(x_0, x_1), H(T(x_0), T(x_1))\} \end{aligned}$$

We consider the following cases:

If the maximum is $d(x_0, x_1)$, then $H(T(x_0), T(x_1)) \leq \alpha d(x_0, x_1)$.

If the maximum is $H(T(x_0), T(x_1))$, then, since $\alpha < 1$, we obtain a contradiction.

From the above cases, it follows that $H(T(x_0), T(x_1)) \leq \alpha d(x_0, x_1)$. Since $D(x_1, T(x_1)) \leq H(T(x_0), T(x_1)) \leq \alpha d(x_0, x_1) < \alpha(1-\alpha)s$, then there exists $x_2 \in T(x_1)$ for which $d(x_1, x_2) < \alpha(1-\alpha)s$.

Furthermore, by triangle inequality one can obtain $d(x_0, x_2) \leq d(x_0, x_1) + d(x_1, x_2) < (1 - \alpha)s + \alpha(1 - \alpha)s = (1 - \alpha^2)s < s$, so $x_2 \in T(x_1) \cap \tilde{B}(x_0, s)$. By induction, we can construct a sequence $(x_n)_{n \in \mathbb{N}}$, with x_n from $T(x_{n-1}) \cap \tilde{B}(x_0, s)$, such that :

$$\begin{cases} x_{n+1} \in T(x_n), \text{ for each } n \in \mathbb{N}, \\ d(x_{n-1}, x_n) \leq \alpha^{n-1}(1 - \alpha)s, \text{ for each } n \in \mathbb{N}^*, \\ d(x_0, x_n) \leq (1 - \alpha^n)s, \text{ for all } n \in \mathbb{N}. \end{cases}$$

It follows that the sequence $(x_n)_{n \in \mathbb{N}}$ is Cauchy, so there exists $x^* \in \tilde{B}(x_0, s)$, such that $x_n \rightarrow x^*$.

As in the proof of (a),(b),(c) and (e), one can show that $x^* \in T(x^*)$. Moreover, since $x_n \in \tilde{B}(x_0, s)$ and $\tilde{B}(x_0, s)$ is closed in X , then $x^* \in \tilde{B}(x_0, s) \subset B(x_0, r)$.

(i) Let $u \in T(x_0)$. Then, by applying the triangle inequality, we get that $d(z, x_0) \leq d(z, u) + d(u, x_0)$, so $d(z, x_0) \leq d(z, u) + \delta(x_0, T(x_0))$. Now, taking $\inf_{u \in T(x_0)}$, it follows that $d(z, x_0) \leq D(z, T(x_0)) + \delta(x_0, T(x_0))$.

We first show that $T(\tilde{B}(x_0, r)) \subset \tilde{B}\left(x_0, \frac{1}{1 - \alpha}r\right)$.

Let $y \in \tilde{B}(x_0, r)$. We will show that $T(y) \subset \tilde{B}\left(x_0, \frac{1}{1 - \alpha}r\right)$.

So, take $z \in T(y)$. The aim is to show that $z \in \tilde{B}\left(x_0, \frac{1}{1 - \alpha}r\right)$, i.e.,

$$d(z, x_0) \leq \frac{1}{1 - \alpha}r.$$

Then $d(z, x_0) \leq D(z, T(x_0)) + \delta(x_0, T(x_0)) < H(T(y), T(x_0)) + (1 - \alpha)r$. So $d(z, x_0) < \alpha M(y, x_0) + (1 - \alpha)r$.

We know that:

$$M(y, x_0) = \max\{d(y, x_0), D(x_0, T(x_0)), D(y, T(y)), \frac{1}{2}[D(y, T(x_0)) + D(x_0, T(y))]\}.$$

We also have $d(y, x_0) \leq r$ and $D(x_0, T(x_0)) \leq \delta(x_0, T(x_0)) < (1 - \alpha)r \leq r$. So, we obtain:

$$M(y, x_0) \leq \max\{r, D(y, T(y)), \frac{1}{2}[D(y, T(x_0)) + D(x_0, T(y))]\}.$$

We employ an analysis on the following cases :

If the maximum from the right hand side is r , then $d(z, x_0) < \alpha r + (1 - \alpha)r = r < \frac{1}{1 - \alpha}r$.

If the maximum is $D(y, T(y))$, then $d(z, x_0) < \alpha D(y, T(y)) + (1 - \alpha)r$. So

$d(z, x_0) < (1 - \alpha)r + \alpha d(y, z) \leq (1 - \alpha)r + \alpha d(y, x_0) + \alpha d(z, x_0)$. This means that $d(z, x_0) < \frac{1}{1 - \alpha}r$.

Finally, if the maximum is $\frac{1}{2} [D(y, T(x_0)) + D(x_0, T(y))]$, then $d(z, x_0) < \frac{\alpha}{2} D(y, T(x_0)) + \frac{\alpha}{2} d(x_0, z) + (1 - \alpha)r$. This implies that $(2 - \alpha)d(z, x_0) < \alpha d(y, x_0) + \alpha d(x_0, T(x_0)) + 2(1 - \alpha)r$ and thus $d(z, x_0) \leq \frac{2 - \alpha^2}{2 - \alpha}r$.

From all the cases, it follows that $d(z, x_0) \leq \max\{\frac{1}{1 - \alpha}r, \frac{2 - \alpha^2}{2 - \alpha}r\} = \frac{1}{1 - \alpha}r$.

This means that $T(\tilde{B}(x_0, r)) \subset \tilde{B}(x_0, \frac{1}{1 - \alpha}r)$. We have that

$$D(x_0, T(x_0)) \leq \delta(x_0, T(x_0)) < (1 - \alpha)r.$$

Taking $X := \tilde{B}(x_0, \frac{1}{1 - \alpha}r)$ and $T : \tilde{B}(x_0, r) \rightarrow P_{cl}(X)$, we apply the conclusion (i) for local version of the fixed point problem for the Ćirić operator on the closed ball. We mention that we have used the fact that $\tilde{B}(x_0, \frac{1}{1 - \alpha}r)$ is closed in the complete metric space (X, d) . Then, there exists $x^* \in F_T \cap \tilde{B}(x_0, r)$. Using the fact that $d(x_0, x^*) \leq r$, we can show that $d(x_0, x^*) < r$.

Suppose to the contrary that $r = d(x_0, x^*)$. Then, we have the following inequalities: $r = d(x^*, x_0) \leq H(T(x^*), T(x_0)) + \delta(x_0, T(x_0)) < \alpha M(x^*, x_0) + (1 - \alpha)r$, where

$$M(x^*, x_0) = \max\{d(x^*, x_0), D(x^*, T(x^*)), D(x_0, T(x_0)), \frac{1}{2} [D(x^*, T(x_0)) + D(x_0, T(x^*))]\}.$$

Notice that $D(x^*, T(x^*)) = 0$, $D(x_0, T(x_0)) \leq \delta(x_0, T(x_0)) \leq (1 - \alpha)r < r$, $D(x_0, T(x^*)) \leq d(x_0, x^*)$ and $D(x^*, T(x_0)) \leq d(x^*, x_0) + D(x_0, T(x_0)) < d(x^*, x_0) + (1 - \alpha)r$.

Then, we get the following cases :

If the maximum from the right hand side is $d(x^*, x_0)$, then $r < \alpha d(x^*, x_0) + (1 - \alpha)r = \alpha r + (1 - \alpha)r = r$, which is false.

If the maximum is $D(x^*, T(x^*))$, then $r < (1 - \alpha)r < r$, which is also false.

If the maximum is $D(x_0, T(x_0)) < (1 - \alpha)r$, then we get $r < (1 - \alpha)r + \alpha(1 - \alpha)r = (1 - \alpha^2)r < r$, also false.

For the last case, if the maximum from the right hand side is $\frac{1}{2} [D(x^*, T(x_0))$

$+D(x_0, T(x^*))]$, then $M(x^*, x_0) \leq \frac{1}{2}H(T(x^*), T(x_0)) + (1 - \alpha)r$. Furthermore, by the condition that the operator is of Ćirić-type, we have that $H(T(x^*), T(x_0)) \leq \alpha M(x^*, x_0)$. So $H(T(x^*), T(x_0)) \leq \frac{\alpha(1 - \alpha)r}{2 - \alpha}$.

It follows that $r = d(x^*, x_0) < H(T(x^*), T(x_0)) + (1 - \alpha)r$, so $r < (1 - \alpha^2)r < r$, which is false.

From all the cases from above, it follows that $d(x^*, x_0) < r$.

(j) We prove that, if V is an open subset of U , then $G(V)$ is open in X . This means that for $x_0 \in U$ and $r_0 > r > 0$, with $B(x_0, r) \subset U$, then $V^0(G(x_0), (1 - \alpha)r) \subset G(B(x_0, r))$.

So, let $y \in V^0(G(x_0), (1 - \alpha)r)$, i.e. $D(y, G(x_0)) < (1 - \alpha)r$. We shall show that $y \in G(B(x_0, r))$. In other words, we shall show that there exists $x^* \in B(x_0, r)$, such that $y \in G(x^*)$, i.e., $y \in x^* - T(x^*)$.

Let us consider the multi-valued operator $F : B(x_0, r) \rightarrow P_{cl}(X)$, defined by $F(x) := y + T(x)$.

If F has a fixed point x^* , then $x^* \in y + T(x^*)$ or $y \in x^* - T(x^*)$. Now, for each $x, z \in B(x_0, r)$, we have that :

$H(F(x), F(z)) = H(y + T(x), y + T(z)) \leq H(T(x), T(z)) \leq \alpha M(x, z)$. Moreover, $D(x_0, F(x_0)) = D(x_0, y + T(x_0)) = D(y, x_0 - T(x_0)) = D(y, G(x_0)) < (1 - \alpha)r$. Then F is a Ćirić operator defined on the open ball $B(x_0, r)$, where $D(x_0, F(x_0)) < (1 - \alpha)r$. Applying the conclusion (h), i.e. the local version involving an open ball, it follows easily that G is open.

(k) For the proof of the Caristi selection of the multi-valued Ćirić operator T , we refer to the work of A. Petruşel and G. Petruşel [20].

(m) Let $\varepsilon > 0$ and consider $y^* \in X$ that satisfies $D(y^*, T(y^*)) \leq \varepsilon$. Then, for each $(x, y) \in \text{Graph}(T)$, we have that $d(x, t^\infty(x, y)) \leq \frac{1}{1 - \alpha}d(x, y)$.

Now, since there exists $(y^*, u^*) = D(y^*, T(y^*))$, we take $x^* := t^\infty(y^*, u^*)$.

This implies that $d(y^*, x^*) = d(y^*, t^\infty(y^*, u^*)) \leq \frac{1}{1 - \alpha}d(y^*, u^*) = \psi(\varepsilon)$,

where $\psi(t) = \frac{t}{1 - \alpha}$.

(n) For the proof of this, we refer to [2].

(o) Let $\varepsilon > 0$. Let's denote $E_\varepsilon(T) := \left\{x \in X \mid \sup_{z \in Tx} d(x, z) \leq \varepsilon\right\}$. Since T is lower semicontinuous, by Lemma 3.3 from [8], we get that for each $\varepsilon > 0$, the set $E_\varepsilon(T)$ is nonempty. Now, let $x, y \in E_\varepsilon(T)$. It follows that:

$$d(x, y) \leq H(\{x\}, T(x)) + H(T(x), T(y)) + H(\{y\}, T(y)).$$

Now, since $x, y \in E_\varepsilon(T)$, then $H(\{x\}, T(x)) \leq \varepsilon$ and $H(\{y\}, T(y)) \leq \varepsilon$. So,

we get that

$$d(x, y) \leq 2\varepsilon + \alpha M(x, y).$$

Then, we have the following cases:

If $M(x, y) = d(x, y)$, then $d(x, y) \leq 2\varepsilon + \alpha d(x, y)$, so $d(x, y) \leq \frac{2\varepsilon}{1-\alpha}$.

If $M(x, y) = D(x, T(x)) = D(\{x\}, T(x)) \leq H(\{x\}, T(x)) \leq \varepsilon$, then $d(x, y) \leq 2\varepsilon + \alpha\varepsilon = \varepsilon(2 + \alpha)$.

Similarly, if $M(x, y) = D(y, T(y)) = D(\{y\}, T(y)) \leq H(\{y\}, T(y)) \leq \varepsilon$, then $d(x, y) \leq 2\varepsilon + \alpha\varepsilon = \varepsilon(2 + \alpha)$.

Finally, if $M(x, y) = \frac{1}{2} [D(x, T(y)) + D(y, T(x))]$, then we infer that:

$$\begin{aligned} D(x, T(y)) &\leq d(x, y) + D(y, T(y)) \\ &= d(x, y) + \inf_{z \in T(y)} d(y, z) \leq d(x, y) + \sup_{z \in T(y)} d(y, z) \\ &\leq d(x, y) + \varepsilon, \end{aligned}$$

since $y \in E_\varepsilon(T)$.

Furthermore,

$$\begin{aligned} D(y, T(x)) &\leq d(x, y) + D(x, T(x)) \\ &= d(x, y) + \inf_{z \in T(x)} d(x, z) \leq d(x, y) + \sup_{z \in T(x)} d(x, z) \\ &\leq d(x, y) + \varepsilon, \end{aligned}$$

since $x \in E_\varepsilon(T)$. Thus, $d(x, y) \leq 2\varepsilon + \alpha\varepsilon + \alpha d(x, y)$.

From all the cases, it follows that

$$d(x, y) \leq \varepsilon \max \left\{ \frac{2+\alpha}{1-\alpha}, \frac{2}{1-\alpha}, 2+\alpha \right\} = \frac{2+\alpha}{1-\alpha} \varepsilon.$$

Now, if the multi-valued Ćirić-type operator T has a strict fixed point, then T has the approximate endpoint property. Let us suppose now that the multi-valued operator T has the approximate endpoint property. We define

$C_n := E_{\frac{1}{n}}(T) = \left\{ x \in X \mid \sup_{y \in T(x)} d(x, y) \leq \frac{1}{n} \right\}$. Then, by our hypothesis, for

each $n \in \mathbb{N}$, C_n is nonempty. Furthermore, for all $n \in \mathbb{N}$, $C_{n+1} \subseteq C_n$.

Also, since T is lower semicontinuous, then C_n are closed, for each $n \in \mathbb{N}$.

Also, we observe that :

$$\delta(C_n) = \delta \left(E_{\frac{1}{n}}(T) \right) \leq \frac{2+\alpha}{1-\alpha} \cdot \frac{1}{n}, \text{ so } \lim_{n \rightarrow \infty} \delta(C_n) = 0.$$

Then, by Cantor's intersection theorem, it follows that $\bigcap_{n \in \mathbb{N}} C_n = \{x_0\}$, so the

conclusion follows easily.

(p) By (d), we have that F_T is closed in (X, d) . Since (X, d) is complete, then F_T is complete with respect to d . Furthermore, let's suppose that F_T is not compact. Then F_T is not precompact. This means that there exist $\delta > 0$ and $(x_k)_{k \in \mathbb{N}} \subset F_T$, such that $d(x_i, x_j) \geq \delta$, for all $i \neq j$.

Denote $\rho := \inf\{R \mid \exists a \in X, \text{ such that } B(a, R) \text{ contains an infinity of } x'_k s\}$.

It is obvious that $\rho \geq \frac{\delta}{2}$, because for each $a \in X$, $B\left(a, \frac{\delta}{2}\right)$ contains at most one x_k .

Furthermore, consider $0 < \varepsilon < (1 - 2\alpha)\rho$ and take $a \in X$, such that the set $J := \{k \mid x_k \in B(a, \rho + \varepsilon)\}$ is infinite. Then, for each $k \in J$, we have

$$D(x_k, T(a)) \leq H(T(x_k), T(a)) \leq \alpha M(x_k, a).$$

Now, we have the following cases:

If $M(x_k, a) = d(x_k, a)$, then $D(x_k, T(a)) \leq \alpha d(x_k, a) \leq \alpha(\rho + \varepsilon)$.

Also, if $M(x_k, a) = D(a, T(a))$, then $D(x_k, T(a)) \leq \alpha d(a, y)$, for $y \in Ta$.

Now, if $M(x_k, a) = \frac{1}{2}D(x_k, T(a)) + \frac{1}{2}D(a, T(x_k))$, then

$$D(x_k, T(a)) \leq \frac{\alpha}{2}D(x_k, T(a)) + \frac{\alpha}{2}d(a, x_k),$$

so $D(x_k, T(a)) \leq \frac{\alpha}{2 - \alpha}d(a, x_k)$. It implies that $D(x_k, T(a)) \leq \alpha(\rho + \varepsilon)$.

So, all the cases from above imply that $D(x_k, a) \leq \max\{\alpha(\rho + \varepsilon), \alpha d(a, y)\}$, where $y \in T(a)$. From all of this, we have two cases to consider :

In the first case, by $D(x_k, T(a)) \leq \alpha d(a, y)$, with $y \in T(a)$, we obtain that $D(x_k, T(a)) \leq \alpha d(a, x_k) + \alpha d(x_k, y)$. Taking $\inf_{y \in T(a)}$, we get, for each $k \in J$,

$$\text{that } D(x_k, T(a)) \leq \frac{\alpha}{1 - \alpha} \cdot (\rho + \varepsilon).$$

Now, the second case is for $D(x_k, T(a)) \leq \alpha(\rho + \varepsilon)$. From these two cases, one can get $D(x_k, T(a)) \leq \max\left\{\alpha, \frac{\alpha}{1 - \alpha}\right\} \cdot (\rho + \varepsilon) = \frac{\alpha}{1 - \alpha} \cdot (\rho + \varepsilon)$. Then

$D(x_k, T(a)) \leq \frac{\alpha}{1 - \alpha} \cdot (\rho + \varepsilon)$, so since $T(a)$ is compact, there exists $y_k \in T(a)$,

such that $d(x_k, y_k) \leq \frac{\alpha}{1 - \alpha}(\rho + \varepsilon)$, for each $k \in J$.

Moreover, since $T(a)$ is compact, then there exists $b \in T(a)$, for which the set $J' := \{k \in J \mid d(y_k, b) < \varepsilon\}$ is infinite. This means that for each $k \in J'$ (since $\alpha < \frac{1}{2}$ and ε was chosen such that $\varepsilon < \rho \cdot (1 - 2\alpha)$), we have that

$$d(x_k, b) \leq d(x_k, y_k) + d(y_k, b) < \frac{\alpha}{1 - \alpha}(\rho + \varepsilon) + \varepsilon < \rho.$$

This contradicts the fact that the ball $B(b, R)$ contains an infinite number of elements x'_k s, where $R = \frac{\alpha}{1-\alpha}\rho + \varepsilon \left(1 + \frac{\alpha}{1-\alpha}\right)$.

(q) Let $F_p^* := \{x \in X \mid D(x, T(x)) < p\}$, for each $p > 0$. Notice that if $x \in F_T$, then $D(x, T(x)) = 0 < p$, for each $p > 0$. So $F_T \subseteq F_p^*$. This implies that $H(F_p^*, F_T) = \rho(F_p^*, F_T) := \sup_{x \in F_p^*} D(x, F_T)$, for all $p > 0$, where ρ denotes

the excess functional.

Moreover, let $x \in F_p^*$ and $\varepsilon > 0$. Because $x \in F_p^*$, then $D(x, T(x)) < p$. So, for $x \in F_p^*$ there exists $x_1 \in T(x)$, for which $d(x, x_1) < (1 + \varepsilon)p$.

For $x_0 = x$ and $x_1 \in T(x) = T(x_0)$, following (b) there exists a sequence of successive approximations $(x_n)_{n \in \mathbb{N}}$, starting from $(x_0, x_1) \in \text{Graph}(T)$, such

that $d(x_n, x^*) \leq \frac{L^n(q)}{1 - L(q)} d(x_0, x_1)$, for each $n \in \mathbb{N}$, where $L(q) := q\alpha$, with

$q \in \left(1, \frac{1}{\alpha}\right)$ and with the property that $x_n \rightarrow x^* \in F_T$ as $n \rightarrow \infty$.

Taking $n = 0$, we obtain $d(x_0, x^*) \leq \frac{1}{1 - L(q)} d(x_0, x_1) \leq \frac{(1 + \varepsilon)p}{1 - L(q)}$. So

$d(x_0, x^*) \leq \frac{(1 + \varepsilon)p}{1 - q\alpha}$. Taking $q \searrow 1$, respectively $\varepsilon \searrow 0$, it follows that

$d(x_0, x^*) \leq \frac{p}{1 - \alpha}$. So, the conclusion follows easily from this inequality. $\square$

We will present now the second result of this article, which is an extended version of strict fixed point principle for multi-valued Ćirić operators. Since all the conclusion from Theorem 2.1 are valid even in the particular case when $(SF)_T \neq \emptyset$, for this case we shall present only the metrical conclusions that are new.

Theorem 2.2 (An extended strict fixed point principle for multi-valued Ćirić operators). *Let (X, d) be a complete metric space and $T : X \rightarrow P_{cl}(X)$ be a multi-valued α -Ćirić type operator. Suppose that $(SF)_T \neq \emptyset$. Then, the following conclusions hold :*

(a) $(SF)_T = F_T = \{x^*\}$;

(b) if $\alpha < \frac{1}{2}$, then T has the Ostrowski property;

(c) the fixed point inclusion $x \in T(x)$ is generalized Ulam-Hyers stable;

(d) the strict fixed point inclusion $\{x\} = T(x)$ is generalized Ulam-Hyers stable;

(e) the fixed point problem is well-posed for T , with respect to D and, respectively, with respect to H ;

(f) if $\alpha < \frac{1}{2}$, then $H(T(x), x^*) \leq \frac{\alpha}{1-\alpha} d(x, x^*)$, for each $x \in X$;

- (g) $d(x, x^*) \leq \frac{1}{1-\alpha} H(x, T(x))$, for each $x \in X$;
 (h) if $G : X \rightarrow P(X)$ is a multi-valued operator with $F_G \neq \emptyset$, and there exists $\eta > 0$, such that $H(T(x), G(x)) \leq \eta$, for all $x \in X$, then $H(F_T, F_G) \leq \eta \cdot \frac{1}{1-\alpha}$.

Proof. (a) Since $(SF)_T \neq \emptyset$, then there exists $x^* \in (SF)_T \subset F_T$. Suppose there exists $y^* \in F_T$. We show that $x^* = y^*$. For this, suppose the contrary that $x^* \neq y^*$. Then:

$$d(x^*, y^*) = D(T(x^*), y^*) \leq H(T(x^*), T(y^*)) \leq \alpha M(x^*, y^*).$$

Since $D(x^*, T(x^*)) = D(y^*, T(y^*)) = 0$, $D(x^*, T(y^*)) \leq d(x^*, y^*)$ and $D(y^*, T(x^*)) = d(x^*, y^*)$, it follows that $M(x^*, y^*) \leq d(x^*, y^*)$. So $d(x^*, y^*) \leq \alpha d(x^*, y^*) < d(x^*, y^*)$. This implies that $d(x^*, y^*) = 0$, so we obtain a contradiction.

Finally, $x^* = y^*$, so $F_T = \{x^*\} = (SF)_T$.

(b) Let $(y_n)_{n \in \mathbb{N}}$ be a sequence, such that $D(y_{n+1}, T(y_n)) \rightarrow 0$. We shall show that $d(y_n, x^*) \rightarrow 0$. Then, we have $d(x^*, y_{n+1}) \leq H(T(x^*), T(y_n)) + D(y_{n+1}, T(y_n)) \leq \alpha M(x^*, y_n) + D(y_{n+1}, T(y_n))$, where

$$\begin{aligned} M(x^*, y_n) &= \\ &= \max \left\{ d(x^*, y_n), D(x^*, T(x^*)), D(y_n, T(y_n)), \frac{1}{2} [D(x^*, T(y_n)) + D(y_n, T(x^*))] \right\} \\ &\leq \max \left\{ d(x^*, y_n), D(y_n, T(y_n)), \frac{1}{2} [d(y_n, x^*) + D(x^*, T(y_n))] \right\}. \end{aligned}$$

Now, we have the following cases :

If the maximum from the right hand side is $d(x^*, y_n)$, then $d(x^*, y_{n+1}) \leq D(y_{n+1}, T(y_n)) + \alpha d(x^*, y_n)$.

If the maximum is $D(y_n, T(y_n)) \leq d(y_n, x^*) + D(x^*, T(y_n))$, then we have $H(x^*, T(y_n)) = H(T(x^*), T(y_n)) \leq \alpha M(x^*, y_n) \leq \alpha d(y_n, x^*) + \alpha H(x^*, T(y_n))$.

So, we get that $H(T(x^*), T(y_n)) \leq \frac{\alpha}{1-\alpha} d(y_n, x^*)$.

It implies that $d(y_{n+1}, x^*) \leq D(y_{n+1}, T(y_n)) + \frac{\alpha}{1-\alpha} d(y_n, x^*)$.

Consider now the case when the maximum is $\frac{1}{2} [d(y_n, x^*) + D(x^*, T(y_n))]$. Then, we obtain $D(x^*, T(y_n)) \leq H(T(x^*), T(y_n)) \leq \alpha M(x^*, y_n)$. Thus $H(T(x^*), T(y_n)) \leq \frac{\alpha}{2} (d(y_n, x^*) + H(T(x^*), T(y_n)))$. This means that

$$H(T(x^*), T(y_n)) \leq \frac{\alpha}{2-\alpha} d(y_n, x^*).$$

Hence $d(y_{n+1}, x^*) \leq D(y_{n+1}, T(y_n)) + \frac{\alpha}{2-\alpha}d(y_n, x^*)$.

Now, since $\beta := \max \left\{ \alpha, \frac{\alpha}{1-\alpha}, \frac{\alpha}{2-\alpha} \right\} = \frac{\alpha}{1-\alpha}$, then from all the cases from above, it follows that $d(y_{n+1}, x^*) \leq D(y_{n+1}, T(y_n)) + \beta d(y_n, x^*) \leq D(y_{n+1}, T(y_n)) + \beta D(y_n, T(y_{n-1})) + \beta^2 d(y_{n-1}, x^*) \leq \dots \leq \beta^{n+1} d(y_0, x^*) + \sum_{k=0}^n \beta^{n-k} D(y_{k+1}, T(y_k))$. Now, since $\beta < 1$, using Cauchy's lemma, we get that $d(y_{n+1}, x^*) \rightarrow 0$.

(c) By (a) we know that $(SF)_T = F_T = \{x^*\}$.

Now, let us consider $x \in X$ and $y \in T(x)$. Then, we have the following:

$d(x, x^*) \leq d(x, y) + H(T(x), T(x^*)) \leq d(x, y) + \alpha M(x, x^*) \leq d(x, y) + \alpha \max \left\{ d(x, x^*), D(x, T(x)), \frac{1}{2}d(x^*, y) + \frac{1}{2}d(x, x^*) \right\}$. Moreover, we consider the following cases:

If $M(x, x^*) = d(x, x^*)$, then $d(x, x^*) \leq \frac{1}{1-\alpha}d(x, y)$.

If $M(x, x^*) = D(x, T(x))$, then

$$d(x, x^*) \leq d(x, y) + \alpha D(x, T(x)) \leq (1 + \alpha)d(x, y)$$

Finally, if $M(x, x^*) \leq \frac{1}{2}d(x^*, y) + \frac{1}{2}d(x, x^*)$, then we have $d(x, x^*) \leq d(x, y) + \frac{\alpha}{2}d(x^*, y) + \frac{\alpha}{2}d(x, x^*)$. So, we get $d(x, x^*) \leq \frac{2+\alpha}{2(1-\alpha)}d(x, y)$. From all the cases we obtain that

$$d(x, x^*) \leq \max \left\{ \frac{1}{1-\alpha}, 1 + \alpha, \frac{2+\alpha}{2(1-\alpha)} \right\} d(x, y) = \frac{2+\alpha}{2(1-\alpha)}d(x, y).$$

Now, let us define $\psi(t) := \frac{2+\alpha}{2(1-\alpha)}t$, so $d(x, x^*) \leq \psi(d(x, y))$. We notice that ψ is continuous in 0, increasing and with $\psi(0) = 0$.

Then, as in (m) of Theorem 2.1, we have the following:

Let $\varepsilon > 0$ and consider $y^* \in X$ that satisfies $D(y^*, T(y^*)) \leq \varepsilon$. Then, for each $(x, y) \in \text{Graph}(T)$, we have $d(x, t^\infty(x, y)) \leq \psi(d(x, y))$.

Now, since there exists $(y^*, u^*) = D(y^*, T(y^*))$, we take $x^* := t^\infty(y^*, u^*)$. This implies that $d(y^*, x^*) = d(y^*, t^\infty(y^*, u^*)) \leq \psi(d(y^*, u^*))$ and the conclusion follows.

(d) Let $\varepsilon > 0$ and $y^* \in X$, such that $H(y^*, T(y^*)) \leq \varepsilon$. Since T is a Ćirić multi-valued operator, from (h) we have that $d(x, x^*) \leq \frac{1}{1-\alpha}H(x, T(x))$, for each $x \in X$. This implies that $d(y^*, x^*) \leq \frac{1}{1-\alpha}H(y^*, T(y^*)) \leq \psi(\varepsilon)$, where $\psi(t) := \frac{t}{1-\alpha}$ satisfies $\psi(0) = 0$ and it is an increasing and continuous

mapping in 0.

(e) The proof of this conclusion is given in [19].

(f) We know that

$$H(T(x), T(x^*)) \leq \alpha \max\{d(x, x^*), D(x, T(x)), \frac{1}{2} [D(x, T(x^*)) + D(x^*, T(x))]\}.$$

We have the following cases:

If the maximum is $d(x, x^*)$, then $H(T(x), T(x^*)) \leq \alpha d(x, x^*)$.

If the maximum is $\frac{1}{2} [D(x, T(x^*)) + D(x^*, T(x))]$, then $H(T(x), T(x^*)) = \frac{\alpha}{2} d(x, x^*) + \frac{\alpha}{2} H(T(x), T(x^*))$ and so $H(T(x), T(x^*)) \leq \frac{\alpha}{2-\alpha} d(x, x^*)$.

If the maximum is $D(x, T(x))$, then we obtain $H(T(x), T(x^*)) \leq \frac{\alpha}{1-\alpha} d(x, x^*)$.

Since $\max\left\{\frac{\alpha}{2-\alpha}, \alpha, \frac{\alpha}{1-\alpha}\right\} = \frac{\alpha}{1-\alpha}$, $H(T(x), x^*) = H(T(x), T(x^*)) \leq \frac{\alpha}{1-\alpha} d(x, x^*)$.

(g) We have the following chain of inequalities $d(x, x^*) \leq H(x, T(x)) + H(T(x), x^*) \leq H(x, T(x)) + \alpha d(x, x^*)$. Thus $d(x, x^*) \leq \frac{1}{1-\alpha} H(x, T(x))$.

(h) Let $x^* \in (SF)_T$ and $y^* \in F_G$. Then, we have

$$d(x^*, y^*) \leq H(G(y^*), x^*) \leq H(G(y^*), T(y^*)) + H(T(y^*), x^*) \leq \eta + \alpha M(y^*, x^*).$$

Now, we have the following cases for $M(y^*, x^*)$:

1) if $M(y^*, x^*) = d(y^*, x^*)$, then $d(y^*, x^*) \leq \frac{\eta}{1-\alpha}$.

2) if $M(y^*, x^*) = D(x^*, T(x^*))$, then $d(y^*, x^*) = 0$.

3) if $M(y^*, x^*) = D(y^*, T(y^*)) \leq H(G(y^*), T(y^*)) \leq \eta$, then $d(y^*, x^*) \leq (1 + \alpha)\eta$.

4) finally, if $M(y^*, x^*) = \frac{1}{2} D(x^*, T(y^*)) + \frac{1}{2} D(y^*, T(x^*))$, then

$H(T(x^*), T(y^*)) \leq \alpha M(y^*, x^*) \leq \frac{\alpha}{2} H(T(y^*), T(x^*)) + \frac{\alpha}{2} d(x^*, y^*)$. Hence, we get that $H(T(x^*), T(y^*)) \leq \frac{\alpha}{2-\alpha} d(x^*, y^*)$. Then $d(x^*, y^*) \leq \eta + \frac{\alpha}{2-\alpha} d(x^*, y^*)$,

which implies that $d(y^*, x^*) \leq \frac{2-\alpha}{2(1-\alpha)} \eta$. It follows that $d(y^*, x^*) \leq \eta \cdot$

$\max\left\{(1 + \alpha), \frac{1}{1-\alpha}, \frac{2-\alpha}{2(1-\alpha)}\right\} = \frac{1}{1-\alpha} \eta$. Using Lemma 1.2 the conclusion follows. $\square$

References

- [1] **J. Andres and L. Górniewicz**, On the Banach contraction principle for multivalued mappings, *Approximation, Optimization and Mathematical Economics (M. Lassonde -ed.)*, Physica-Verlag, Heidelberg, (2001), 1-23
- [2] **Boriceanu M.**, Fixed point theory for multivalued generalized contraction on a set with two b-metrics, *Stud. Univ. Babeş-Bolyai*, **54** (3), (2009)
- [3] **L. Ćirić**, Fixed points for generalized multi-valued contractions, *Mat. Vesnik.*, **9**, (1972), 265–272
- [4] **N.V. Dung and A. Petruşel**, On iterated functions systems consisting of Kannan maps, Reich maps, Chatterjea type maps, and related results, *J. Fixed Point Theory Appl.*, **9**, (2017), 2271–2285
- [5] **G. Chifu and G. Petruşel**, Fixed point results for multivalued Hardy-Rogers contractions in b-metric spaces, *Filomat*, **31** (8), (2017), 2499–2507
- [6] **A.A. Harandi**, Endpoints of set-valued contractions in metric spaces, *Nonlinear Anal.*, **72**, (2010), 132–134
- [7] **A.D. Rogers and G.E. Hardy**, A generalization of fixed point theorem of Reich, *Canad. Math. Bull.*, **16**, (1973), 201–208
- [8] **N. Hussain, A.A. Harandi, and Y.J. Cho**, Approximate endpoints for set-valued contractions in metric spaces, *Fixed Point Theory Appl.*, **2010:614867**, (2010), 1–13
- [9] **J.R. Jachymski**, Caristi's fixed point theorem and selections of set-valued contractions, *J. Math. Anal. Appl.*, **227**, (1998), 55–67
- [10] **T.A. Lazăr, A. Petruşel, and N. Shahzad**, Fixed points for non-self operators and domain invariance theorems, *Nonlinear Anal.*, **70**, (2009), 117–125
- [11] **T. Lazăr, D. O'Regan, and A. Petruşel**, Fixed points and homotopy results for Ćirić-type multivalued operators on a set with two metrics, *Bull. Korean Math. Soc.*, **45** (1), (2008), 67–73
- [12] **T. Lazăr, G. Moş, G. Petruşel, and S. Szentesi**, The theory of Reich's fixed point theorem for multivalued operators, *Fixed Point Theory Appl.*, **10**, (2010)
- [13] **V.L. Lazăr**, Fixed point theory for multivalued φ -contractions, *Fixed Point Theory Appl.*, **50**, (2011)
- [14] **S.B. Nadler Jr**, Multi-valued contraction mappings, *Pacific J. Math.*, **30**, (1969), 475–488
- [15] **T.P. Petru, A. Petruşel, and J.C. Yao**, Ulam-Hyers stability of operatorial equations and inclusions via nonself operators, *Taiwanese J. Math.*, **15** (5), (2011), 2195–2212
- [16] **A. Petruşel**, Some variants of the contraction principle for multi-valued operators, generalizations and applications, *to appear*
- [17] **A. Petruşel**, Ćirić type fixed point theorems, *Stud. Univ. Babeş-Bolyai*, **59** (2), (2014), 233–245

- [18] **A. Petruşel and I.A. Rus**, The theory of a metric fixed point theorem for multi-valued operators, *Fixed Point Theory and its Applications, Proc. Ninth International Conference on Fixed Point Theory and Applications, Changhua, Taiwan, (L.J. Lin, A. Petruşel, H.K. Xu - Eds.), Yokohama Publ.*, (2010), 161–175
- [19] **A. Petruşel, I.A. Rus, and J.C. Yao**, Well-posedness in the generalized sense of the fixed point problems for multivalued operators, *Taiwanese J. Math.*, **11** (3), (2007), 903–914
- [20] **A. Petruşel and G. Petruşel**, Selection theorems for multivalued generalized contractions, *Math. Morav.*, **9**, (2005), 43–52
- [21] **A. Petruşel, I.A. Rus, and M.A. Serban**, Basic problems of the metric fixed point theory and the relevance of a metric fixed point theorem for a multivalued operator, *J. Nonlinear Convex Anal.*, **15** (3), (2014), 493–513
- [22] **B. Prasad and R. Sahni**, Endpoints of multivalued contraction operators, *ISRN Mathematical Anal.*, **2013**, (2013), Article ID 542302, 7 pages
- [23] **J.S. Raymond**, Multivalued contractions, *Set-Valued Anal.*, **2**, (1994), 559–571
- [24] **S. Reich**, Fixed points of contractive functions, *Boll. Un. Math. Ital.*, **4** (5), (1972), 26–42

Cristian Daniel Alecsa

Department of Mathematics, Babeş-Bolyai University
M. Kogălniceanu Street, nr. 1
Cluj-Napoca
Romania

E-mail: cristian.alecsa@math.ubbcluj.ro

Tiberiu Popoviciu Institute of Numerical Analysis
Romanian Academy
Fântânele Street nr. 57
Cluj-Napoca
Romania

E-mail: cristian.alecsa@ictp.acad.ro

Adrian Petruşel

Department of Mathematics, Babeş-Bolyai University
M. Kogălniceanu Street, nr. 1
Cluj-Napoca
Romania

Academy of Romanian Scientists
Independenței Street, nr. 54
Bucharest
Romania

E-mail: petrusel@math.ubbcluj.ro