Best Approximation and Unique Extension of Lipschitz Functions

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1. INTRODUCTION

In the sequel, X will always denote a metric space with the metric d, x_0 a fixed point from X, and Y a subset of X such that $x_0 \in Y$. If f is a real-valued function defined on X, denote

$$\|f\|_Y = \sup\{|f(x) - f(y)|/d(x, y) \colon x, y \in Y, x \neq y\}. \tag{1.1}$$

A Lipschitz function on X is a function $f \colon X \to R$ such that $\|f\|_X < \infty$. Denote by $\mathrm{Lip}_0 X$ the Banach space of all Lipschitz functions on X which vanish at x_0, with the norm $\|f\| = \|f\|_X$. Put also

$$Y^{\perp} = \{f \colon f \in \mathrm{Lip}_0 X, f|_Y = 0\}. \tag{1.2}$$

A Lipschitz extension of a function $f \in \mathrm{Lip}_0 Y$ is a function $F \in \mathrm{Lip}_0 X$ such that $F|_Y = f$ and $\|F\|_X = \|f\|_Y$. It is known (see, e.g., [2]) that every $f \in \mathrm{Lip}_0 Y$ has a Lipschitz extension in $\mathrm{Lip}_0 X$.

For a subset Y of X and $x \in X$ we put

$$d(x, Y) = \inf\{d(x, y) \colon y \in Y\}. \tag{1.3}$$

Now, let E be a normed linear space, G a nonempty subset of E, x an element from E, and

$$P_G(x) = \{y \in G \colon \|x - y\| = d(x, G)\}. \tag{1.4}$$

An element from $P_G(x)$ is called a best approximation to x from G. If M is a subset of E we say that G is M-proximinal if $P_G(x) \neq \varnothing$, for all $x \in M$. If $P_G(x)$ contains exactly one element for every $x \in M$, then G is called M-chebyshevian. If the set G is E-proximinal (respectively E-chebyshevian) then we say, simply, that G is proximinal (respectively chebyshevian).

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We say that a linear subspace Z of E has the property (U) if every continuous linear functional on Z has a unique Hahn–Banach extension to E (i.e., linear and norm preserving) [6]. Let us denote by E^* the conjugate space of E and by $Z^\perp$ the annihilator of the subspace Z in E^*, i.e.,

$$Z^\perp = \{\varphi \in E^*; \ \varphi \mid_Z = 0\}. \tag{1.5}$$

Phelps [6] showed that the subspace Z of E has property (U) if and only if its annihilator $Z^\perp$ is chebyshevian. This result can be extended to Lipschitz functions:

THEOREM 1 ([5, Lemma 2]). *Let X be a metric space, x_0 in X, and $Y \subseteq X$ such that $x_0 \in Y$. The space $Y^\perp$ is chebyshevian for $f \in \mathrm{Lip}_0 X$ if and only if $f \mid_Y \in \mathrm{Lip}_0 Y$ has a unique Lipschitz extension in $\mathrm{Lip}_0 X$.*

We also need the following lemma.

LEMMA 1. *Every best approximation to $f \in \mathrm{Lip}_0 X$ from $Y^\perp$ is of the form $f - F$, where F is a Lipschitz extension of $f \mid_Y$ to X.*

Proof. Suppose F is a Lipschitz extension of $f \mid_Y$ to X. Then, by [5, Theorem 2 and Lemma 1], we get

$$\|f - (f - F)\|_X = \|F\|_X = \|f\|_Y = d(f, Y^\perp).$$

Conversely, if $g \in Y^\perp$ is a best approximation to f, then $\|f - g\|_X = d(f, Y^\perp) = \|f\|_Y$ and $(f - g) \mid_Y = f \mid_Y$. Therefore $F = f - g$ is a Lipschitz extension of $f \mid_Y$.

2. MAIN THEOREM

A metric space X is called uniformly discrete if there exists a number $\delta > 0$, such that $d(x, y) \geqslant \delta$ for all $x, y \in X$ with $x \neq y$. The following theorem appears in [5], in the hypothesis that Y has an accumulation point in X. The main result is:

THEOREM 2. *Let X, x_0, and Y be as in Theorem 1. Suppose, further, that Y is nonuniformly discrete. If every $f \in \mathrm{Lip}_0 Y$ has a unique Lipschitz extension, then $\overline{Y} = X$ (or equivalently $Y^\perp = \{0\}$).*

Proof. Since Y is nonuniformly discrete, for every $n \in N$, there exist $x_n, y_n \in Y$, $x_n \neq y_n$ such that $d(x_n, y_n) < 1/n$. Defining $f_n: X \to R$ by

$$f_n(x) = d(x, x_n) - d(x, y_n) - d(x_0, x_n) + d(x_0, y_n), \qquad n = 1, 2, 3, \dots.$$

we have

$$f_n(x_0) = 0, \qquad n = 1, 2, 3,...,$$

$$-2d(x_n, y_n) \leqslant f_n(x_n) = -d(x_n, y_n) - d(x_0, x_n) + d(x_0, y_n)$$

$$\leqslant 0, \qquad n = 1, 2, 3,...,$$

$$0 \leqslant f_n(y_n) = d(x_n, y_n) - d(x_0, x_n) + d(x_0, y_n)$$

$$\leqslant 2d(x_n, y_n), \qquad n = 1, 2, 3,...,$$

$$\|f_n\|_X = \sup\{|d(x, x_n) - d(x, y_n) - d(y, x_n) + d(y, y_n)|/d(x, y):$$

$$x, y \in Y, x \neq y\} \leqslant 2, \qquad n = 1, 2, 3,...,$$

so that $f_n \in \mathrm{Lip}_0 X$ for $n = 1, 2, 3,... $.

Let $a_n = d(x_0, y_n) - d(x_0, x_n)$, and suppose that the set $I = \{n \in N: a_n \leqslant 0\}$ is infinite, say $I = \{n_j : j \in N\}$. Then, we have $f_{n_j}(x_0) = 0, f_{n_j}(x_{n_j}) < 0,$ $f_{n_j}(y_{n_j}) \geqslant 0, j = 1, 2, 3,... $. Now, we consider the sequence $\{\psi_j\}$ of functions $\psi_j : f_{n_j}(X) \to [0, 1]$ defined by

$$\psi_j(t) = 1, \qquad\qquad t < f_{n_j}(x_{n_j}),$$

$$= t/f_{n_j}(x_{n_j}), \qquad f_{n_j}(x_{n_j}) \leqslant t < 0 = f_{n_j}(x_0),$$

$$= 0, \qquad\qquad t \geqslant 0,$$

for $j = 1, 2, 3,... $. Putting $q_j = \psi_j \circ f_{n_j}$, we have

$$\|q_j\|_Y \geqslant |\psi_j(f_{n_j}(x_{n_j})) - \psi_j(f_{n_j}(y_{n_j}))|/d(x_{n_j}, y_{n_j}) \geqslant n_j .$$

By [5, Corollary 2] it follows that

$$d(x, Y) \leqslant (\sup\{\psi_j(f_{n_j}(y)): y \in Y\} - \inf\{\psi_j(f_{n_j}(y)): y \in Y\})/(2\|q_j\|_Y)$$

$$= 1/(2\|q_j\|_Y) \leqslant 1/n_j \to 0,$$

so that $x \in \overline{Y}$, for all $x \in X$, that is $\overline{Y} = X$.

By Theorems 1 and 2, we have

COROLLARY 1.   *Suppose that Y is nonuniformly discrete. Then $Y^\perp$ is chebyshevian in $\mathrm{Lip}_0 X$ if and only if $Y^\perp = \{0\}$.*

We can also prove the following result.

THEOREM 3.   *Let X, x_0, and Y be as in Theorem 1. If $(Y^\perp)^\perp$ has the property (U) then every $f \in \mathrm{Lip}_0 Y$ has a unique Lipschitz extension $F \in \mathrm{Lip}_0 X$.*

Proof.   Follows from [8, Corollary 3.1.b)] and the above Theorem 1.

COROLLARY 2.   *Let X, x_0, and Y be as in Theorem 1. Suppose that Y is nonuniformly discrete. If $(Y^\perp)^\perp$ has the property (U), then $\overline{Y} = X$ (or equivalently $Y^\perp = \{0\}$).*

3. EXAMPLES

(a) Let $X = [0, 1]$, $d(x, y) = |x - y|$, $x_0 = 0$, and $Y = \{0, 1\}$. Then every $f \in \mathrm{Lip}_0\{0, 1\}$ has a unique Lipschitz extension $F \in \mathrm{Lip}_0[0, 1]$, namely, $F(x) = f(1)x$. This example shows that the supposition that Y is non-uniformly discrete is essential in Theorem 2.

(b) Let $f \in \mathrm{Lip}_0[0, 1]$ and let Y be the set of points $0 = x_0 < x_1 < \cdots < x_{n+1} = 1$. Then, we have:

THEOREM 4. *The following conditions are equivalent:*

(1°) $Y^\perp$ *is f-chebyshevian,*

(2°) $\|f\|_Y = |[x_k, x_{k+1}; f]|$, $k = 0, 1, 2,..., n$,

where $[x_k, x_{k+1}; f] = (f(x_{k+1}) - f(x_k))/(x_{k+1} - x_k)$.

Proof. (1°) $\Rightarrow$ (2°) Obviously,

$$L(x) = [x_k, x_{k+1}; f](x - x_k) + f(x_k), \qquad x \in (x_k, x_{k+1}), \quad k = 0, 1,..., n, \tag{3.1}$$

is a Lipschitz extension of $f|_Y$ and $\|f\|_Y \geqslant |[x_k, x_{k+1}; f]|$, $k = 0, 1,..., n$. Suppose that k_0, $0 \leqslant k_0 < n$ is such that $\|f\|_Y > |[x_{k_0}, x_{k_0+1}; f]|$. We have to consider the following cases:

(i) $f(x_{k_0}) < f(x_{k_0+1})$,

(ii) $f(x_{k_0}) > f(x_{k_0+1})$,

(iii) $f(x_{k_0}) = f(x_{k_0+1})$.

If condition (i) holds, put $z_1 = x_{k_0} + (f(x_{k_0+1}) - f(x_{k_0}))/\|f\|_Y$ and define the function $F_1: [0, 1] \to R$ by

$$\begin{aligned} F_1(x) &= L(x), & x &\in [0, 1] - [x_{k_0}, x_{k_0-1}], \\ &= f(x_{k_0}) - \|f\|_Y (x - x_{k_0}), & x &\in (x_{k_0}, z_1), \\ &= f(x_{k_0}), & x &\in [z_1, x_{k_0+1}). \end{aligned} \tag{3.2}$$

It is easy to see that F_1 is a Lipschitz extension of $f|_Y$, distinct from L, and then, by Theorem 1, $Y^\perp$ is not f-chebyshevian.

In case (ii) the proof proceeds similarly. If condition (iii) holds, put $z_2 = (2x_{k_0} + x_{k_0+1})/3$ and define

$$\begin{aligned} F_2(x) &= L(x), & x &\in [0, 1] - [x_{k_0}, x_{k_0+1}], \\ &= f(x_{k_0}) + \|f\|_Y (x - x_{k_0}), & x &\in (x_{k_0}, z_2], \\ &= f(x_{k_0+1}) - \|f\|_Y (x - x_{k_0+1}), & x &\in (z_2, x_{k_0-1}). \end{aligned} \tag{3.3}$$

Then F_2 is a Lipschitz extension of $f|_Y$, different from L. By Theorem 1, $Y^\perp$ is not f-chebyshevian.

$(2°) \Rightarrow (1°)$ If $|[x_k, x_{k+1}; f]| = \|f\|_Y$ for $k = 0, 1, 2,..., n$, then the function L defined by (3.1) is the only Lipschitz extension of $f|_Y$.

A consequence of Theorem 4 is:

COROLLARY 3.   *Let Y be the set of points* $0 = x_0 < x_1 < \cdots < x_{n+1} = 1$, $f \in \text{Lip}_0[0, 1]$ *and*

$$K = \{h: h \in \text{Lip}_0 [0, 1], h(x_k) = f(x_k), k = 0, 1, 2,..., n + 1\}. \quad (3.4)$$

Then $Y^\perp$ is K-chebyshevian if and only if $Y^\perp$ is f-chebyshevian.

(c)   Let $C^1[0, 1]$ be the space of all continuously differentiable functions on $[0, 1]$ and let Y be the set of points $0 = x_0 < x_1 < \cdots < x_{n+1} = 1$. Put

$$Z = C^1[0, 1] \cap \text{Lip}_0[0, 1], \qquad W = C^1[0, 1] \cap Y^\perp. \quad (3.5)$$

For $f \in Z$, we have

$$\|f\|_{[0,1]} = \max\{|f'(x)|: x \in [0, 1]\}. \quad (3.6)$$

Let us define the function set S by

$$S = \{h: h \in Z, [x_k, x_{k+1}; h][x_{k+1}, x_{k+2}; h]$$

$$\neq -\|h\|_Y^2, k = 0, 1, 2,..., n - 1\}. \quad (3.7)$$

We need the following two lemmas:

LEMMA 2.   *Let $[p, q] \subset R$, $f(x) = ax + b$, $a, b \in R$, $a > 0$, and $M > a$. Then there exists a function $g \in C^1[p, q]$ such that $f(p) = g(p)$, $f(q) = g(q)$, $f'(p) = M$ ($f'(q) = M$), $f'(q) = g'(q)$ ($f'(p) = g'(p)$) and $\max\{|g'(x)|: x \in [p, q]\} = M$.*

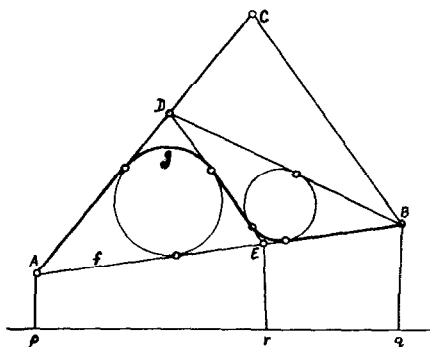*Proof.*   The proof of the lemma is obvious from Fig. 1:

FIGURE 1

(AC) $s_1(x) = f(p) + M(x - p)$,

(BC) $s_2(x) = f(q) - M(x - q)$,

(DE) $s_3(x) = f(r) - M(x - r)$, $r \in (p, q)$.

LEMMA 3.   *If* $h \in S$, *then* $h \mid_Y$ *has at least one Lipschitz extension* $H \in Z$.

Proof.   Let $h \in S$ and $k_0 \in N$, $0 \leqslant k_0 < n + 1$, such that

$$[x_{k_0}, x_{k_0+1} ; h] = \| h \|_Y . \tag{3.8}$$

By the definition of S, we have

$$- \| h \|_Y < [x_{k_0-1}, x_{k_0} ; h] \leqslant \| h \|_Y .$$

$$- \| h \|_Y < [x_{k_0+1}, x_{k_0+2} ; h] \leqslant \| h \|_Y .$$

Applying Lemma 2 to the intervals $[x_{k_0-1}, x_{k_0}]$ and $[x_{k_0+1}, x_{k_0+2}]$, twice it follows that there exists a function H_1 in $C^1[x_{k_0-1}, x_{k_0+2}]$ such that $\max\{| H_1{}'(x)| : x \in [x_{k_0-1}, x_{k_0+2}]\} = \| h \|_Y$ and which interpolates the function h at the points x_{k_0-1}, x_{k_0}, x_{k_0+1}, x_{k_0+2} .

Applying Lemma 2 to the intervals $[x_i, x_{i+1}]$, $i = 0, 1,..., k_0 - 2$, $k_0 + 2,..., n$, we get a function $H \in Z$, which is a Lipschitz extension of $h \mid_Y$ to $[0, 1]$.

If $[x_{k_0}, x_{k_0+1} ; h] = -\| h \|_Y$ we can proceed analogously.

THEOREM 5.   *The subspace* W *is* S *proximinal and for each* $h \in S$ *the following equality holds*:

$$d(h, W) = d(h, Y^\perp). \tag{3.9}$$

Proof.   Let $h \in S$. By Lemma 3, $h \mid_Y$ has a Lipschitz extension $H \in Z$. Then, $h - H \in W$, and this is a best approximation to h, from $Y^\perp$.

But then

$$d(h, Y^\perp) \leqslant d(h, W) \leqslant \| h - (h - H)\|_X = d(h, Y^\perp),$$

so that

$$\| h - (h - H)\|_X = d(h, W) = d(h, Y^\perp).$$

Remark 1.   Let $f \in Z - S$; that is, there exists $0 \leqslant k_1 < n + 1$ such that

$$[x_{k_1-1}, x_{k_1} ; f][x_{k_1}, x_{k_1+1} ; f] = - \| f \|_Y^2 .$$

In this case, it is possible that no Lipschitz extension to f exists in Z; e.g., for $f(x) = -4x^2 + 4x$, $Y = \{0, \frac{1}{2}, 1\}$ we have

$$[0, \tfrac{1}{2} ; f][\tfrac{1}{2}, 1 ; f] = -4$$

and the only Lipschitz extension of $f|_Y$ is

$$F(x) = 2x, \qquad\qquad x \in [0, \tfrac{1}{2}),$$
$$ = -2(x - 1), \qquad x \in [\tfrac{1}{2}, 1],$$

which, obviously, does not belong to Z.

By Lemmas 2 and 3, every $h \in S$ has a best approximation in W, namely, $h - H$, where H is a Lipschitz extension of h, such that $H \in Z$. We can show that every best approximation is of this form (Lemma 1). It follows that W is chebyshevian for $h \in S$ if and only if $h|_Y$ has a unique Lipschitz extension in Z. A class of such functions is given by

$$S_1 = \{h: h \in S, h(x_k) = h(1)\, x_k, k = 0, 1, 2,..., n + 1\}. \qquad (3.11)$$

THEOREM 6. W is S_1-chebyshevian.

Proof.   If $h \in S_1$, then the unique Lipschitz extension of h in Z is $H(x) = h(1)x$. Therefore $h(x) - h(1)x$ is the only element of best approximation for h in W.

Remark 2.   J. Favard and recently de Boor [1] considered a problem analogous to that in Example (c).

(d)   Finally, let X be a metric space of finite diameter (i.e., $\sup\{d(x, y): x, y \in X\} < \infty$), x_0 a fixed element in X, and Y a subset of X such that $x_0 \in Y$. Let $f \in \mathrm{Lip}_0 X$ and let $G(f)$ be the set of best approximation to f from $Y^\perp$. We can define on $\mathrm{Lip}_0 X$ the uniform norm $\|\cdot\|_u: \mathrm{Lip}_0 X \to R$ by

$$\|f\|_u = \sup\{|f(x)|: x \in X\}, \qquad f \in \mathrm{Lip}_0 X. \qquad (3.12)$$

Obviously, the set $G(f) \subset Y^\perp$ is closed, convex, and bounded, for every $f \in \mathrm{Lip}_0 X$. We consider the following problems: Find $g_*, g^* \in G(f)$ such that

$$\|f - g_*\|_u = \inf\{\|f - g\|_u: g \in G(f)\}, \qquad (3.13)$$

and

$$\|f - g^*\|_u = \sup\{\|f - g\|_u: g \in G(f)\}; \qquad (3.14)$$

i.e., find the nearest and the farthest point to f in $G(f)$, in the uniform norm.

Since every element in $G(f)$ is of the form $f - F$, where F is a Lipschitz extension of $f|_Y$ it follows that the problems (3.13) and (3.14) are equivalent to the following problems: Find two Lipschitz extensions F_* and F^* of $f|_Y$ such that

$$\|F_*\|_u = \inf\{\|F\|_u: F \text{ is a Lipschitz extension of } f|_Y\} \qquad (3.13')$$

and

$$\|F^*\|_u = \sup\{\|F\|_u: F \text{ is a Lipschitz extension of } f|_Y\}. \qquad (3.14')$$

THEOREM 6. *The infimum* (3.13) *is attained for every* $g_* = f - F_*$ *such that* F_* *is a Lipschitz extension of* $f|_Y$ *and* $\|F_*\|_u = \|f|_Y\|_u$. *The set of these extensions is nonempty.*

Proof. If F is a Lipschitz extension of $f|_Y$ then

$$\|F\|_u \geqslant \sup\{|F(y)|: y \in Y\} = \sup\{|f(y)|: y \in Y\} = \|f|_Y\|_u.$$

Therefore, if $\|F_*\|_u = \|f|_Y\|_u$ then $\inf\{\|F\|_u: F$ is a Lipschitz extension of $f|_Y\} = \|F_*\|_u = \|f|_Y\|_u$. Now, if F is a Lipschitz extension of $f|_Y$, we define a new Lipschitz function F_* by

$$
\begin{aligned}
F_*(x) &= \|f|_Y\|_u & &\text{if} & F(x) &> \|f|_Y\|_u, \\
&= F(x) & &\text{if} & -\|f|_Y\|_u &\leqslant F(x) \leqslant \|f|_Y\|_u, & (3.15) \\
&= -\|f|_Y\|_u & &\text{if} & F(x) &< -\|f|_Y\|_u.
\end{aligned}
$$

It is easy to see that F_* is a Lipschitz extension of $f|_Y$ such that $\|F_*\|_u = \|f\|_u$.

THEOREM 7. *The supremum* (3.14) *is attained for* $f - F_1^*$ *or* $f - F_2^*$ *or for both of these functions, where*

$$F_1^*(x) = \inf\{[f(y) + \|f|_Y d(x, y)]: y \in Y\}, \qquad (3.16)$$

and

$$F_2^*(x) = \sup\{[f(y) - \|f|_Y d(x, y)]: y \in Y\}. \qquad (3.17)$$

Proof. By [2], F_1^* and F_2^* are Lipschitz extensions of $f|_Y$ and obviously, for every Lipschitz extension F of $f|_Y$ we have

$$F_2^*(x) \leqslant F(x) \leqslant F_1^*(x), \qquad x \in X.$$

From these inequalities, it follows that

$$\|F\|_u \leqslant \max(\|F_1^*\|_u, \|F_2^*\|_u).$$

Remark 3. Dunham [3] has considered a problem similar to the problem in (d) in the case when $G(f)$ has the betweenness property (see [3] for definition). In (d) the set $G(f)$, being convex, has the betweenness property. We found explicitly the nearest and the farthest points of f in $G(f)$.

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