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CONTENTS

(59) $I(a) < I(1) = 0$, auquel cas (59), avec (44) et la proposition 1 donnent

(63) $$\alpha_4'(\beta) > 0.$$

Or, l'expression de $\alpha_4(\beta)$ de l'énoncé de la proposition 4, avec (51), (53) et la règle de l'Hospital donnent

(64) $$\lim_{\beta \to 0+} \alpha_4(\beta) = -\infty.$$

L'expression mentionnée de $\alpha_4(\beta)$ et (54') donnent $\lim_{\beta \to \beta_0 -} \alpha_4(\beta) = -1/(17-24\ell)$, ce qui, avec (64) et (63) démontre la proposition énoncée.

<u>Remarque</u> 4. Nous ne savons pas si l'hypothèse (61) est vraie. Elle concorde, en tout cas, avec les valeurs $f_1'''(-\frac{1}{2}) = 8(3 - 14\ell + 14\ell^2)$ et $f_1'''(1) = 8(3 - 5\ell + \ell^2)$ données par (2), parce qu'on a $f_1'''(-\frac{1}{2}) > f_1'''(1)$, du fait que l'inégalité s'écrit $\ell > \frac{9}{13} = 0,692...$ est devient par suite évidente.

B I B L I O G R A P H I E

1 Petru T. Mocanu, Dumitru Ripeanu, Ioan Şerb, <u>The order of starlikeness of certain integral operators</u>, Mathematica, Revue d'Analyse Numérique et de Théorie de l'Approximation, Mathematica, 23 (46) ,2, (1981), 225 - 230.

ABOUT THE DETERMINATION OF EXTREMA OF A HÖLDER FUNCTION

by

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Let (X,d) be a metric space and $\alpha \in (0,1]$. A function $f: X \to R$ is called <u>Hölder of class α on X</u> if there exists $K \geq 0$ such that

$$|f(x) - f(y)| \leq K.d^{\alpha}(x,y) , \quad (1)$$

for all $x,y \in X$.

Put

$$\|f\|_{\alpha,X} = \sup\{|f(x) - f(y)|/d^{\alpha}(x,y) : x,y \in X, \ x \neq y\} \quad (2)$$

Then $\|f\|_{\alpha,X}$ is the smallest constant K for which the inequality (1) holds and it is called the <u>Hölder norm</u> of f .

Denote by $\Lambda_\alpha(X,d)$ the set of Hölder functions of class α on X [3]. Then $\Lambda_\alpha(X,d)$ is a vector lattice, that is, it is closed under the operations of addition, multiplication by scalars and formation of supremum and infimum of two of its elements.

For a nonvoid subset Y of X , the Hölder norm $\|f\|_{\alpha,Y}$ and the space $\Lambda_\alpha(Y,d)$ are defined similarly .

THEOREM 1. <u>Let</u> (X,d) <u>be a metric space</u> , $Y \subset X$ <u>and</u> $\alpha \in (0,1]$. <u>If</u> $f \in \Lambda_\alpha(Y,d)$ <u>then the functions</u>

$$F_1(x) = \inf\{ f(y) + \|f\|_{\alpha,Y} . d^{\alpha}(x,y) : y \in Y\} , \quad x \in X$$

<u>and</u> (3)

$$F_2(x) = \sup\{ f(y) - \|f\|_{\alpha,Y} . d^{\alpha}(x,y) : y \in Y\} , \quad x \in X$$

<u>are extensions of</u> f , <u>i.e.</u>

a) $F_1|_Y = F_2|_Y = f$,

and

b) $\|F_1\|_{\alpha,X} = \|F_2\|_{\alpha,X} = \|f\|_{\alpha,Y}$.

Theorem 1 follows from Corollary 1.2 in [3] (see also [1],[2]).

Let (X,d) be a compact metric space and let f be in $\Lambda_\alpha(X,d)$. If Y is a subset of X and $q > \|f|_Y\|_{\alpha,Y}$ (here $f|_Y$ denotes the restriction of f to Y), then the functions

$$U(x) = \inf \left\{ f(y) + q.d^\alpha(x,y) : y \in Y \right\} , \quad x \in X$$

and

$$u(x) = \sup \left\{ f(y) - q.d^\alpha(x,y) : y \in Y \right\} , \quad x \in X \tag{4}$$

are extensions of $f|_Y$ which belong to $\Lambda_\alpha(X,d)$ and have the norms q (see [2]). If F_1 and F_2 denotes the functions defined by (3) , then

$$F_2(x) \le f(x) \le F_1(x) \quad , \quad x \in X \tag{5}$$

and for $q > \|f|_Y\|_{\alpha,X}$ we have

$$u(x) \le F_2(x) \le f(x) \le F_1(x) \le U(x) \quad , \quad x \in X \tag{6}$$

A <u>maximum</u> (respectively a <u>minimum</u>) point of a function $f : X \to R$ is a point $x^* \in X$ such that

$$f(x^*) \geq f(x) \quad (\text{ respectively } f(x^*) \le f(x)) \tag{7}$$

for all $x \in X$.

For a bounded real function f on X put

$$M_f = \sup \left\{ f(x) : x \in X \right\} \quad , \quad E_f = \left\{ x \in X : f(x) = M_f \right\} \tag{8}$$

$$m_f = \inf \left\{ f(x) : x \in X \right\} \quad , \quad e_f = \left\{ x \in X : f(x) = m_f \right\} \tag{9}$$

Let now (X,d) be a compact metric space and let f be a function in $\Lambda_\alpha(X,d)$.

We define now inductively two sequences $(x_n)_{n\geq 0}$ and $(M_n)_{n\geq 0}$ of points in X and of real numbers, respectively, as follows :

Let $q > \|f\|_{\alpha,X}$ be fixed and let x_0 be a fixed point in X . Let

$$U^0(x) = f(x_0) + q.d^\alpha(x,x_0) \quad , \quad x \in X \tag{10}$$

the greatest extension of f obtained from (4) for $Y = \{x_0\}$ and let

$$M_0 = \sup \left\{ U^0(x) : x \in X \right\} .$$

Suppose now that for a natural number $n \geq 1$, the points $x_0, x_1, \dots, x_{n-1}$ and the numbers $M_0, M_1, \dots, M_{n-1}$ were defined . Let U^{n-1} be the greatest extension of $f|_Y$ obtained from (4) for $Y = \{x_0, x_1, \dots, x_{n-1}\}$. Put $M_n = \sup \{ U^{n-1}(x) : x \in X \}$ and let x_n be a point in X such that $U^{n-1}(x_n) = M_n$.

The properties of the so defined sequences $(x_n)_{n\geq 0}$ and $(M_n)_{n\geq 0}$ are described in the following theorem :

THEOREM 2. <u>Let (X,d) be a compact metric space and let</u> $f \in \Lambda_\alpha(X,d)$. <u>For a fixed</u> $q > \|f\|_{\alpha,X}$ <u>let the sequences</u> $(x_n)_{n\geq 0}$ <u>and</u> $(M_n)_{n\geq 0}$ <u>be defined as above</u> . <u>Then</u>

a) $\lim_{n \to \infty} M_n = M_f$;

b) $\lim_{n \to \infty} \inf \left\{ d(x,x_n) : x \in E_f \right\} = 0$;

c) The sequence $(f(x_n))_{n\geq 0}$ has the number M_f as a limit point .

Proof. Since $U^n \leq U^{n-1}$ for $n = 1,2,\ldots$, it follows that the sequence $(M_n)_{n\geq 0}$ is nondecreasing . By (6) $M_n = U^{n-1}(x_n) \geq \geq f(x_n) \geq \min\{f(x) : x\in X\}$ so that the sequence $(M_n)_{n\geq 0}$ is also bounded . Therefore there exists $M = \lim M_n$. By (6) $f(x) \leq U^n(x) \leq M_n$, for all $x\in X$ and $n\in N$ so that

$$f(x) \leq M \quad , \text{ for all } x\in X \tag{11}$$

The metric space X being compact, the sequence $(x_n)_{n\geq 0}$ contains a subsequence $(x_{n_k})_{k\geq 0}$ convergent to a point $z\in X$. Since the function f is continuous it follows that

$$f(x_{n_k}) \to f(z) \quad , \quad k\to\infty \tag{12}$$

But, for $k\geq 1$,

$$\left|U^{n_k-1}(z) - M_{n_k-1}\right| = \left|U^{n_k-1}(z) - U^{n_k-1}(x_{n_k})\right| \leq q.d^{\alpha}(z,x_{n_k}) \to 0$$

for $k\to\infty$, and $M_{n_k-1} \to M$ for $k\to\infty$, so that

$$U^{n_k-1}(z) \to M \quad , \text{ for } k\to\infty \tag{13}$$

By the relation

$$\left|U^{n_k}(z) - f(x_{n_k})\right| = \left|U^{n_k}(z) - U^{n_k}(x_{n_k})\right| \leq q.d^{\alpha}(z,x_{n_k}) \to 0 \ , \ k\to\infty,$$

and by (12) it follows that

$$U^{n_k}(z) \to f(z) \quad , \quad k\to\infty \ , \tag{14}$$

Therefore , if in the inequalities

$$f(z) \leq U^{n_k-1}(z) \leq U^{n_{k-1}}(z) \quad , \quad k\geq 1$$

we let $k\to\infty$ one obtains $f(z) \leq M \leq f(z)$, so that $f(z) = M$.

Taking into account (11) it follows that

$$M = f(z) = \max\{f(x) : x\in X\} \ .$$

To prove b) , observe that if contrary, then there exist $\varepsilon > 0$ and an infinite subset J of N such that

$$\inf\{d(x,x_j) : x\in E_f\} > \varepsilon \tag{15}$$

for all $j\in J$. The space X being compact there exists a subsequence $(x_{j_k})_{k\geq 0}$ of $(x_j)_{j\in J}$ converging to a point $y\in X$. But then , repeating the above arguments will follow that $y\in E_f$, which contradicts (15) .

The affirmation c) follows from (12) .

Remarks. 1) In the case $X = [a,b]$ and $\alpha = 1$ a similar result in proved in [6].

2) If the extensions U^n are replaced by the extensions u^n and $m_n = \inf\{u^n(x) : x\in X\}$, $u^n(x_{n+1}) = m_n$, then one obtains a procedure to find the minimum m_f of a function $f\in\Lambda_{\alpha}(X,d$

Example. Let $X = [0,1]$, $d(x,y) = |x - y|$ and

$$f(x) = x.\sin\frac{1}{x} \quad , \text{ if } x\in(0,1]$$

$$= 0 \quad , \text{ if } x = 0$$

It is known that $f \in \Lambda_\alpha(1,d)$ if and only if $\alpha \in (0, \frac{1}{2}]$ (see [7], Problem 153) and in this case

$$\|f\|_{\alpha,X} \leq \left[1 + 2.\ln(1 + 2\pi) + 2\pi\right]^{1/2} < 4 .$$

We apply now Theorem 2 to find the global maximum of the function f on $[0, 1/\pi]$ for $\alpha = 1/2$.

Step 0 . Take $x_0 = 1/2\pi$.

The greatest extension of $f|_{\{x_0\}}$ to $[0, 1/\pi]$ is

$$U^0(x) = f(x_0) + 4|x - x_0|^{1/2}$$

The set of points of local maximum of the function U^0 is

$$D_0 = \left\{ (0, 4/\sqrt{2\pi}) , (1/\pi , 4/\sqrt{2\pi}) \right\} .$$

Step 1 . Take $x_1 = 1/\pi$, and let U^1 be the greatest extension of $f|_{\{x_0, x_1\}}$ to $[0, 1/\pi]$, that is

$$U^1(x) = \min\left\{f(x_k) + 4\,|x - x_k|^{1/2} ;\ k \in \{0,1\}\right\}$$

The points of local maximum of the function U^1 is

$$D_1 = \left\{(x_{01}, U^1(x_{01})) , (0, U^1(0))\right\}$$

where x_{01} is the solution of the equation

$$U^0(x) = U^1(x) ,$$

belonging to $[x_0, x_1]$ and let $x_2 = 0$.

Step 2 . Let U^2 be the greatest extension of $f|_{\{x_0, x_1, x_2\}}$ to $[0, 1/\pi]$. Let

$$D_2 = \left\{(x_{12}, U^2(x_{12})),\ (x_{01}, U^2(x_{01}))\right\}$$

be the set of points of local maximum of the function U^2 and let $x_3 = x_{01}$.

Step n . Let U^n be the greatest extension of the function $f|_{\{x_0, x_1, \dots, x_n\}}$ to $[0, 1/\pi]$ and let x_{n+1} be the greatest of the abscisses of the poits of the global maximum of U^n .

To stop the algorithm one can proceeds in two ways :

a) the algorithm stops after a fixed number of iterations ;

b) the algorithm stops when the difference between the global maximum $U^n(x_M)$ of the function U^n and $f(x_M)$ is smaller than a fixed number $\varepsilon > 0$.

We have programmed this algorithm in the language BASIC with $n = 300$ iterations.

The program is the following :

```
1 OPEN'LP:'FOROUTPUT ASFILEITO WRITE
10 INPUT X%=X% + 1 : Y% = 2*X%
20 DIM X(Y%),F1(X%),F2(X%)
25 M=1 : P% = 0
30 DEF FN F(X)
31 IF X=0 THEN FNF=0 ELSE 33
32 GO TO 35
33 FNF=X*SIN(1./X)
35 FNEND
40 DEF FN P(X,U)=FNF(U) + 3.355*SQR(ABS(X!U))
```

```
50  DEF FN S(X,Y)
60  A = Y + X
70  D = (FNF(Y) - FNF(X))/3.355: D = D*D : E = Y - X
75  D3 = D*(E+E-D)
80  S1 = (A+SQR(D3))/2.
90  IF S1>= X1  AND  S1<= Y  GO TO  110
100  S1 = A - S1
110  FNS = S1
120 FNEND
130 X(1) = 0 : X(2) = 1./PI : X(3) =1./(2.*PI)
140 I% = 3 : J% = 1 : F2(J%) = FNP(X(1),X(3)) : F1(J%) = X(1)
150 J% = J% +1 : F2(J%) = FNP(X(2), X(3)) : F1(J%) = X(2)
160 I% = 1 : S = F2(1)
170 FOR  K% = 2 TO J%
180 IF  S >=F2(K%) THEN 190 ELSE I%=K% : S=F2(K%)
190 NEXT  K%
200 D1=X(1)-X(2) : V1%=0 : V2%=0 : D2=-D1:C=F1(I%)
210 FOR K%=1 TO I%
220 T=X(K%)-C
230 IF T<0 THEN 240 ELSE 250
240 IF T<D1 THEN 250 ELSE D1=T : V1%=K%
250 IF T>0 THEN 1010 ELSE 1020
1010 IF T D2 THEN 1020 ELSE D2=T : V2%=K%
1020 NEXT K%
1030 IF V1%=0 THEN 1070
1040 M%=I% : I%=I% + 1
1050 X(I%)=FNS(X(V1%),C)
```



```
1060 F1(I%)=X(I%) : F2(I%)=FNP(X(I%),C):M1=ABS(F2(I%)-  &
     -FNF(F1(I%))) : IF M1<M THEN M=M1ELSE1070:P%=I%
1070 IF V2%=0 THEN 1140
1080 I%=I% + 1
1090 X(I%)=FNS(C,X(V2%))
1100 IF V1%=0 THEN 1110 ELSE 1120
1110 F1(I%)=X(I%)  : F2(I%)=FNP(X(I%),C) :M1=ABS(F2(I%)-FNF(F1I%))): &
     IF M1<MTHENM=M1ELSE1140:P%=I% :GOTO 1140
1120 J%=J%+1
1125 IF J%=X%-1 OR I%=Y%-1 THEN 1150
1130 F1(J%)=X(I%) : F2(J%)=FNP(X(I%),C):M1=ABS(F2(J%)-FNF(F1(J%))): &
     IF M1<MTHENM1ELSE1135:P%=J%
1135 IF J%=X% OR I%=Y% THEN 1150
1140 IF ABS(F2(J%)-FNF(X(I%)))<1.E-3 THEN 1150 ELSE 160
1150 PRI #1, : PRI #1,'NR. MAXIM DE ITERATII : '; X%-1
1151 PRI #1,: PRI #1,' S-AU FACUT   ' ; J%; ' ITERATII'
1155PRI #1,:PRI #1,'X';TAB(11);'F(X)';TAB(21);'P(X)';TAB(33);'X'; &
     TAB(43);'F(X)';TAB(53);'P(X)';
1156 PRI #1,TAB(65);'X';TAB(75);'F(X)';TAB(85);'P(X)';TAB(97);'X'; &
     TAB(107);'F(X)';TAB(117);'P(X)':PRI #1,
1170 PRI #1USING '#.###### #.###### #.# #####      ',F1(K%); &
     FNF(F1(K%));F2(K%);FOR K%=1 TO J%-1
1180PRI #1,: PRI #1,: PRI #1,'SOL APROXIMATIVA';'X=';F1(P%); &
     'F(X)=';FNF(F1(P%));'P(X)=';F2(P%); 'DELTA='; &
     ABS(F2(P%)-FNF(F1(P%)))
1184U=1./F1(P%)
1185PRI #1,:PRI #1,"VALOAREA DERIVATEI : F'(X)=";SIN(U)-U*COS(U)
1190 END
```

One can show that in this case the sequence $(x_n)_{n\geq 0}$ converges to the maximum point $x^* = 0.12944$ (the solution of the equation $tg(1/x) = 1/x$, $x \in (1/3\pi , 1/2\pi)$).

The maximum of f is $M_f = f(x^*) = 0.128374$. After 300 iteration we have obtained the following results :

$$x_{300} = 0.129982$$

$$f(x_{300}) = 0.128309$$

$$M_{229} = U^{299}(x_{300}) = 0.144181$$

The errors are in this case

$$M_f - f(x_{300}) = 0.000065$$

$$x_{300} - x^* = 0.000537$$

R E F E R E N C E S

1. G. ARONSSON , Extension of functions satisfying Lipschitz conditions , Arkiv för Mathematik 6, 28 (1967), 551 - 561 .
2. J. CZIPSER and L. GEHER , Extension of functions satisfying a Lipschitz condition, Acta.Math.Acad.Sci.Hungar 6 (1955), 213-220.
3. R. LEVY and M.D. RICE , The Approximation of Uniformly Continuous Mappings by Lipschitz and Hölder Mappings, (preprint),1980.
4. E.J. McSHANE , Extension of range of functions, Bull.Amer.Math. Soc. 40 (1934), 837-842.
5. C. MUSTATA , Best Approximation and Unique Extension of Lipschitz Functions, Journal Approx. Theory 19 (1977), 222-230.
6. B. SHUBERT , A sequential method seeking the global maximum of a function , SIAM J. Numer. Anal. 9 (1972), 379 - 388.
7. The Otto Dunkel Memorial Problem Book, New York (1957).

NORMAL CONE VALUED METRICS AND NONCONVEX VECTOR MINIMIZATION PRINCIPLE

by A.B. Németh

0. Introduction. The extension in (N1-2) of the nonconvex minimization principle in (E) does not be formally a generalization. In order to formulate a theorem which contains as particular cases both the classical formulation due to EKELAND (E) as well as its extension for ordered spaces in (N2), we have to introduce cone valued metrics. It turns out that these metrics offer useful technical facilities in various problems of the nonlinear functional analysis. The generalization of the classical metric spaces in this direction was initiated by KUREPA (KU) and then used by KANTOROVIC, VULICH and PINSKER (KVP), SCHRÖDER (SchR), COLLATZ (CO), ANTONOVSKII, BOLTJANSKII and SARYMSAKOV (ABS) etc. Are noteworthy the applications in fixed point theory. We have got an excellent guide for surprising this orientation in the monography (R) of RUS and the literature cited there. The first relevant results in this direction seem to be these due to PEROV (PE) and PEROV and KIDENKO (PK) (see also (R)). The regular cone valued metrics present for us a special interest. Their importance for the fixed point theory was discovered by KIESELFELD and LAKSHMIKANTHAM who have developed in (KL1-3) various aspects of fixed point theory for spaces with metric having values in a regular cone of a Banach space.

In our next considerations occur naturally some theoretical problems. First of all we have to verify if our principal result is or not a generalization of that in (N2). Further, we have to characterise the topological spaces which are metrizable in this