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ON A SURJECTIVITY THEOREM

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It was proved in [3] that if X and Y are two Banach spaces and $F: X \to Y$ is a multifunction such that $F(X)$ is a closed subset of Y, then the multifunction F is surjective if and only if for every $b \in F(X)$ the tangent cone in b to the set $F(X)$ is Y.

In this note we shall give necesarry and sufficient conditions for the surjectivity of a multifunction $F: X \to Y$, when X and Y are arbitrary metric spaces.

For a metric space Z and a fixed point $z_0 \in Z$ denote by $\mathrm{Lip}_0(Z,R)$ the set of all real-valued Lipschitz functions on Z which vanish at z_0, i.e.

$$(1)\quad \mathrm{Lip}_0(Z,R) = \{ f: Z \to R \text{ , } f \text{ is Lipschitz and } f(z_0) = 0 \} .$$

Recall that a function $f: Z \to R$ is called <u>Lipschitz</u> if there exists $M \geq 0$ such that

$$(2)\quad |f(z_1) - f(z_2)| \leq M\, d(z_1,z_2) \quad ,$$

for all $z_1, z_2 \in Z$, where d denotes the metric of Z. The smallest number M for which (2) holds is called the <u>Lipschitz norm</u> of f and is denoted by $\| f \|$. The norm is given by :

$$(3)\quad \| f \| = \sup \{ |f(z_1) - f(z_2)| / d(z_1,z_2) \ : \ z_1, z_2 \in Z \text{ , } z_1 \neq z_2 \} .$$

The space $Lip_o(Z,R)$ equipped with the norm (3) is a Banach space .

If $Z_1 \subset Z$ is such that $z_o \in Z_1$ and $f \in Lip_o(Z_1,R)$ then a function $\bar{f}: Z \to R$, $\bar{f} \in Lip_o(Z,R)$ is called an *extension* of f if

$$(4) \qquad \bar{f}\big|_{Z_1} = f \quad \text{and} \quad \|\bar{f}\| = \|f\| .$$

In general, every $f \in Lip_o(Z_1,R)$ has a least one extension $\bar{f} \in Lip_o(Z,R)$. Concerning the properties of the set of all extensions of f see [4] , [5] .

A subset A of the metric space (Z,d) is called *uniformly discrete* if there exists $\delta > 0$ such that $d(x,y) \geq \delta$ for all $x,y \in A$, $x \neq y$.

For $Z_1 \subset Z$ denote by $Z_1^{\perp}$ the subspace of $Lip_o(Z,R)$ defined by

$$(5) \qquad Z_1^{\perp} = \{ f \in Lip_o(Z,R) \ , \ f\big|_{Z_1} = 0 \} .$$

Obviously , $Z_1^{\perp}$ is a closed subspace of the Banach space $Lip_o(Z,R)$.

A subset Y of a normed space X is called proximinal if every $x \in X$ has a best approximation element in Y , i.e. for every $x \in X$ there exists $y_o \in Y$ such that

$$(6) \qquad \|x - y_o\| = \inf\{\|x - y\| \ : \ y \in Y\} .$$

If every $x \in X$ has exactly one element of best approximation in Y , then the set Y is called a *Cebysevian subset* of X .

Let now X and Y be a metric space and $F: X \to Y$ a multifunction. It was proved in [4] that $F(X)^{\perp}$ is always a proximinal subset of Y and that $F(X)^{\perp}$ is Cebysevian if and only if

$F(X)^{\perp} = \{0\}$ (Lemma 1 and Corollary 1 in [4])

The main result of this Note is the following theorem :

THEOREM 1. *Let X,Y be two metric spaces , let $F: X \to Y$ be a multifunction such that $F(X)$ is closed and non-uniformly discrete and let $y_o \in F(X)$ be fixed . Then the following conditions are equivalent :*

(a) *F is surjective* (i.e. $F(X) = Y$) ;

(b) *Every $f \in Lip_o(F(X),R)$ has exactly one extension $\bar{f} \in Lip_o(Y,R)$* ;

(c) *The subspace $F(X)^{\perp}$ is Cebysevian in* $Lip_o(Y,R)$.

Proof. If F is surjective then $F(X) = Y$ so that $Lip_o(F(X),R) = Lip_o(Y,R)$ and $F(X)^{\perp} = Y^{\perp} = \{0\}$ which show that the implications (a) $\Rightarrow$(b) and (a) $\Rightarrow$(c) hold true .

Suppose now that condition (b) is verified then, by [4, Th.2] it follows that $\overline{F(X)} = Y$ and since $F(X)$ is closed in Y it follows that $F(X) = Y$, i.e. (b) $\Rightarrow$(a) .

The equivalence (b)$\Leftrightarrow$(c) follows by Lemma 1 and Corollary 1 from [4]. Theorem 1 is completely proved .

If X and Y are Banach spaces then condition (b) is equivalent also to the condition :

(d) For every $y \in F(X)$ the tangent cone $T_yF(X)$ to $F(X)$ in y is Y ([3], Theorem 3) .

COROLLARY 1. *Let X,Y be two Banach spaces and let $F: X \to Y$ be a multifunction such that $F(X)$ is closed. Then*

$$(a) \Leftrightarrow (d) \ ; \quad (b) \Leftrightarrow (c) \quad \text{and} \quad (d) \Rightarrow (b) \ ;$$

If, furthermore $F(X)$ is not uniformly discrete then (b) $\Rightarrow$(d) *too* .

Example. The hypothesis that $F(X)$ is not uniformly discrete is essential in Theorem 1.

Let $X = Y = [0,1]$ and $F: X \to Y$ defined by

$$F(x)=\begin{cases} 0 & , \; x \in [0, 1/2] \\ 1 & , \; x \; (1/2 , 1] \end{cases}$$

Then $F(X) = \{0,1\}$ is closed and uniformly discrete. Let

$$\mathrm{Lip}_0([0,1] ,R) = \{ f :[0,1] \to R, \; f \text{ is Lipschitz and } f(0) = 0\}$$

Every function $f \in \mathrm{Lip}_0(\{0,1\}, R)$ has a unique extension $\bar{f}$ defined by $\bar{f}(x) = f(1)x$, $x \in [0,1]$. Indeed

$$\bar{f}|_{\{0,1\}} = f \quad \text{and} \quad \|\bar{f}\| = \|f\| = |f(1)| \quad .$$

Therefore (b) does not imply (a) .

Also

$$F(\{0,1\})^{\perp} = \{f \in \mathrm{Lip}_0([0,1] ,R) , \; f(1) = 0\}$$

and for every $f \in \mathrm{Lip}_0([0,1] ,R)$ the element g_0 of best approximation for f in $F(\{0,1\})^{\perp}$ is

$$g_0(x) = f(x) - f(1)x \quad , \; x \in [0,1] \; .$$

This element is unique since $g_0(x) = f(x) - \bar{f}(x)$ and $\bar{f}$ is the only extension of f (see [4], Lemma 1). Therefore (c) does not imply (a) .

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