„BABEŞ—BOLYAI" UNIVERSITY
FACULTY OF MATHEMATICS
RESEARCH SEMINARIES

SEMINAR OF

FUNCTIONAL ANALYSIS AND NUMERICAL METHODS

Preprint Nr. 1 , 1985

CLUJ-NAPOCA
ROMANIA

C O N T E N T S

"BABES-BOLYAI" UNIVERSITY, Faculty of Mathematics
Research Seminaries
Seminar of Functional Analysis and Numerical Methods
Preprint Nr.1, 1985, pp. 84 - 92.

ON THE EXTENSION OF HÖLDER FUNCTIONS

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1. Let (X,d) and (Y,ρ) be two metric spaces. For $\alpha \in (0,1]$ a function $f: X \to Y$ is called <u>Hölder of class α</u> if there exists $M \geq 0$ such that

$$(1) \qquad (F(x),f(y)) \leq M\,(d(x,y))^{\alpha} ,$$

for all $x,y \in X$.

Denote by $\Lambda_\alpha(X,Y)$ the set of all Hölder functions of class α from X to Y.

If Y is a metric linear space then, equiped with the pointwise operations of addition and multiplication by scalars, $\Lambda_\alpha(X,Y)$ become a linear space. If $Y = R$ then $\Lambda_\alpha(X,R)$ is also a lattice (the order is defined pointwisely too).

For $f \in \Lambda_\alpha(X,R)$ put

$$(2) \qquad \|f\|_\alpha = \sup\{\,|f(x) - f(y)|/(d(x,y))^{\alpha} : x,y \in X,\ x \neq y\}\,,$$

the smallest number $M \geq 0$ for which the inequality

$$(3) \qquad |f(x) - f(y)| \leq M\,(d(x,y))^{\alpha} ,$$

holds for all $x,y \in X$.

Obviously $\|f\|_\alpha \geq 0$ and $\|f\|_\alpha = 0$ if and only if $f = \text{const.}$, for all $f \in \Lambda_\alpha(X,R)$.

Let $x_0 \in X$ be fixed and let

$$(4) \qquad \Lambda_\alpha(x_0,X,R) = \{f \in \Lambda_\alpha(X,R)\ ,\ f(x_0) = 0\}\,.$$

Then $\Lambda_\alpha(x_0,X,R)$ is a subspace of $\Lambda_\alpha(X,R)$ and the functional defined by (2) is a norm on this subspace and is called the <u>Hölder norm</u> of f.

We say that two functions $f,g \in \Lambda_\alpha(X,R)$ are equivalent if $f - g = \text{const.}$ and we shall denote this by $f \sim g$.

It is immediate that the quatient space of $\Lambda_\alpha(X,R)$ by this equivalence relation is isomorphic to $\Lambda_\alpha(x_0,X,R)$.

A very important problem in the theory of Hölder functions is the extension problem. More exactly, let (X,d), (Y,ρ) be two metric spaces and let $Z \subset X$. The <u>extension problem</u> is the following : for $f \in \Lambda_\alpha(Z,Y)$ find $F \in \Lambda_\alpha(X,Y)$ such that

$$(5) \qquad f = F|_Z \quad \text{and} \quad \|f\|_\alpha = \|F\|_\alpha .$$

The function F is called a <u>norm preserving</u> extension of f.

For $\alpha = 1$ (the case of Lipschitz functions) the problem was extensively studied . The existence of a norm preserving extension for every $f \in \Lambda_\alpha(Z,Y)$ depends on the properties of the sets Z and Y. A positive solution for the extension problem in the case $Y = R$ and for X arbitrary was given by Mc SHANE [14] and by G. MINTY [11] in the case when X and Y are Hilbert spaces.

If X and Y are arbitrary metric spaces (even Banach spaces) the extensions is not always possible as was shown by B. GRÜNBAUM [5] and S.O. SCHÖNBECK [12], [13].

T.M. FLETT [4] proved that if X and Y are normed spaces and $Z \subset X$ is convex, closed, bounded of diameter δ and contains a ball of radius $r > 0$ then for every $f \in \Lambda_1(Z,Y)$ there exists

$F \in \Lambda_1(X,Y)$ such that $F|_Z = f$ and $\|F\|_1 = \frac{\delta}{r} \cdot \|f\|_1$.

If every function $f \in \Lambda_\alpha(Z,Y)$ has an extension $F \in \Lambda_\alpha(X,Y)$ it is natural to ask if this extension is unique or not. It was shown that the question of the unicity of the norm preserving extension is closely related to some approximation problems in the space $\Lambda_\alpha(X,Y)$ (see [7] , [8] , [10]) .

2. In the following we shall denote $Lip(X,Y) = \Lambda_1(X,Y)$. If X is a Banach space and S is a closed ball of radius $r > 0$ in X then as was shown by T.M. FLETT [4] there exists a function $F \in Lip(X,X)$ such that

$$\|F\|_1 = 2\, \|f\|_1 \tag{6}$$

where $f = F|_S$.

THEOREM 1. <u>Let</u> X <u>be a Banach space and let</u> $f \in Lip(X,X)$. <u>Suppose that the following conditions hold true</u> :

a) <u>There exists a convex, closed, bounded set</u> C <u>of diameter</u> ρ <u>and containing a ball of radius</u> $\delta > 0$ <u>such that</u>

$$\|f|_C\|_1 < \frac{\delta}{\rho} \quad ;$$

b) <u>Every extension</u> F <u>of</u> f <u>verifies</u>

$$\|f - F\|_1 < 1 - \|f|_C\|_1 \cdot \frac{\rho}{\delta} . \tag{7}$$

<u>Then there exists a unique</u> $x^* \in X$ <u>such that</u> $f(x^*) = x^*$.
(<u>The function</u> f <u>has a unique fix point</u> $x^* \in X$) .

<u>Proof</u>. Let $f \in Lip(X,X)$ and $C \subset X$ such that condition a) is verified. By the above quated result of Flett there exists $F \in Lip(X,X)$ such that $F|_C = f|_C$ and $\|F\|_1 = \|f|_C\|_1 \frac{\rho}{\delta}$. Then

$$\|f\|_1 = \|f - F + F\|_1 \le \|f - F\|_1 + \|F\|_1 < 1 - \|f|_C\|_1 \frac{\rho}{\delta} + \|f|_C\|_1 \cdot \frac{\rho}{\delta} = 1.$$

Since $\|f(x) - f(y)\| \le \|f\|_1 \|x - y\|$ for all $x,y \in X$ it follows that f is a contraction on X and by Banach contraction principle there exists a unique $x^* \in X$ such that $f(x^*) = x^*$. Theorem is proved .

COROLLARY 1. <u>Let</u> X <u>be a Banach space and</u> $f \in Lip(X,X)$. <u>Suppose that there exists a closed ball</u> $S \subset X$ <u>of radius</u> $\delta > 0$ <u>such that every extension</u> F <u>of</u> $f|_S$ <u>verifies the condition</u> :

$$\|f - F\|_1 < 1 - 2\, \|f|_S\|_1 . \tag{8}$$

<u>Then</u> f <u>has a unique fix point in</u> X .

<u>Proof</u>. The diameter of S is $\rho = 2\delta$ and by (8) $\|f|_S\|_1 < \frac{1}{2}$ so that the condition a) and b) from Theorem 1 are verified.

<u>Remark</u> 1. Let C be as in Theorem 1 and $f \in Lip(X,X)$. If $\|f|_C\|_1 = 0$ then $\|f(x) - f(y)\| = 0$ for all $x,y \in C$ and $f(x) = f(y) = z \in X$ for all $x,y \in C$. Since $F|_C = f$ and $\|F\|_1 = 0 \frac{\rho}{\delta} = 0$ it follows that $F(x) = z$ for all $x \in X$. Therefore the condition (7) from Theorem 1 becomes

$$\|f\|_1 < 1 \ , \tag{7'}$$

i.e. f is a contraction on X .

If $\|f|_C\|_1 = 0$ then $f = const.$ on C and the extension $F \in Lip(X,X)$ is unique .

3. We consider the following problem : for a metric space X, a subset M of X and a function $f \in \Lambda_\alpha(M,R)$ find

$$\min \{ f(y) \ : \ y \in M \} . \tag{9}$$

In concrete problems the set M is usually determined by some restrictions and the function f is replaced by the function $\overline{F}$

defined by

$$\bar{F}(x) = \begin{cases} f(x) , & x \in M \\ +\infty , & x \in X \setminus M \end{cases}$$

Obviously $\min \{ f(y) \ : \ y \in M \} = \min \{ \bar{F}(x) \ : \ x \in X \}$.

HIRIART-URRUTY [6] proved that if X is a Banach space $M \subset X$ is closed and $f \in Lip(M,R)$, then the problem

$$\min \{ f(y) \ : \ y \in M \}$$

can be replaced by the problem :

$$\min \{ F_1(x) \ : \ x \in X \} ,$$

where $F_1(x) = \inf_{y \in M} [f(y) + \|f\|_1 \cdot \| x - y \|] , \quad x \in X .$

In this note we shall give some similar results in the case of a metric space X and for a function $f \in \Lambda_\alpha(M,R)$, $0 < \alpha \leq 1$.

Let X be a metric space, let M be a closed subset of X and let $f \in \Lambda_\alpha(M,R)$. By a result in [9] the function F_1 defined by

(1o) $$F_1(x) = \inf_{y \in M} [f(y) + \|f\|_\alpha (d(x,y))^\alpha] , \quad x \in X$$

is in $\Lambda_\alpha(X,R)$ and

$$F_1|_M = f , \quad \|F_1\|_\alpha = \|f\|_\alpha .$$

A point $y_0 \in M$ is called a minimum (maximum) for f if

$$f(y_0) \leq f(y) \qquad (\ f(y_0) \geq f(y) \ )$$

for all $y \in M$.

THEOREM 2. <u>Let</u> X <u>be a metric space,</u> M <u>a closed subset of</u> X <u>and</u> $f \in \Lambda_\alpha(M,R)$. <u>Then</u> $y_0 \in M$ <u>is a minimum point for</u> f <u>on</u> M <u>if and only if</u> y_0 <u>is a minimum point for</u> F_1 <u>on</u> X .

<u>Proof</u>. Let y_0 be a minimum point for f on M and let F_1 be defined by (1o). For every $x \in M$ we have

$$F_1(x) = f(x) \geq f(y_0) = F_1(y_0) .$$

If $x \notin M$, the set M being closed, there exists $\delta > 0$ such that $d(x,y) \geq \delta > 0$ for all $y \in M$. Therefore

$$F_1(x) = \inf_{y \in M} [f(y) + \|f\|_\alpha (d(x,y))^\alpha] \geq$$

$$\geq \inf_{y \in M} [f(y) + \|f\|_\alpha \delta^\alpha] = \|f\|_\alpha \delta^\alpha + f(y_0) > f(y_0) ,$$

so that y_0 is a minimum point for F_1 on X .

Conversely, suppose that y_0 is a minimum point for F_1 on X . If we would show that $y_0 \in M$ then, as $F_1|_M = f$, it would follow that y_0 is a minimum point for f on M .

Suppose , on the contrary, that $y_0 \notin M$. Then, since M is closed ,

$$d(y_0,M) = \inf \{ d(y_0,y) \ : \ y \in M \} = q > 0 .$$

By the definition of F_1 we have

$$F_1(y_0) = \inf_{y \in M} [f(y) + \|f\|_\alpha (d(y_0,y))^\alpha] ,$$

so that, for every $\varepsilon > 0$, there exists $y_\varepsilon \in M$ such that

$$F_1(y_0) + \varepsilon > f(y_\varepsilon) + \|f\|_\alpha (d(y_0,y_\varepsilon))^\alpha .$$

For $\varepsilon_n = \frac{\|f\|_\alpha q}{n}$, denoting $y_n = y_{\varepsilon_n}$, one obtains

$$f(y_{\varepsilon_n}) = F_1(y_n) \geq F_1(y_0) > f(y_n) + \|f\|_\alpha (d(y_0,y_n))^\alpha - \frac{\|f\|_\alpha q}{n}$$

which implies

$$\|f\|_\alpha \cdot ((d(y_0,y_n))^\alpha - \frac{q}{n}) \leq 0 .$$

If $\|f\|_\alpha = 0$ then $f = $ const on M and y_0 (as every other point in M). will be a minimum point for f on M.

If $\|f\|_\alpha > 0$ then

$$(d(y_0,y_n))^\alpha - \frac{q}{n} \leq 0$$

so that

$$0 < q \leq d(y_0,y_n) \leq \left(\frac{q}{n}\right)^{\frac{1}{\alpha}}.$$

Letting $n \to \infty$ in the inequality $0 < q \leq \left(\frac{q}{n}\right)^{\frac{1}{\alpha}}$ one obtains a contradiction. Theorem 2 is proved.

Let $f \in \Lambda_\alpha(M,R)$ and let

$$F_2(x) = \sup_{y \in M} \left[f(y) - \|f\|_\alpha (d(x,y))^\alpha \right] , \quad x \in X$$

The function F_2 has the properties :

$$F_2|_M = f \quad \text{and} \quad \|F_2\|_\alpha = \|f\|_\alpha ,$$

(see [9]).

THEOREM 3. Let X be a metric space, M a closed subset of X and $f \in \Lambda_\alpha(M,R)$. Then $y_0 \in M$ is a maximum point for f on M if and anly if y_0 is a maximum point for F_2 on X.

The proof of this Theorem is simillar to the proof of Theorem 2.

Remark 2. If X is a metric linear space, M a closed convex subset of X and $f \in \Lambda_\alpha(M,R)$ is convex, then f has minimum point on M. The function F_1, defined by (1o), has the same minimum on X as f on M. Furthermore the function F is convex too (see [8]).

If f is a concave function on M, then the function F_2 is concave too on X and the maximum of F_2 on X equals the maximum of f on M.

REFERENCES

1. ARONSSON, G., Extension of functions satisfying Lipschitz conditions, Arkiv für Mathematik 6 (1967), 551 - 561.
2. CZIPSER, J., GEHER, L., Extension of functions satisfying a Lipschitz condition, Acta Math. Sci. Hungar 6 (1955), 213 - 220.
3. DANZER, L., GRUNBAUM, B., KLEE, V., Helly's theorem and its relatives, Proc. Sympos. Pure Mathem. 7, Amer. Math. Soc., Providence, R.I. (1963), 101-180.
4. FLETT, T.M., Extension of Lipschitz functions, J. London Math. Soc. 7 (1974), 604 - 608.
5. GRÜNBAUM, B., A generalization of Theorems of Kirszbraum and Minty, Proc. Amer. Math. Soc, 13 (1962), 812 - 814.
6. HIRIART-URRUTY, J.B., Extension of Lipschitz functions, Preprint 1980 (23 pp.).
7. MUSTĂȚA, C., Best Approximation and Unique Extension of Lipschitz Functions, Journal of Approx. Theory 19, 3 (1977), 222 - 230.
8. MUSTĂȚA, C., COBZAȘ, S., Norm Preserving Extension of Convex Lipschitz Functions, Journal of Approx. Theory 24, 3 (1978), 236 - 244.
9. MUSTĂȚA, C., About the determination of extrema of a Hölder functions, Seminar of Funct.Anal. Numerical Methods, Babeș-Bolyai University, Fac. of Math. Preprint Nr. 1 (1983), 107 - 116.

10. MUSTĂȚA, C., Asupra unicității prelungirii funcțiilor Hölder reale , Seminarul itinerant de Ecuații Functionale, Aproximare și Convexitate, Cluj-Napoca, 1980 .

11. MINTY, G.J., On the extension of Lipschitz, Lipschitz-Hölder contnuous, and monotone functions , Bull. Amer. Math. Soc. 76 (1970), 334 - 339 .

12. SCHÖNBECK, S.O., Extension of nonlinear contractions , Bull. Amer. Math . Soc. 72 (1966), 99 - 101 .

13. SCHÖNBECK, S.O., On the extension of Lipschitz maps , Arkiv för Mathematik 7 (1967 - 1969), 201 - 209.

14. McSHANE, E.J., Extension of range of functions , Bull. Amer. Math. Soc., 40 (1934), 837 - 842 .

15. VALENTINE, F.A., A Lipschitz condition preserving extension for a vector function , Amer. J. Math. 67 (1945) , 83 - 93 .

"BABES-BOLYAI" UNIVERSITY, Faculty of Mathematics
Research Seminaries
Seminar on Functional Analysis and Numerical Methods
Preprint Nr. 1, 1985, pp. 93 - 116.

ON SOME UNIVERSAL SUBDIFFERENTIABILITY PROPERTIES OF ORDERED VECTOR SPACES

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Introduction and definitions. Besides many studies of convex operators with values in order complete vector lattices (see for example (V), (L), (IL), (AK), (KU1), (Z3), (K), (B3) and (P)), ZOWE (Z1), (Z2), FEL'DMAN (F), and recently BORWEIN (B1), (B2) and the author (N1), (N2) have considered problems on subdifferentiability of convex operators with values in more general ordered vector spaces. The main result in (N2), which constitutes the complete characterization of ordered vector spaces admitting strictly monotone functionals in order to every convex operator with values in them have pleasant subdifferentiability properties, gives the idea to consider other less restrictive conditions on subdifferentiability from this point of view. More precisely this approach is the following : to consider ordered vector spaces with some universality property formulated in terms of subdifferentiability and then to characterize these spaces in other terms of the theory of ordered vector spaces, as well as to estabilish interrelations of various subdifferentiability like properties. Our paper constitutes an attempt in this direction. A program of this kind is very general and since the subdifferentiability