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# ON THE EXTENSION OF LIPSCHITZ FUNCTIONS

Costică Mustăţa

1. Introduction. Let $A$ be a subset of the interval $[a,b] \subset R$. A function $f: A \to R$ is called Lipschitz if there exists $L \geq 0$ such that

(1) $$|f(x) - f(y)| \leq L\cdot|x - y| \quad ,$$

for all $x,y \in A$. The smallest number $L$ for which the inequality (1) holds is called the Lipschitz norm of $f$ and is denoted by $\|f\|_L$. The Lipschitz norm of $f$ can be calculated also by the formula

(2) $$\|f\|_L = \sup\left\{|f(x) - f(y)| / |x - y| : x,y \in A,\ x \neq y\right\} .$$

Denote by Lip $A$ the set of all real valued Lipschitz functions on $A$, i.e.

(3) $$\text{Lip } A = \left\{ f,\ f: A \to R,\ f \text{ is Lipschitz} \right\} .$$

With the usual ( i.e. pointwise ) operations of addition and multiplication by scalars, Lip $A$ is a vector space .

A Lipschitz extension of $f$ to $[a,b]$ is a Lipschitz function $F: [a,b] \to R$ such that

(4) $F|_A = f$ and $\|F\|_L = \|f\|_L$ .

By a result of Mc SHANE [4] every function in Lip A has at least one Lipschitz extension in Lip [a,b] . More exactly, the following two functions

(5) $F_1(x) = \sup \{ f(y) - \|f\|_L \cdot |x-y| \ : \ y \in A \}$

and

(6) $F_2(x) = \inf \{ f(y) + \|f\|_L \cdot |x-y| \ : \ y \in A \}$ .

are Lipschitz extensions of f to [a,b] . Denoting by E(f;[a,b]) the set of all Lipschitz extensions of f to [a,b] , i.e.

(7) $E(f;[a,b]) = \{ F \in \mathrm{Lip}\,[a,b] \ : \ F|_A = f \text{ and } \|F\|_L = \|f\|_L \}$

the following assertions hold true

(a) $F_1(x) \le F(x) \le F_2(x)$ , $x \in [a,b]$ , for all $F \in E(f;[a,b])$ ;

(b) E(f;[a,b]) is a convex subset of Lip [a,b] ;

(c) The functions $F_1$ and $F_2$ are extreme points of E(f; [a,b]) .

By the definition of the Lipschitz norm ( or by (2) ), $\|f\|_L = 0$ if and only if f = constant and therefore $\|\cdot\|_L$ is not actually a norm on Lip A but it is a norm on the space $\mathrm{Lip}_0 A$ of all functions in Lip A vanishing at a fixed point $x_0 \in A$ . The space $\mathrm{Lip}_0 A$ with the Lipschitz norm is a dual Banach space (see [3] ).

2. <u>Lipschitz extensions from finite subsets of</u> [a,b] .

Let C[a,b] be the space of all real valued continuous functions on [a,b] and let

(8) $M = \{x_0, x_1, \ldots, x_n\}$ , $a \le x_0 < x_1 < \ldots < x_n \le b$ ,



be a finite subset of [a,b] . Then obviously, the restriction $f|_M$ of a function $f \in C[a,b]$ to M is in Lip M and

(9) $\|f|_M\|_L = \max\{ |f(x_i) - f(x_j)| / |x_i - x_j| \ : \ i,j = 0,1,\ldots,n, \ i \ne j \}$

Let $U: C[a,b] \to \mathrm{Lip}\,M$ be the restriction operator, i.e.

(10) $U(f) = f|_M$ , for $f \in C[a,b]$ .

By the above quoted result of Mc SHANE , $f|_M$ has at least a Lipschitz extension $F \in \mathrm{Lip}\,[a,b]$ . Let $V: \mathrm{Lip}\,M \to \mathcal{P}(\mathrm{Lip}[a,b])$ the extension operator defined by

(11) $V(g) = E(g;[a,b])$ , $g \in \mathrm{Lip}\,M$ ,

and let $W: C[a,b] \to \mathcal{P}(\mathrm{Lip}[a,b])$ be the composition of U and V

(12) $W = V \circ U$ .

In general W is a multivalued operator (point to set). A function $f \in C[a,b]$ such that $f \in W(f)$ is called a <u>fix point</u> of W . Obviously , the set of fix points of the operator W is non-void. Indeed, if $g \in \mathrm{Lip}\,M$ and f is a Lipschitz extension of g to [a,b] , then $f \in W(f)$ . The fix points of the operator W are characterized in Theorem 2.1 below.

For $x,y \in [a,b]$, $x \ne y$ and $f \in C[a,b]$ , put

(13) $[x,y;f] = (f(x) - f(y))/(x - y)$

and

(14) $\ell(x,y;f)(t) = [x,y;f].(t-x) + f(x)$ , $t \in [a,b]$ .

2.1 <u>THEOREM. Let</u> $f \in C[a,b]$ <u>and let</u> M <u>be the set</u> (8) . <u>Then</u> $f \in W(f)$ <u>if and only if there exists an index</u> $k \in \{0,1,\ldots,n-1\}$ <u>such that</u>

(15) $\sup\{ |[x,y;f]| \ : \ x,y \in [a,b], \ x \ne y \} = |[x_k, x_{k+1} \ ; f]|$ .

<u>Proof</u>. If $f \in W(f)$ then $f \in E(f|_M ; [a,b])$ and

$$\|f\|_L = \|f|_M\|_L = \max\{|[x_j, x_{j+1} ; f]| : j=0,1,\dots,n-1\}$$

so that, there exists $k \in \{0,1,\dots,n-1\}$ such that $\|f|_M\|_L = |[x_k, x_{k+1} ; f]|$, and the relation (15) holds.

Conversely, if the relation (15) holds for a index $k \in \{0,1,\dots,n-1\}$, then

$$|[x_k, x_{k+1} ; f]| = \sup\{|[x,y;f]| : x,y \in [a,b] , x \neq y\} \geq$$

$$\geq \max\{|[x_j, x_{j+1} ; f]| : j = 0,1,\dots,n-1\} = \|f|_M\|_L .$$

Therefore, $\|f|_M\|_L = |[x_k, x_{k+1} ; f]| = \|f\|_L$, which shows that $f \in W(f)$.

Theorem 2.1 has some corollaries.

2.2 COROLLARY. <u>If the relation</u> (15) <u>holds for an index</u> $k \in \{0,1,\dots,n-1\}$ <u>then</u>

$$f(x) = \ell(x_k, x_{k+1} ; f)(x) \quad \text{for all} \quad x \in [x_k, x_{k+1}] .$$

<u>Proof</u>. If $f(x') \neq \ell(x_k, x_{k+1} ; f)(x')$, for an $x' \in (x_k, x_{k+1})$ then

$$\max\{|[x_k, x' ; f]| , |[x', x_{k+1} ; f]|\} > |[x_k, x_{k+1} ; f]| =$$

$$= \sup\{|[x,y;f]| : x,y \in [a,b] , x \neq y\} ,$$

which is a contradiction.

2.3 COROLLARY. <u>If</u> $f \in W(f)$ <u>then</u> $f(x) = F(x)$, $x \in [x_k, x_{k+1}]$, <u>for all</u> $F \in W(f)$, <u>where</u> $k \in \{0,1,\dots,n-1\}$ <u>is the index for which relation</u> (15) <u>is true</u>.

<u>Proof</u>. If $F \in W(f)$ then

$$\sup\{|[x,y;F]| : x,y \in [a,b] , x \neq y\} = \|F\|_L = \|f|_M\|_L = |[x_k, x_{k+1} ; F]|$$



and Corollary 2.3 follows from Corollary 2.2 .

2.4 COROLLARY. <u>If</u> $a = x_0$ <u>and</u> $x_n = b$ , <u>then</u>

(a) $\sup\{|[x,y;f]| : x,y \in [a,b], x \neq y\} = |[x_0, x_n ; f]|$, <u>implies</u> $f(x) = \ell(x_0, x_n ; f)(x)$, <u>for all</u> $x \in [a,b]$ , <u>and</u>

(b) $\sup\{|[x,y;f]| : x,y \in [a,b] , x \neq y\} = |[x_k, x_{k+1} ; f]|$ , $k = 0,1,\dots,n-1$ , <u>implies</u> $f(x) = \ell(x_k, x_{k+1} ; f)(x)$ , $x \in [x_k, x_{k+1}]$ , $k = 0,1,\dots,n-1$ .

Combining Corollaries 2.3 and 2.4 it follows

2.5 COROLLARY. <u>If</u> $f \in Lip[a,b]$, $x_0 = a$, $x_n = b$ <u>and</u> $\|f\|_L = |[x_k, x_{k+1} ; f]|$ , $k = 0,1,\dots,n-1$ , <u>then</u> $W(f) = \{f\}$ .

## 3. <u>Faces and extreme points</u>.

Let $X$ be a normed space and $B(X) = \{x \in X : \|x\| \leq 1\}$ its closed unit ball. A subset $A \subseteq B(X)$ is called an <u>extremal subset</u> ( a <u>face</u> ) of $B(X)$ if $\alpha f_1 + (1 - \alpha) f_2 \in A$ for $f_1, f_2 \in B(X)$ and a number $\alpha$ , $0 < \alpha < 1$ , implies $f_1, f_2 \in A$ . If $A$ contains exactly one point $f$ , then $f$ is called an <u>extreme point</u> of $B(X)$ .

Let $M$ be a finite set of real numbers

$$M = \{x_0, x_1, \dots, x_n\} , \quad x_0 < x_1 < \dots < x_n$$

and

$$Lip_0 M = \{f , \; f : M \to \mathbb{R} , \; f \text{ is Lipschitz and } f(x_0) = 0\} .$$

3.1 THEOREM. <u>The function</u> $f \in B(Lip_0 M)$ <u>is an extreme point of</u> $B(Lip_0 M)$ <u>if and only if</u>

$$\|f\|_L = 1 = |[x_k, x_{k+1} ; f]| , \tag{16}$$

<u>for</u> $k = 0,1,\dots,n-1$ .

<u>Proof</u>. Suppose that relation (16) do not hold and let $k \in \{0,1,\dots,n-1\}$ and $\varepsilon > 0$ be such that

$$|[x_k, x_{k+1}; f]| = 1 - \varepsilon .$$

Let

$$f_1(x) = \begin{cases} f(x) , & x \in \{x_o, x_1, \ldots, x_k\} \\ f(x) + \delta , & x \in \{x_{k+1}, \ldots, x_n\} \end{cases}$$

and

$$f_2(x) = \begin{cases} f(x) , & x \in \{x_o, x_1, \ldots, x_k\} \\ f(x) - \delta , & x \in \{x_{k+1}, \ldots, x_n\} \end{cases}$$

where $\delta = (\varepsilon/2)(x_{k+1} - x_k)$.

Because

$$\frac{|f(x_{k+1}) + \delta - f(x_k)|}{x_{k+1} - x_k} \le 1 - \varepsilon/2 < 1 \quad \text{and}$$

$$\frac{|f(x_{k+1}) - \delta - f(x_k)|}{x_{k+1} - x_k} \le 1 - \varepsilon/2 < 1 , \text{ it follows that}$$

$\|f_1\|_L = 1 = \|f_2\|_L$ . But $f = (1/2)(f_1 + f_2)$ , so that $f$ is not an extreme point of $B(\mathrm{Lip}_o M)$ .

Suppose now that condition (16) is fulfilled and there exist two functions $g_1, g_2 \in B(\mathrm{Lip}_o M)$ such that $g_1 \ne f \ne g_2$ and $f = (1/2)(g_1 + g_2)$ . Let $x_i$ be the smallest element of $M$ for which $g_1(x_i) \ne f(x_i)$ . As $g_1(x_o) = f(x_o) = 0$ and $g_2(x_o) = 0 = f(x_o)$ , it follows $i \ge 1$ .

Case I. $f(x_i) > f(x_{i-1})$ and $g_1(x_i) > f(x_i)$. In this case

$$\frac{g_1(x_i) - g_1(x_{i-1})}{x_i - x_{i-1}} > \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} = 1 ,$$

which implies $\|g_1\|_L > 1$ , i.e. $g_1 \notin B(\mathrm{Lip}_o M)$ .



Case II. $f(x_i) > f(x_{i-1})$ and $g_1(x_i) < f(x_i)$. In this case $g_2(x_i) = 2 f(x_i) - g_1(x_i) > f(x_i)$ and

$$\frac{g_2(x_i) - g_2(x_{i-1})}{x_i - x_{i-1}} > \frac{f(x_i) - [2 f(x_{i-1}) - g_1(x_{i-1})]}{x_i - x_{i-1}} = $$

$$= \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} = 1 ,$$

so that $\|g_2\|_L > 1$ , i.e. $g_2 \notin B(\mathrm{Lip}_o M)$ .

In the remaining cases, i.e. $f(x_i) < f(x_{i-1})$ and $g_1(x_i) > f(x_i)$ or $f(x_i) < f(x_{i-1})$ and $g_1(x_i) < f(x_i)$ , we have similarly $\|g_2\|_L > 1$ , respectively $\|g_1\|_L > 1$ . The obtained contradictions show that $f$ must be an extreme point of $B(\mathrm{Lip}_o M)$ .

3.2 Remark. Taking in (16) all possible signs, it follows that the unit ball of the space $\mathrm{Lip}_o M$ has exactly $2^{n+1}$ extreme points .

3.3 COROLLARY. Let $f \in \mathrm{Lip}\,[a,b]$ , $M = \{x_o, x_1, \ldots, x_n\}$ , $a = x_o < x_1 < \ldots < x_n = b$ , and $f(x_o) = 0$ . If

$$\|f\|_L = 1 = |[x_k, x_{k+1}; f]|$$

for all $k = o,1,\ldots,n-1$ , then $f$ is an extreme point of the unit ball of $\mathrm{Lip}_o\,[a,b]$ .

3.4 Remark. If $f$ is an extreme point of $B(\mathrm{Lip}_o M)$ and the set $M$ is as in Corollary 3.3 , then the unique Lipschitz extension $F$ of $f$ , $F(x) = \sup\{ f(x_i) - |x - x_i| : i=o,1,\ldots,n\}$ $= \inf\{ f(x_i) + |x - x_i| : i=o,1,\ldots,n\}$ , $x \in [a,b]$ , is an extreme point of $B(\mathrm{Lip}_o\,[a,b])$ .

## 4. Best approximation of Lipschitz functions.

Let

$$M = \{x_o, x_1, \ldots, x_n\} \quad , \quad a \le x_o < x_1 < \ldots < x_n \le b \; ,$$

and let

(17) $\quad \mathrm{Lip}_o[a,b] = \{f \; , \; f:[a,b] \to R \; , \; f$ is Lipschitz and $f(x_o) = 0\}$.

Let also

$$M^{\perp} = \{f \in \mathrm{Lip}_o[a,b] \quad : \quad f|_M = 0\} .$$

Since every function $f \in \mathrm{Lip}_o M$ has at least one extension $F \in \mathrm{Lip}_o[a,b]$ it follows that every function $f \in \mathrm{Lip}_o[a,b]$ has a best approximation (nearest point) in $M^{\perp}$, i.e. there exists $g_o \in M^{\perp}$ such that $\|f - g_o\|_L = \inf\{\|f - g\|_L \; : \; g \in M^{\perp}\}$. It was shown ( see [5] ) that $g_o \in M^{\perp}$ is an element of best approximation for $f \in \mathrm{Lip}_o[a,b]$ by elements from $M^{\perp}$ if and only if $g_o = f - F$ , for $F \in W(f)$ .

Taking into account the precedings results it follows :

4.1 THEOREM. Let $f \in \mathrm{Lip}_o[a,b]$ and $M = \{x_o, x_1, \ldots, x_n\}$ , $a \le x_o < x_1 < \ldots < x_n \le b$ .

(a) If the relation (15) holds for an index $k \in \{o, 1, \ldots, n-1\}$ then all the best approximation elements for $f$ in $M^{\perp}$ vanish on the interval $[x_k, x_{k+1}]$ ;

(b) If $x_o = a$ , $x_n = b$ and $\|f\|_L = |[x_o, x_n ; f]|$ , then 0 is the only best approximation element for $f$ in $M$ ;

(c) If $\|f\|_L = |[x_k, x_{k+1} ; f]|$ , for all $k = o, 1, \ldots, n-1$ , then 0 is the only best approximation element for $f$ in $M^{\perp}$.

Proof. Assertion (a) follows from Corollary 2.3 , taking into account that the best approximation elements $g_o$ for $f$ in $M^{\perp}$ have the form

$$g_o = f - F \quad , \quad F \in W(f) .$$

Assertions (b) and (c) follow from Corollary 2.4 .



4.2 Remark. Let $M$ be the set (8) and let

$$\mathrm{Lip}_o[a,b] = \{f:[a,b] \to R \; , \; f(x_o) = 0 \; , \; f \text{ is Lipschitz}\} .$$

If $f \notin W(f)$ , then, by Theorem 2.1 , it follws

$$\|f\|_L > \|f|_M\|_L \; .$$

If, further

(i) $\qquad f(x) \le F_1(x) \quad , \quad x \in [a,b]$

where $F_1$ is given by (5) ( with $A = M$ ) , then all best approximation elements of $f$ in $M^{\perp}$ are non-positive, and if

(ii) $\qquad f(x) \ge F_2(x) \quad , \quad x \in [a,b]$

with $F_2$ given by (6) (with $A = M$ ) , then all the elements of best approximation for $f$ in $M^{\perp}$ are non-negative .

Obviously, there exist functions $f \in \mathrm{Lip}_o[a,b]$ verifying the conditions (i) and (ii) and which are not fixed points of $W$ .

For example, taking $g \in \mathrm{Lip}_o[a,b]$ , the function

$$f(x) = \sup\{g(x_k) - L.|x - x_k| \; : \; x_k \in M \; , \; k = o, 1, \ldots, n\}$$

with $L > \|g|_M\|_L$ , verifies the condition (i) and the function

$$h(x) = \inf\{g(x_k) + L.|x - x_k| \; : \; x_k \in M \; , \; k = o, 1, \ldots, n\}$$

with $L > \|g|_M\|_L$ verifies the condition $h(x) \le F_2(x)$ , $x \in [a,b]$ ( i.e. condition (ii) ) .

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# KNOWN AND NEW EQUIVALENT FORMS OF THE ARCHIMEDIAN PROPERTY OF ORDERED VECTOR SPACES

A. B. Németh

The Archimedian property of the ordered vector spaces, its equivalent or weaken forms become important in some recent results on vectorial optimization (see e.g., (F) , (N1) (B) (N2) etc.). This suggests us to gather the various equivalent forms, to estabilish new equivalences and to present them together in order to facilite further references.

The Archimedian property involves in its statement two elements of the space, hence it is typically finite dimensional: it holds for the whole space if and only if it holds for its finite dimensional subspaces. It is in fact a geometrical property concerning straight lines. The Archimedian property is equivalent with the lineally closedness of the positive cone of the ordered vector space. The two different approaches : Archimedian property and the lineally closedness subzist in the literature perhaps since the first concerns with vector spaces ordered by cones while the second term is used merely in the case when the vector spaces are ordered by wedges (see e.g. (SY), (D), (F) etc.).