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ERROR ESTIMATION IN THE APPROXIMATION OF FUNCTIONS BY INTERPOLATION CUBIC SPLINES

C. IANCU, C. MUSTĂŢA

1. In this Note we give estimations for the error of approximation of a continuous function $f:[a, b] \to R$ by interpolation cubic splines with respect to a given division Δ_x of the interval $[a, b]$.

Let $f:[a, b] \to R$ be a function and let

$$\Delta_x : a = x_0 < x_1 < \ldots < x_n = b \tag{1}$$

be a division of the interval $[a, b]$.

Put

$$f_i = f(x_i), \quad i = 0, 1, 2, \ldots, n. \tag{2}$$

and let us denote by $\mathrm{Sp}(3, \Delta_x)$ the set of all cubic spline s corresponding to the partition Δ_x and having the properties:

(i) the restriction of s to every interval $[x_{i-1}, x_i]$ is a polynomial of degree at most 3, for $i = 1, 2, \ldots, n$;

(ii) $s \in C^2[a, b]$, i.e. s is continuously two times differentiable on $[a, b]$;

(iii) $s(x_i) = f_i$, $i = 0, 1, 2, \ldots, n$ i.e. s interpolates the function f on the knots in Δ_x.

Put also

$$\begin{aligned} h_i &= x_i - x_{i-1}, \quad i = 1, 2, \ldots, n \\ m_i &= s'(x_i) \quad , \quad i = 0, 1, 2, \ldots, n \\ M_i &= s''(x_i) \quad , \quad i = 0, 1, 2, \ldots, n. \end{aligned} \tag{3}$$

For s in $\mathrm{Sp}(3, \Delta_x)$, the restriction of the second derivative s'' of s to the interval $[x_{i-1}, x_i]$ is a polynomial of degree at most 1, so that

$$s''(x) = M_{i-1} + \frac{M_i - M_{i-1}}{x_i - x_{i-1}}(x - x_{i-1}), \tag{4}$$

$$x \in [x_{i-1}, x_i], \quad i = \overline{1, n}$$

Taking into account the conditions

$$\begin{aligned} s(x_{i-1}) &= f_{i-1} \\ s'(x_{i-1}) &= m_{i-1} \end{aligned} \tag{5}$$

the relation (4) gives:

$$s(x) = \frac{M_i - M_{i-1}}{6h_i}(x - x_{i-1})^3 + \frac{M_{i-1}}{2}(x - x_{i-1})^2 + \tag{6}$$

$$+ m_{i-1}(x - x_{i-1}) + f_{i-1}$$

for $x \in [x_{i-1}, x_i]$ and $i = 1, 2, \ldots, n$.

PROPOSITION. *Every function* $s \in \mathrm{Sp}(3, \Delta_x)$, *given by formula* (6), *is uniquely determined by the conditions*:

(i) $s(x_i) = f_i$, $i = 1, 2, \ldots, n$

(ii) $s'(x_i) = m_i$, $i = 1, 2, \ldots, n$

(iii) $m_0 = p$, $M_0 = q$, p, q — given real numbers.

Proof. Conditions (i) and (ii) in the Proposition can be rewritten in the form

$$M_i = 6\frac{f_i - f_{i-1}}{h^2_i} - \frac{6}{h_i} \cdot m_{i-1} - 2M_{i-1} \tag{7}$$

$$m_i = 3\frac{f_i - f_{i-1}}{h_i} - 2m_{i-1} - \frac{h_i}{2}M_{i-1}, \quad i = 1, 2, \ldots, n$$

By condition (iii) system (7) is compatible and has a unique solution $m_1, m_2, \ldots, m_n$; $M_1, M_2, \ldots, M_n$. System (7) can be recursively solved starting from the condition (iii) $m_0 = p$, $M_0 = q$.

2. *Estimation of the approximation error.* In some papers (see e.g. [2], [3] and the papers quoted there) are given evaluations of the uniform norms $\|s - f\|$ and $\|s' - f'\|$ for f satisfying some sufficiently restrictive conditions.

(a) In the following we shall evaluate the uniform norm

$$\|s - f\|, \tag{8}$$

supposing that f is a Lipschitz function on $[a, b]$, i.e. there exists a number $K \geq 0$ (called a Lipschitz constant) such that

$$|f(x) - f(y)| \leq K|x - y|, \tag{9}$$

for all $x, y \in [a, b]$.

The number

$$\|f\|_L = \sup\{|f(x) - f(y)|/|x - y| : x, y \in [a, b], x \neq y\} \tag{10}$$

is the smallest Lipschitz constant for f and is called the *Lipschitz norm* of f on the interval $[a, b]$.

The space of all Lipschitz function on $[a, b]$ is denoted by Lip $[a, b]$. The Lipschitz norm of the restriction of f to the division Δ_x is given by

$$\|f|_{\Delta x}\|_L = \max\{|[x_{i-1}, x_i; f]| : i = 1, 2, \ldots, n\} \tag{11}$$

where $[x_{i-1}, x_i; f] = (f(x_i) - f(x_{i-1}))/(x_i - x_{i-1})$ is the divided difference of the function f on the knots x_{i-1}, x_i.

In the sequel we shall need the following extension result of McShane [4]: Let X be a matric space, Y a subset of X and let $f : Y \to R$ be a Lipschitz function. Then there exists a Lipschitz function $F : X \to R$ such that $F|_Y = f$ and $\|F\|_L = \|f\|_L$. In [6] it was proved that for every $f \in \mathrm{Lip}\, Y$ and every $K \geqslant \|f\|_L$ there exists an extension $F : X \to R$ of f such that $\|F\|_L = K$.

By this result, if $f \in \mathrm{Lip}\,[a, b]$, then the restriction $f|_{\Delta x}$ of f to Δ_x has at least one extension $F \in \mathrm{Lip}\,[a, b]$ such that $\|F\|_L = \|f\|_L$. It is obvious that such an extension is f itself, but the following two functions

$$(12)\qquad \begin{aligned} F_1(x) &= \sup\{f(x_k) - \|f\|_L \cdot |x - x_k| : k = 0, 1, 2, \ldots, n\} \\ F_2(x) &= \inf\{f(x_k) + \|f\|_L \cdot |x - x_k| : k = 0, 1, \ldots, n\} \end{aligned}$$

are also extensions of f with norm $\|f\|_L$, i.e.

$$(13)\qquad \|F_1\|_L = \|F_2\|_L = \|f\|_L \text{ and } F_1|_{\Delta x} = F_2|_{\Delta x} = f|_{\Delta x}$$

and every extension F of $f|_{\Delta x}$ such that $\|F\|_L = \|f\|_L$ verifies $F_1 \leqslant F \leqslant F_2$ (see [5]). In particular

$$(14)\qquad F_1(x) \leqslant f(x) \leqslant F_2(x),$$

for all $x \in [a, b]$.

From (14) it follows

$$s(x) - F_1(x) \geqslant s(x) - f(x) \geqslant s(x) - F_2(x), \quad x \in [a, b]$$

so that

$$(15)\quad \min\{\|s - F_1\|, \|s - F_2\|\} \leqslant \|s - f\| \leqslant \max\{\|s - F_1\|, \|s - F_2\|\}$$

Taking into account the fact that the functions F_1 and F_2 given by (12) are piecewise linear, the calculation of the norms $\|s - F_1\|$ and $\|s - F_2\|$ reduces to the calculation of the norm of third degree polynomials on compact subintervals of $[a, b]$.

If

$$(16)\qquad \begin{aligned} a_i &= \|(s - F_1)|_{[x_{i-1}, x_i]}\| \text{ and} \\ b_i &= \|(s - F_2)|_{[x_{i-1}, x_i]}\| \end{aligned}$$

for $i = 1, 2, \ldots, n$, then

$$\|s - F_1\| = \max\{a_i : i = 1, 2, \ldots, n\}$$

and

$$\|s - F_2\| = \max\{b_i : i = 1, 2, \ldots, n\}$$

In order to calculate the numbers a_i, b_i, $i = 1, 2, \ldots, n$, we have to distinct three cases:

Case 1. $f_{i-1} < f_i$.
In this case, for $x \in [x_{i-1}, x_i]$ we have

$$F_1(x) = \begin{cases} f_{i-1} - \|f\|_L(x - x_{i-1}), & x \in [x_{i-1}, \underline{x}] \\ f_i + \|f\|_L(x - x_i), & x \in (\underline{x}, x_i] \end{cases}$$

where

$$\underline{x} = \frac{x_i + x_{i-1}}{2} + \frac{f_{i-1} - f_i}{2\|f\|_L}$$

and

$$F_2(x) = \begin{cases} f_{i-1} + \|f\|_L(x - x_{i-1}), & x \in [x_{i-1}, \bar{x}] \\ f_i - \|f\|_L(x - x_i), & x \in (\bar{x}, x_i] \end{cases}$$

where

$$\bar{x} = \frac{x_i + x_{i-1}}{2} - \frac{f_{i-1} - f_i}{2\|f\|_L}$$

Since

$$x_{i-1} < \underline{x} < \bar{x} < x_i$$

we have

$$a_i = \max \{\|(s - F_1)|_{[x_{i-1}, \underline{x}]}\|, \|(s - F_1)|_{[\underline{x}, \bar{x}]}\|, \|(s - F_1)|_{[\bar{x}, x_i]}\|\}$$

and

$$b_i = \max \{\|(s - F_2)|_{[x_{i-1}, \underline{x}]}\|, \|(s - F_2)|_{[\underline{x}, \bar{x}]}\|, \|(s - F_2)|_{[\bar{x}, x_i]}\|\}$$

Case 2. $f_{i-1} > f_i$.

In this case $x_{i-1} < \bar{x} < \underline{x} < x_i$ and therefore the norms of $s - F_1$ and $s - F_2$ are calculated on the intervals $[x_{i-1}, \bar{x}]$, $[\bar{x}, \underline{x}]$, $[\underline{x}, x_i]$.

Case 3. $f_{i-1} = f_i$.

In this case $\underline{x} = \bar{x} = (x_{i-1} + x_i)/2$ and the norms of $s - F_1$ and $s - F_2$ are calculated on the intervals $[x_{i-1}, (x_i + x_{i-1})/2]$, $[(x_i + x_{i-1})/2, x_i]$.

In concrete situations, the numbers a_i and b_i can be easily calculated. We do not enter into details, but let us mention that, in general, can be obtained evaluations from above of the norms occurring in the expressins of a_i and b_i, depending only on m_i, m_{i-1}, M_i, M_{i-1}, h_i and $\|f\|_L$.

Concerning the exactity of the evaluations (15) we show that in the set of all real valued Lipschitz functions g on $[a, b]$ with norm $\|g\|_L = \|f\|_L$ and such that $g(x_i) = f_i$, $i = 0, 1, 2, \ldots, n$, there exists two functions $\bar{f}$ and $\underline{f}$ such that the evaluations (15) are the best possible in this set.

Let

$$(17) \quad E(f|_{\Delta_x}; [a, b]) = \{g \in \mathrm{Lip}\,[a, b] : g(x_i) = f(x_i), \quad i = 0, 1, 2, \ldots, n, \quad \|g\|_L = \|f\|_L\}$$

Obviously, the functions F_1 and F_2 defined by (12) belong to $E(f|_{\Delta_x}; [a, b])$ and if $g \in E(f|_{\Delta_x}; [a, b])$ then

$$F_1(x) \leqslant g(x) \leqslant F_2(x), \quad x \in [a, b].$$

For every interval $[x_{i-1}, x_i]$, $i = 1, 2, \ldots, n$, let us define the function $\bar{f}_i$ in the following way:

$$(18) \qquad \bar{f}_i = \begin{cases} F_1|_{[x_{i-1}, x_i]} & \text{if } a_i = \max\{a_i, b_i\} \\ F_2|_{[x_{i-1}, x_i]} & \text{if } b_i = \max\{a_i, b_i\} \end{cases}$$

Let the function $\bar{f} : [a, b] \to R$ be defined by

$$\bar{f}\,|_{[x_{i-1}, x_i]} = \bar{f}_i, \quad i = 1, 2, \ldots, n. \tag{19}$$

Then $\bar{f} \in \mathrm{Lip}\,[a, b]$ and, as can be easily seen from the definition of the function $\bar{f}$,

$$\|\bar{f} - s\| \leqslant \|g - s\| \tag{20}$$

for every function $g \in E(f|_{\Delta_x}; [a, b])$.

Similarly, the function $\underline{f} : [a, b] \to R$ defined by

$$\underline{f}\,|_{[x_{i-1}, x_i]} = \underline{f}_i, \quad i = 1, 2, \ldots, n \tag{21}$$

where

$$\underline{f}_i = \begin{cases} F_1\,|_{[x_{i-1}, x_i]} \text{ if } a_i = \min\,\{a_i, b_i\} \\ F_2\,|_{[x_{i-1}, x_i]} \text{ if } b_i = \min\,\{a_i, b_i\} \end{cases} \tag{22}$$

for $i = 1, 2, \ldots, n$, verifies the inequality

$$\|g - s\| \leqslant \|\underline{f} - s\| \tag{23}$$

for every function $g \in E(f|_{\Delta_x}; [a, b])$.

(b) *Evaluation of the norm* $\|f' - s'\|$.

In the following we shall suppose $f \in C^1\,[a, b]$. In this case $f \in \mathrm{Lip}[a, b]$ and

$$\|f\|_L = \max\,\{|f'(x)| : x \in [a, b]\}. \tag{24}$$

The formulae (12) become

$$F_1(x) = \sup\,\{f(x_i) - \max_{x \in [a, b]} |f'(x)| \cdot |x - x_i| : i = 0, 1, \ldots, n\}$$

and

$$F_2(x) = \inf\,\{f(x_i) + \max_{x \in [a, b]} |f'(x)| \cdot |x - x_i| : i = 0, 1, \ldots, n\}$$

These functions are in $\mathrm{Lip}\,[a, b]$ but, in general, they do not belong to $C^1[a, b]$. They are differentiable on (a, b) excepting (eventually) the points in Δ_x and the points of the form

$$x = \frac{x_i + x_{i-1}}{2} + \frac{f_{i-1} - f_i}{2\,\|f\|_L}, \quad x = \frac{x_i + x_{i-1}}{2} + \frac{f_{i-1} - f_i}{2\|f\|_L}$$

If $f_{i-1} < f_i$, then the functions $s - F_1$ and $s - F_2$ are continuously differentiable on every interval $(x_{i-1}, \underline{x})$, $(\underline{x}, \bar{x})$, $(\bar{x}, x_i)$. We have

$$s'(x) - f'(x) = \frac{M_i - M_{i-1}}{2h_i}(x - x_{i-1})^2 + M_{i-1} \cdot (x - x_{i-1}) + m_{i-1} - f'(x) \tag{25}$$

for all $x \in [x_{i-1}, x_i]$.

Since

$$-\|f\|_L \leqslant -f'(x) \leqslant \|f\|_L, \quad x \in [x_{i-1}, x_i]$$

it follows

$$s'(x) - \|f\|_L \leqslant s'(x) - f'(x) \leqslant s'(x) + \|f\|_L, \; x \in [x_{i-1}, x_i],$$

so that

$$|s'(x) - f'(x) \leqslant \max \{\| s' + \|f\|_L \|, \; \| s' - \|f\|_L \|\}$$

for every $x \in [x_{i-1}, x_i]$, where the norms occurring in the right member of the above inequality are calculated on the interval $[x_{i-1}, x_i]$, $i = 1, 2, \ldots n$.

Denoting

$$(26) \qquad \begin{aligned} c_i &= \| s' + \|f\|_L \| \\ d_i &= \| s' - \|f\|_L \| \end{aligned}$$

where the norms are again calculated on $[x_{i-1}, x_i]$, $i = 1, 2, \ldots, n$, we find that the inequalities

$$(27) \qquad \min_{i=\overline{1,n}} \{c_i, d_i\} \leqslant \| s' - f' \| \leqslant \max_{i=\overline{1,n}} \{c_i, d_i\}$$

hold true on the interval $[a, b]$.

Denoting by

$$(28) \qquad x_0 = \frac{x_i M_i - 2x_i M_{i-1} + x_{i-1} M_{i-1}}{M_i - M_{i-1}}$$

the root of the equation $s''(x) = 0$ in the interval $[x_{i-1}, x_i]$ one gets

$$(29) \quad c_i = \begin{cases} \max \left\{ |s'(x_0) + \|f\|_L|, \; |m_{i-1} + \|f\|_L|, \; \left| \frac{h_i}{2}(M_i + M_{i-1}) + m_{i-1} + \|f\|_L \right| \right\}, & \text{if } x_0 \in (x_{i-1}, x_i) \\ \max \left\{ |m_{i-1} + \|f\|_L|, \; \left| \frac{h_i}{2}(M_i + M_{i-1}) + m_{i-1} + \|f\|_L \right| \right\} & \text{if } x_0 \notin [x_{i-1}, x_i] \end{cases}$$

and, respectively,

$$(30) \quad d_i = \begin{cases} \max \left\{ |s'(x_0) - \|f\|_L|, \; | m_{i-1} - \|f\|_L|, \; \left| \frac{h_i}{2}(M_i + M_{i-1}) + m_{i-1} - \|f\|_L \right| \right\} & \text{if } x_0 \in (x_{i-1}, x_i) \\ \max \left\{ |m_{i-1} - \|f\|_L|, \; \left| \frac{h_i}{2}(M_i + M_{i-1}) + m_{i-1} - \|f\|_L \right| \right\} & \text{if } x_0 \notin [x_{i-1}, x_i] \end{cases}$$

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Institute of Mathematics
Oficiul Poştal 1, C.P. 68
3400 Cluj-Napoca, Romania