"BABEŞ-BOLYAI" UNIVERSITY
Faculty of Mathematics and Physics
Research Seminars
Seminar on Mathematical Analysis
Preprint Nr.7, 1988, pp. 65 - 74.

M - IDEALS IN METRIC SPACES

Costică Mustăţa

Abstract. Considering a metric space and its Lipschitz dual one defines the notion of M-ideal and HB-subspace of a metric space (with respect to its Lipschitz dual). One obtains some results analogous to those in the theory of M-ideals and HB-subspaces in a normed space. The results in the paper are based on an extension theorem of McShane [2], [8] and on a uniqueness theorem which is similar to one of R.R. Phelps [10], [11].

Introduction. The notion of M-ideal, introduced by E.M. Alfsen and E. Effross [1], has many applications in functional analysis as, for example, to the problem of best approximation of continuous linear operators by compact operators (see [6]). Following E.M. Alfsen and E. Effross [1], an M-ideal in a normed space X is a closed subspace Y of X whose annihilator $Y^{\perp}$ admits a complement G in X^* such that $\|g+h\| = \|g\| + \|h\|$, $g \in G$, $h \in Y^{\perp}$. J. Hennefeld [4], [5] defined a more general notion : A closed subspace Y of a normed space X is called an HB-subspace if $Y^{\perp}$ has a complement G in X^* such that for every $f \in X^*$, $\|f\| \geq \|h\|$, $\|f\| > \|g\|$, whenever $f = g + h$, with $g \in G$, $h \in Y^{\perp}$, $h \neq 0$. It is immediate that every M-ideal is an HB-subspace and the converse is not true : The space $K(\ell_1, \ell_1)$ of compact linear operators from ℓ_1 to ℓ_1 is an HB-subspace of $L(\ell_1, \ell_1)$, the space of all continuous linear operators from ℓ_1 to ℓ_1, which is not an M-ideal (see [4], [5]).

The aim of this paper is to transpose these notions to a general metric space X (without any linear structure) by appealing to the Lipschitz dual of X, i.e. a Banach space of Lipschitz functions on X. The notions of M-ideal and HB-subspace cannot be extended automatically to a metric linear space by using its linear dual, for two reasons : first, this dual may be trivial, as is the case for the space $L^p[0,1]$, $0<p<1$, and second, there is no norm on the dual of a metric linear space.

Results. Let (X,d) be a metric space, x_0 a fixed point in X and let Y be a subset of X such that $x_0 \in Y$. If X is a metric linear space we take always $x_0 = 0$. A function $f : Y \to R$ is called a Lipschitz function on Y if there exists $K \geq 0$ such that

$$(1) \qquad |f(x) - f(y)| \leq K\, d(x,y) ,$$

for all $x, y \in Y$.

Denote by $Lip_0 Y$ the following set

$$(2) \qquad Lip_0 Y = \{f : Y \to R ,\ f \text{ is a Lipschitz function on } Y ,\ f(x_0) = 0\}.$$

Equiped with the usual operations of addition and multiplication by scalars, $Lip_0 Y$ is a linear space and the application $\|\ \|_Y : Lip_0 Y \to R$, defined by

$$(3) \qquad \|f\|_Y = \sup \{|f(x) - f(y)|/d(x,y) \ : \ x,y \in Y ,\ x \neq y\},$$

is a norm on $Lip_0 Y$. It is easily seen that $\|f\|_Y$ is the smallest of the numbers $K \geq 0$ for which the inequality (1) holds. The space $(Lip_0 Y, \|\ \|_Y)$ is a Banach space (even a dual Banach space, see [9]) and we call it the Lipschitz dual of Y.

For $Y = X$ the space $Lip_0 X$ and the norm $\|\ \|_X$ are defined similarly.

In the following, one supposes always that the subset Y of X contains x_0 - the fixed element of X.

The following Hahn-Banach type extension theorem for Lipschitz functions was proved by Mc Shane [8] (see also [2]) :

THEOREM 1. Let (X,d) be a metric space, x_0 a fixed point in X and let Y be a subset of X such that $x_0 \in Y$. Then every function $f \in Lip_0 Y$ has a norm-preserving extension in $Lip_0 X$, i.e. there exists $F \in Lip_0 X$ such that $F|_Y = f$ and $\|F\|_X = \|f\|_Y$.

In fact, Mc Shane [8], proved this theorem in the case of the space $Lip\ Y$ and $Lip\ X$ of all Lipschitz functions on Y and X, respectively, but the above formulation is more appropriate for our needs. We shall call sometimes briefly any function F, as given in Theorem 1, an extension of f.

In general, the extension of a function $f \in Lip_0 Y$ to X is not unique. The functions

$$(4) \qquad \begin{aligned} F_1(x) &= \inf \{ f(y) + \|f\|_Y\, d(x,y) \ : \ y \in Y\}, \\ F_2(x) &= \sup \{ f(y) - \|f\|_Y\, d(x,y) \ : \ y \in Y\} \end{aligned}$$

are two extensions of f and they are extremal elements of the convex set $E(f;Y)$ of all extensions of f. Every extension F of f verifies the inequalities :

$$(5) \qquad F_2(x) \leq F(x) \leq F_1(x) ,\quad x \in X .$$

Therefore, the function $f \in Lip_0 Y$ has a unique extension in $Lip_0 X$ if and only if $F_1 = F_2$.

DEFINITION 1. The subset Y of X is said to have property (U) if every function $f \in Lip_0 Y$ has a unique extension in $Lip_0 X$.

Necessary and sufficient conditions in order that a subset Y of X have property (U) and relations of this property with the problem of best approximation in $Lip_0 X$ by elements in $Y^{\perp}$ are given in [10].

For $Y \subseteq X$ denote by $Y^{\perp}$ its annihilator in $\mathrm{Lip}_0 X$, i.e.

(6) $$Y^{\perp} = \{ F \in \mathrm{Lip}_0 X \ , \ F|_Y = 0 \} \ .$$

Obviously, $Y^{\perp}$ is a closed subspace of $\mathrm{Lip}_0 X$.

DEFINITION 2. A subset V of a normed space Z is called <u>proximinal</u> if for every $z \in Z$ there exists $v_0 \in V$ such that

(7) $$\| z - v_0 \| = d(z,V) \ ,$$

where $d(z,V) = \inf \{ \| z - v \| : v \in V \}$ denotes the distance from z to V. An element v_0 satisfying (7) is called a <u>best approximation element</u> of z by elements in V. If every $z \in Z$ has a unique best approximation element in V then the set V is called <u>Chebyshevian</u>.

R.R. Phelps [11] obtained some results concerning the relations between the property (U) and the unicity of best approximation in the dual of a normed space. As was shown in [10] similar results hold also in the Lipschitz case :

THEOREM 2. ([10]). <u>A subset Y of a metric space X has property (U) if and only if its annihilator $Y^{\perp}$ is Chebyshevian subspace of</u> $\mathrm{Lip}_0 X$.

The proof of this theorem is based on the following lemma, which will be used in the sequel :

LEMMA 1. ([10]). <u>Let X be a metric space, Y a subset of X and $Y^{\perp}$ the annihilator of Y in $\mathrm{Lip}_0 X$. If $F \in \mathrm{Lip}_0 X$ then</u> $d(F, Y^{\perp}) = \| F|_Y \|_Y$ <u>and an element $g_0 \in Y^{\perp}$ is a best approximation element for F by elements in $Y^{\perp}$ if and only if $g_0 = F - F_0$, where F_0 is a norm preserving extension of $F|_Y$ to X.</u>

The property (U) can be characterized also in terms of some decompositions of the Lipschitz dual $\mathrm{Lip}_0 X$ of X. To give this characterization we need first some definitions and notations.

Let $r : \mathrm{Lip}_0 X \to \mathrm{Lip}_0 Y$ denote the <u>restriction operator</u>, defined by :

(8) $$r(F) = F|_Y \ , \ F \in \mathrm{Lip}_0 X \ ,$$

and let $e : \mathrm{Lip}_0 Y \to \mathcal{P}(\mathrm{Lip}_0 X)$ denote the <u>extension operator</u>, defined by :

(9) $$e(f) = E(f;Y) \ , \ f \in \mathrm{Lip}_0 Y \ ,$$

where $E(f;Y)$ denotes the set of all norm preserving extensions of f to X. Let $w : \mathrm{Lip}_0 X \to \mathcal{P}(\mathrm{Lip}_0 X)$ be the composition of the operators r and e, i.e.

(10) $$w = e \circ r \ .$$

Then, for $F \in \mathrm{Lip}_0 X$, we have $w(F) = e(r(F)) = E(r(F);Y)$ and $r(G) = r(F)$, $\| G \|_X = \| r(F) \|_Y$, for all $G \in E(r(F);Y)$.

In general, the operator w is multi-valued and $w(F)$ is a convex subset of the ball of radius $\| r(F) \|_Y$ and center 0 in $\mathrm{Lip}_0 X$.

We can now state the theorem of characterization of property (U):

THEOREM 3. <u>If Y is a subset of a metric space X then the following assertions are equivalent</u> :

1^0 Y <u>has property</u> (U) ;

2^0 <u>Every function $F \in \mathrm{Lip}_0 X$ can be uniquely written in the form</u>

(11) $$F = H + g \ , \ H \in w(F) \ , \ g \in Y^{\perp} \ ,$$

<u>and</u> $\| F \|_X > \| H \|_X$, <u>whenever</u> $g \neq 0$;

3^0 $\mathcal{G} = \{ H \in \mathrm{Lip}_0 X \ , \ \| H \|_X = \| r(H) \|_Y \}$ <u>is the only subset of</u>

$\mathrm{Lip}_0 X$ such that every $F \in \mathrm{Lip}_0 X$ can be uniquely written in the form $f = H + g$, $H \in \mathcal{G}$, $g \in Y^{\perp}$ and $\|F\|_X > \|H\|_X$ if $g \neq 0$.

Proof. $1^{\circ} \Longrightarrow 2^{\circ}$. If the set Y has property (U) then the extension operator e, defined by (9), is single-valued and so is the operator w defined by (10). For $F \in \mathrm{Lip}_0 X$ the function $w(F) \in \mathrm{Lip}_0 X$ is the only norm preserving extension of $r(F)$ to X, i.e. $w(F)|_Y = r(F)$ and $\|w(F)\|_X = \|r(F)\|_Y$. It follows that $g = F - w(F) \in Y^{\perp}$ and $F = w(F) + g$ is the unique decomposition of F with $g \in Y^{\perp}$. By the definition (3) of Lipschitz norm we have $\|F\|_X \geq \|r(F)\|_Y$. The equality $\|F\|_X = \|r(F)\|_Y$ implies that F is also a norm preserving extension of $r(F)$ and, by the unicity of the extension it follows $F = w(F)$, so that $g = F - w(F) = 0$. Hence $\|F\|_X > \|r(F)\|_Y$ if $g \neq 0$.

$2^{\circ} \Longrightarrow 3^{\circ}$. Let $F \in \mathrm{Lip}_0 X$ and let $F = H + g$ the decomposition of F given in 2°. As $H \in w(F) = e(r(F))$ it follows that $r(F) = r(H)$ and $\|H\|_X = \|r(F)\|_Y$, i.e. $H \in \mathcal{G}$. The condition $\|F\|_X > \|H\|_X$, for $g \neq 0$, follows from the similar condition from 2°.

$3^{\circ} \Longrightarrow 1^{\circ}$. Let $F \in \mathrm{Lip}_0 X$ and let $F = H + g$, $H \in \mathcal{G}$, $g \in Y^{\perp}$ be the unique decomposition of F given in 3°. Then $F - g = H$, $r(H) = r(F)$ and, by Lemma 1, $\|F - g\|_X = \|H\|_X = \|r(H)\|_Y = \|r(F)\|_Y = d(F, Y^{\perp})$, which shows that g is an element of best approximation for F by elements in $Y^{\perp}$. If g_1 is an other element of best approximation for F by elements in $Y^{\perp}$ then, appealing again to Lemma 1, there exists $H_1 \in e(r(F))$ such that $g_1 = F - H_1$ and $\|H_1\|_X = \|F - g_1\|_X = d(F, Y^{\perp}) = \|r(F)\|_Y = \|r(H_1)\|_Y$, which shows that $H_1 \in \mathcal{G}$. Taking into account the unicity assumption in 3°, it follows $g = g_1$. Therefore, $Y^{\perp}$ is a Chebyshevian subspace of $\mathrm{Lip}_0 X$ and, by Theorem 2, the set Y has property (U).

Theorem 3 is completely proved.

DEFINITION 3. A subset Y of a metric space X is called an M-ideal if its annihilator $Y^{\perp}$ has a complement $\mathcal{G}$ in $\mathrm{Lip}_0 X$ such that $\|F\|_X = \|G\|_X + \|H\|_X$, whenever $F = G + H$, with $G \in \mathcal{G}$ and $H \in Y^{\perp} \setminus \{0\}$. The subset Y is said to have property (HB) if $Y^{\perp}$ has a complementary subspace $\mathcal{G}$ of $\mathrm{Lip}_0 X$ such that $\|F\|_X \geq \|H\|_X$, $\|F\|_Y > \|G\|_X$, whenever $F = G + H$, with $G \in \mathcal{G}$ and $H \in Y^{\perp} \setminus \{0\}$, for every function $F \in \mathrm{Lip}_0 X$.

THEOREM 4. If a subset Y of a metric space X has the property (HB) then Y has the property (U).

Proof. Suppose that Y has the property (HB) and has not the property (U). Then there exists a function $f \in \mathrm{Lip}_0 Y$ having two distinct extensions F_1, F_2 $\mathrm{Lip}_0 X$ and the subspace $Y^{\perp}$ has a complementary subspace $\mathcal{G}$ in $\mathrm{Lip}_0 X$ such that the condition in Definition 3 is fulfilled, implying $F_i = G_i + H_i$, with $G_i \in \mathcal{G}$ and $H_i \in Y^{\perp}$ for $i = 1,2$. As $F_1 - F_2 \in Y^{\perp}$ it follows that $G_1 - G_2 = F_1 - F_2 - (H_1 - H_2) \in Y^{\perp}$, hence $G_1 = G_2 = G$ ($\mathcal{G} \cap Y^{\perp} = \{0\}$, as $\mathcal{G}$ and $Y^{\perp}$ are complementary subspaces of $\mathrm{Lip}_0 X$). Therefore $F_1 = G + H_1$ and $F_2 = G + H_2$. Now, if $H_1 \neq 0$ then $\|F_1\|_X > \|G\|_X$ so that $\|f\|_Y = \|F_1\|_X > \|G\|_X \geq \|r(G)\|_Y = \|f\|_Y$. If $H_1 = 0$ then $H_2 \neq H_1 = 0$, hence $G = F_1$ and, by Definition 3, the equality $F_1 = F_2 + H_2$ implies $\|f\|_Y = \|F_1\|_X > \|F_2\|_X = \|f\|_Y$. The obtained contradictions shows that the set Y cannot have the propetty (HB). Theorem 4 is completely proved.

THEOREM 5. If the subset Y of X has the property (HB) and $F \in \mathrm{Lip}_0 X$, $F \neq 0$, then $F \in \mathcal{G}$ if and only if $\|r(F)\|_Y = \|F\|_X$, where $\mathcal{G}$ is the complementary subspace of $Y^{\perp}$ given in Definition 3.

Proof. Let $F \in \mathcal{G}$, $F \neq 0$, and let G be a norm preserving extension of $r(F)$ to X. Let $G = G_1 + H_1$, $G_1 \in \mathcal{G}$, $H_1 \in Y^{\perp}$ be the decomposition of G given by Definition 3. Supposing $H_1 \neq 0$, one obtains the contradiction $\|G\|_X > \|G_1\|_X \geq \|r(G_1)\|_Y = \|r(F)\|_Y = \|G\|_X$. Therefore $H_1 = 0$ and $G = G_1 \in \mathcal{G}$. As $\mathcal{G}$ is a subspace of $\mathrm{Lip}_0 X$ it follows $F - G \in \mathcal{G}$. But $F - G$ is in $Y^{\perp}$ too, because $r(F) = r(G)$, so that $F - G \in \mathcal{G} \cap Y^{\perp} = \{0\}$, i.e. $F = G \in \mathcal{G}$.

Conversely, suppose that $F \in \mathrm{Lip}_0 X$, $F \neq 0$, is such that $\|r(F)\|_Y = \|F\|_X$ (>0). Let $F = G + H$ with $G \in \mathcal{G}$, $H \in Y^{\perp}$. If $H \neq 0$ then $\|F\|_X > \|G\|_X$ and the equality $r(F) = r(G)$ gives the contradiction $\|r(F)\|_Y = \|F\|_X > \|G\|_X = \|r(G)\|_Y = \|r(F)\|_Y$, which shows that $H = 0$ and $F = G \in \mathcal{G}$.

THEOREM 6. <u>If the subset Y of a metric space X has the property (HB) then the subspace $\mathcal{G}$ (given in Definition 3) is isometrically isomorphic to the space $\mathrm{Lip}_0 Y$.</u>

<u>Proof</u>. By Theorem 4 the subset Y has the property (U), so that the restriction r_1 of the restriction operator r to $\mathcal{G}$ is single-valued and linear. By Theorem 5, $\|r_1(G)\|_Y = \|G\|_X$, for all $G \in \mathcal{G}$, showing that r_1 is an isometry.

THEOREM 7. <u>If the subset Y of a metric space X has the property (HB), then the extension operator $e : \mathrm{Lip}_0 Y \to \mathrm{Lip}_0 X$ is linear.</u>

<u>Proof</u>. Let $\tilde{e} : \mathrm{Lip}_0 Y \to \mathcal{G}$ be the inverse of the restriction operator $r_1 = r|_{\mathcal{G}} : \mathcal{G} \longrightarrow \mathrm{Lip}_0 Y$ which, by Theorem 6, is an isometrical isomorphism between $\mathcal{G}$ and $\mathrm{Lip}_0 Y$. Then e is linear and $e = j \circ \tilde{e}$ where $j : \mathcal{G} \longrightarrow \mathrm{Lip}_0 X$, denotes the imbedding operator of $\mathcal{G}$ into $\mathrm{Lip}_0 X$.

REFERENCES

1 ALFSEN, E.M., EFFROSS, E., <u>Structure in real Banach spaces</u>, Ann. of Math. 96 (1972), 98 - 173.

2 CZIPSER, J., GÉHER, L., <u>Extension of Functions satisfying a Lipschitz conditions</u>, Acta Math. Acad. Sci. Hungar 6 (1955), 213 - 220.

3 FAKHOURY, H., <u>Sélections linéaires associées au Théorème de Hahn - Banach</u>, J. of Functional Analysis 11 (1972), 436 - 452.

4 HENNEFELD, J., <u>M - ideals, HB - subspaces and Compact Operators</u>, Indiana Univ. Math. J. 28 (6) (1979), 927 - 934.

5 HENNEFELD, J., <u>A note on M - ideals in B(X)</u>, Proc. Amer. Mat. Soc. 98 (1) (1980), 89 - 92.

6 HOLMES, R.B., SCRANTON, B., WARD, J.D., <u>Approximation from the space of compact operators and other M - ideals</u>, Duke Math. J. 42 (1975), 259 - 269.

7 HOLMES, R.B., <u>Geometric Functional Analysis and its Applications</u>, Springer - Verlag, New York - Heidelberg - Berlin, 1975.

8 Mc SHANE, E.J., <u>Extension of range of functions</u>, Bull. Amer. Math. Soc. 40 (1934), 837 - 842.

9 JOHNSON, J.A., <u>Banach Spaces of Lipschitz Functions and vector - valued Lipschitz Functions</u>, Trans. Amer. Math. Soc. 148 (1970), 147 - 169.

10 MUSTĂŢA, C., <u>Best Approximation and Unique Extension of Lipschitz Functions</u>, J. Approx. Theory 19 (3) (1977), 222 - 230.

11 PHELPS, R.R., <u>Uniqueness of Hahn - Banach Extension and Unique Best Approximation</u>, Trans. Amer. Math. Soc. 95 (1960), 238 - 255.

12 OJA, E., <u>On the uniqueness of the norm preserving extension of linear functional in the Hahn - Banach Theorem</u>, Proc. Acad. Science Estonian SSR 33 (4) (1984), 422 - 433 (Russian)

Institutul de Calcul
Oficiul Poştal 1
C. P. 68
3400 Cluj-Napoca
Romania

This note is in final form and no version of it is or will be submitted for publication elsewhere.