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AN APPLICATION OF A THEOREM OF Mc SHANE

by
Costică Mustăţa

Let (X,d) be a metric space and let Y be a nonvoid subset of X. A function $f: Y \to R$ is called Lipschitz on Y if there exists a number $L \geq 0$ such that

(1) $$|f(x) - f(y)| \leq L\, d(x,y) \quad ,$$

for all $x,y \in Y$. A number L satisfying (1) is called a Lipschitz constant for f on Y. Denote by Lip Y the set of all real-valued Lipschitz function on Y and by $K_Y(f)$ the set of all Lipschitz constant for the function f on Y. It is easy to check that $K_Y(f) = [L_f\ , \infty\ )$, where

(2) $$L_f = \sup \left\{ |f(x) - f(y)| / d(x,y) \ :\ x,y \in Y\ ,\ x \neq y \right\} .$$

If $Y = X$ then Lip X denotes the set of all real-valued Lipschitz functions on X.

Mc Shane [6] proved the following extension theorem for Lipschitz functions :

THEOREM 1. Let (X,d) be a metric space and let Y be a nonvoid subset of X. If $f \in$ Lip Y and $L \geq 0$ is a Lipschitz constant for f on Y then there exists $F \in$ Lip X such that $F|_Y = f$ and $|F(x) - F(y)| \leq L\, d(x,y)$, for all $x,y \in X$.

A function F , as given in Theorem 1 , is called an <u>extension of f with preservation of Lipschitz constant</u> , or, briefly , a <u>Lipschitz extension</u> of f .

If X is a compact metric space denote by $C(X)$ the Banach space of all real-valued continuous functions on X . As usually, we consider the uniform norm on $C(X)$, i.e.

(3) $$\|h\|_X = \sup\{|h(x)| : x\in X\} , \quad h\in C(X) .$$

Then $\mathrm{Lip}\,X$ is a subspace of $C(X)$ which is dense in $C(X)$ with respect to the uniform topology (i.e. the topology generated by the norm (2)) (see [1]).

Let now $f\in C(X)$ and suppose that f is Lipschitz on X . For $h\in C(X)$ consider the problem of the evaluation of the norm

(4) $$\|f - h\|_X ,$$

knowing a Lipschitz constant $L\in K_X(f)$ of f and the values of f on a subset Y of X .

Obviously, $f|_Y\in \mathrm{Lip}\,Y$ and $L_f|_Y\leq L_Y$, implying $K_X(f)\subseteq K_Y(f_{|Y})$. By Theorem 1 if $L\in K_Y(f|_Y)$ then $f|_Y$ has a Lipschitz extension F on X , such that L is a Lipschitz constant for F , too (i.e. $L\in K_X(F)$).

It was shown in [4], [6] that for $L\in K_Y(f|_Y)$, the following functions :

$$F_s(x) = \inf\{f(y) + L\,d(x,y) : y\in Y\} , \quad x\in X ,$$

(5) and

$$F_i(x) = \sup\{f(y) - L\,d(x,y) : y\in Y\} , \quad x\in X ,$$

are Lipschitz extensions of $f|_Y$ having L as Lipschitz constant.

For $L\in K_Y(f|_Y)$, denote by

(6) $$E_L(f|_Y) = \{F\in \mathrm{Lip}\,X ,\ F|_Y = f|_Y \text{ and } L\in K_X(F)\} ,$$

the set of all Lipschitz extensions admitting L as Lipschitz constant.

Concerning the extensions given by the formulae (5) one can prove the following result :

LEMMA 1. <u>Let X be a metric space , Y a nonvoid subset of X , $f\in \mathrm{Lip}\,X$ and $L\in K_X(f)$. Then every $F\in E_L(f|_Y)$ verifies the inequalities</u> :

(7) $$F_i(x) \leq F(x) \leq F_s(x) , \quad x\in X .$$

<u>Proof</u>. If L is a Lipschitz constant for f on X then L is also a Lipschitz constant for $f|_Y$ on Y , so that the functions F_i and F_s are well defined .

Now, let $F\in E_L(f|_Y)$ and suppose that there exists $u\in X\setminus Y$ such that $F_s(u) < F(u)$. Then , by the definition (5) of F_s , it follows the existence of an element $y\in Y$ such that

$$f(y) + L\,d(x,u) < F(u) .$$

But F and f agree on Y , so that $F(y) = f(y)$ and

$$F(u) - F(y) > L\,d(x,u) ,$$

contradicting the fact that L is a Lipschitz constant for F . The inequality $F_i\leq F$ is proved in a similar way .

<u>Remark</u> 1. If $L\in K_X(f)$ then $f\in E_L(f|_Y)$ so that, by Lemma 1,

(8) $$F_i(x) \leq f(x) \leq F_s(x) , \quad x\in X .$$

The following theorem shows that the set of Lipschitz extensions has some compactness properties :

THEOREM 2. <u>If X is a compact metric space, Y a nonvoid closed subset of X , $f\in \mathrm{Lip}\,X$ and $L\in K_X(f)$, then the set</u>

$E_L(f|_Y)$ is compact with respect to the uniform topology of $C(X)$.

We intend to apply Arzela - Ascoli compactness theorem and we shall divide the proof into three lemmas.

LEMMA 2. The set $E_L(f|_Y)$ is uniformly bounded in $C(X)$.

Proof. By Lemma 1

$$\|F\|_X \le \max\{\|F_1\|_X , \|F_2\|_X\} ,$$

for all $F \in E_L(f|_Y)$.

LEMMA 3. The set $E_L(f|_Y)$ is a (uniformly) echicontinuous subset of $C(X)$.

Proof. For $\varepsilon > 0$ let $\delta = \varepsilon/(L+1)$. Then

$$|F(x) - F(y)| \le L\, d(x,y) < \frac{L.\varepsilon}{L+1} < \varepsilon ,$$

for all $x,y \in X$ with $d(x,y) < \delta$ and all $F \in E_L(f|_Y)$.

LEMMA 4. The set $E_L(f|_Y)$ is closed in $C(X)$, with respect to the uniform topology of $C(X)$.

Proof. Suppose that $(F_n)_{n\ge 1}$ is a sequence in $E_L(f|_Y)$ converging uniformly to a function $F \in C(X)$. We have to show that F is also in $E_L(f|_Y)$. But

$$|F(x) - F(y)| \le |F(x) - F_n(x)| + |F_n(x) - F_n(y)| +$$
$$+ |F_n(y) - F(y)| \le 2\|F - F_n\| + L.d(x,y) ,$$

for all $x,y \in X$ and all $n \in N$. Leting $n \to \infty$, one obtains

$$|F(x) - F(y)| \le L.d(x,y) ,$$

for all $x,y \in X$. As $F_n(y) = f(y)$, for all $n \in N$ and all $y \in Y$, and $F_n(y) \to F(y)$, for $n \to \infty$, it follows $F(y) = f(y)$, for all $y \in Y$, i.e. $F \in E_L(f|_Y)$.

Proof of Theorem 2. Taking into account Lemmas 2,3 and 4, the theorem follows from the Arzela - Ascoli compactness theorem.

COROLLARY 1. Let X be a compact metric space, Y a nonvoid closed subset of X, $f \in Lip\, X$, $L \in K_X(f)$ and $h \in C(X)$. Then there exist at least two functions $F_1, F_2 \in E_L(f|_Y)$ such that :

$$\|F_1 - h\|_X \le \|F - h\|_X \le \|F_2 - h\|_X \quad (9)$$

for all $F \in E_L(f|_Y)$. In particular

$$\|F_1 - h\|_X \le \|f - h\|_X \le \|F_2 - h\|_X . \quad (10)$$

Proof. The Corollary follows by the compactity of the set $E_L(f|_Y)$ and the continuity of the function $d(h,F) = \|h - F\|_X$, $F \in E_L(f|_Y)$.

COROLLARY 2. Let X be a compact metric space, Y a nonvoid closed subset of X, $f \in Lip\, X$, $L \in K_X(f)$ and $h \in C(X)$. Then

$$\min\{\|h - F_1\|_X , \|h - F_2\|_X\} \le \|h - f\|_X \le \max\{\|h - F_1\|_X , \|h - F_2\|_X\} \quad (11)$$

where the functions F_1 and F_2 are defined by (5).

Proof. Corollary 2 follows from the inequalities (8) and Corollary 1.

COROLLARY 3. Let X be a compact metric space, Y a nonvoid closed subset of X, $f \in Lip\, X$, $L \in K_X(f)$ and $h \in C(X)$. Then there exist at least two function $F_1, F_2 \in E_L(f|_Y)$ such that

$$\|h - F_1\|_X = \min\{\|h - F_1\|_X , \|h - F_2\|_X\}$$
(12) and
$$\|h - F_2\|_X = \max\{\|h - F_1\|_X , \|h - F_2\|_X\}$$

Proof. By Corollary 1, there exist two functions $F_1, F_2 \in E_L(f|_Y)$ verifying the inequalities (9), for all $F \in E_L(f|_Y)$. But then $\|F_1 - h\|_X \le \|F_2 - h\|_X$ and $\|F_s - h\|_X \le \|F_2 - h\|_X$, so that

$$\max\left\{\|h - F_i\|_X ,\ \|h - F_s\|_X\right\} \le \|F_2 - h\|_X .$$

On the other hand, by (8), it follows

$$\|F_2 - h\|_X \le \max\left\{\|F_i - h\|_X ,\ \|F_s - h\|_X\right\} ,$$

which shows that the second equality in (12) is true. The first one is proved similarly.

Remark 2. Corollary 3 shows that the evaluations (11) are exact in the class $E_L(f|_Y)$.

A δ- net in a metric space (X,d) is a subset Z of X such that for every $x \in X$ there exists $z \in Z$ such that $d(x,z) < \delta$.

THEOREM 3. If X is a compact metric space, Y is a δ-net in X, $f \in \mathrm{Lip}\, X$ and $L \in K_Y(f)$, then

$$(13) \qquad \left| \|F_s - h\|_X - \|F_i - h\|_X \right| \le 2L\delta .$$

Proof. Let $x_0 \in X$. As Y is a δ-net in X there exists $y_0 \in Y$ such that $d(x_0,y_0) < \delta$, so that

$$F_i(x_0) - F_s(x_0) = \inf\left\{f(y) + L\, d(x_0,y) \ :\ y \in Y\right\} - \sup\left\{f(y) - L\, d(x_0,y) \ :\ y \in Y\right\} =$$

$$= \inf\left\{f(y) + L\, d(x_0,y) \ :\ y \in Y\right\} + \inf\left\{-f(y) + L\, d(x_0,y) \ :\ y \in Y\right\}$$

$$\le f(y_0) + L\, d(x_0,y_0) - f(y_0) + L\, d(x_0,y_0) = 2L\, d(x_0,y_0) \le 2L\delta .$$

As x_0 was arbitrarily choosen in X, it follows

$$2L\delta \ge \|F_s - F_i\|_X \ge \left| \|F_s - h\|_X - \|F_i - h\|_X \right| .$$

EXAMPLES.

1^0 Let $X = [a,b]$, $d(x,y) = |x - y|$ and let Y be a division Δ_X of the interval $[a,b]$, $\Delta_X : a = x_0 < x_1 < \dots < x_n = b$. Let f be a Lipschitz function on $[a,b]$ with Lipschitz constant L and let $f_i = f(x_i)$, $i = 0,1,\dots,n$. It was shown in [5] that there exists a unique interpolation cubic spline s having the form:

$$s(x) = \frac{M_i - M_{i-1}}{6\, h_i}(x - x_{i-1})^3 + \frac{M_{i-1}}{2}(x - x_{i-1})^2 + m_i(x - x_{i-1}) + f_{i-1} ,$$

for $x \in [x_{i-1}, x_i]$, $i = 1,2,\dots,n$ and such that $s(x_i) = f_i$, $s'(x_i) = m_i$, $s''(x_i) = M_i$, $i = 1,2,\dots,n$; $m_0 = p$, $M_0 = q$.

The evaluation (11) give:

$$(14) \qquad \min\left\{\|s - F_i\|_X ,\ \|s - F_s\|_X\right\} \le \|s - f\| \le \max\left\{\|s - F_i\|_X ,\ \|s - F_s\|_X\right\}.$$

By Corollary 3 the evaluations (14) are exact in the class of Lipschitz functions on $[a,b]$ admitting a Lipschitz constant L and taking the values f_i on the knots x_i, $i = 0,1,\dots,n$.

All of the terms involved in the inequalities (14) can be efectively calculated, as F_i and F_s are polygonal lines on every subinterval $[x_{i-1}, x_i]$ and $s - F_i$, $s - F_s$ are polynomials of degree at most 3 on every such subinterval (see [5]).

2^0 Let $X = [0,1]$ and for given $a,b \in \mathbb{R}$ and $K \ge 0$ consider the following set of Lipschitz functions:

$$\mathcal{F} = \left\{ f \in \mathrm{Lip}[0,1] \ , \ f(0) = a \ , \ f(1) = b \ , \ L_f \le K \right\}$$

where L_f is the smallest Lipschitz constant of f given by (2).

Let $B_n : \mathcal{F} \longrightarrow C[0,1]$ be the Bernstein polynomial (restricted to $\mathcal{F}$) defined by

$$B_n(f)(x) = \sum_{k=0}^{n} C_n^k \, f\left(\frac{k}{n}\right) (1-x)^{n-k} \cdot x^k \ , \quad x \in [0,1] \ , \ f \in \mathcal{F} \ .$$

B.M. Brown, D. Elliot and D.F. Paget [3] proved that $B_n(\mathcal{F}) \subseteq \mathcal{F}$, for all $n = 1,2, \ldots$.

Taking into account the formulae (5) and applying Lemma 1 it follows that

$$F_i(x) = \max \left\{ a - L_f x \ , \ b - L_f (1-x) \right\} \le$$

$$\le B_n(f)(x) \le \min \left\{ a + L_f x \ , \ b + L_f (1-x) \right\} = F_s(x)$$

for $x \in [0,1]$, implying

$$\frac{a+b}{2} - \frac{L_f}{2} \le B_n(f)(x) \le \frac{a+b}{2} + \frac{L_f}{2}$$

for all $x \in [0,1]$, and consequently

$$\frac{a+b}{2} - \frac{K}{2} \le B_n(f)(x) \le \frac{a+b}{2} + \frac{K}{2} \ , \quad x \in [0,1]$$

(because $L_f \le K$, for all $f \in \mathcal{F}$).

Therefore

$$\| B_n(f) \| \le \max \left\{ \frac{|a+b+K|}{2}, \frac{|a+b-K|}{2} \right\} ,$$

for all $f \in \mathcal{F}$.

But, the set $\mathcal{F}$ contains the functions F_i and F_s and consequently it contains all the Bernstein polynomials , $B_n(F_i)$ and $B_n(F_s)$, $n = 1,2, \ldots$.

The function F_i being convex, the polynomials $B_n(F_i)$ are convex too , and

$$B_n(F_i)(x) \ge F_i(x) \ , \quad x \in [0,1] \ .$$

Analogously , F_s is concave , the polynomials $B_n(F_s)$ are concave too , and

$$B_n(F_s)(x) \le F_s(x) \ , \quad x \in [0,1] \ ,$$

(see e.g. [8] , [9]).

Because

$$\lim_{n \to \infty} \| B_n(F_i) - F_i \| = 0$$

and

$$\lim_{n \to \infty} \| B_n(F_s) - F_s \| = 0$$

$$\max_{x \in [0,1]} F_s(x) = \frac{a+b}{2} + \frac{K}{2}$$

$$\min_{x \in [0,1]} F_i(x) = \frac{a+b}{2} - \frac{K}{2}$$

it follows the formulae :

$$\lim_{n \to \infty} \sup_{f \in \mathcal{F}} \| B_n(f) \| = \max \left\{ \frac{|a+b+K|}{2} , \frac{|a+b-K|}{2} \right\} .$$

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Institutul de Calcul
Oficiul Poştal 1
C.P. 68
3400 Cluj-Napoca, Romania

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