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AN ABSTRACT KOROVKIN TYPE THEOREM AND APPLICATIONS

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REZUMAT. — O teoremă abstractă de tip Korovkin şi aplicaţii. În lucrare se obţin teoreme de tip Korovkin pentru spaţiul $C(X)$, unde X este un spaţiu metric compact (Teoremele 2 şi 3). Se aplică rezultatele obţinute pentru cazul cînd X este o submulţime compactă a unui spaţiu prehilbertian şi se dau delimitări ale diferenţei $\| B_n(f) - f \|$, unde B_n este operatorul lui Bernstein-Lototsky-Schnabl.

The well known Korovkin's theorem (see e.g. [1]) asserts that if $(L_n)_{n\geq 1}$ is a sequence of positive linear operators, acting from $C[a, b]$ to $C[a, b]$ and such that $(L_n(e_k))_{n\geq 1}$ converges uniformly to e_k, for $k = 0, 1, 2$, where $e_k(t) = t^k$, $t \in [a, b]$, then the sequence $(L_n(f))_{n\geq 1}$ converges uniformly to f, for every $f \in C[a, b]$.

This theorem was extended and generalized in many directions. One direction is to replace the above mentioned system of test functions by other systems of functions, which led to the theory of so called Korovkin subspaces. Another direction is to consider functions defined on more general compact spaces than the interval $[a, b]$, first of all on compact subsets of $\mathbf{R}^m$.

The aim of this paper is to give Korovkin type theorems for the space $C(X)$, where X is a compact metric space. As application, supposing that X is a compact convex subset of a Hilbert space, one obtains evaluations of the order of approximation by the Bernstein — Lototsky — Schnabl operator, similar to those given in [4].

If (X, d) is a compact metric space, denote by $C(X) = C(X, \mathbf{R})$ the space of all real-valued continuous functions defined on X and by Lip (X) the subspace of $C(X)$ formed of all real-valued Lipschitz functions defined on X. Equiped, as usually, with the uniform norm $\|f\| = \sup \{|f(x)| : x \in X\}$, $f \in C(X)$, the space $C(X)$ is a Banach space.

Our first result is a density theorem:

THEOREM 1. *The subspace* Lip (X) *is dense in* $C(X)$, *with respect to the uniform norm.*

Proof. The assertion of the theorem will follow from the Stone—Weierstrass theorem if we shall show that Lip (X) is a subalgebra of $C(X)$ containing the constant functions and separating the points of X.

If $f, g \in$ Lip (X) then

$$|(f \cdot g)(x) - (f \cdot g)(y)| \leq |f(x)| \cdot |g(x) - g(y)| + |g(y)| \cdot |f(x) - f(y)| \leq \leq (\|f\| \cdot K_g + \|g\| \cdot K_f) \cdot d(x, y),$$

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for all $x, y \in X$, where K_f and K_g are Lipschitz constants for f and g, respectively. Therefore $f \cdot g \in \mathrm{Lip}\,(X)$ and since Lip (X) is a subspace of $C(X)$ it follows that Lip (X) is a subalgebra of the algebra $C(X)$.

As the constant functions are obviously in Lip (X) to finish the proof we have only to show that the algebra Lip (X) separates the points of X. For $x, y \in X$, $x \neq y$ let $f: X \to \mathbf{R}$ be defined by $f(z) = d(z, y)$, $z \in X$. Then

$$|f(z_1) - f(z_2)| = |d(z_1, y) - d(z_2, y)| \leqslant d(z_1, z_2), \quad z_1, z_2 \in X,$$

which shows that f is in Lip (X), $f(y) = d(y, y) = 0$ and $f(x) = d(x, y) > 0$. Theorem is proved.

A *Markov operator* L on $C(X)$ is a positive linear operator $L: C(X) \to C(X)$ such that $L(e_0) = e_0$, where $e_0(x) = 1$, $x \in X$, i.e. L preserves the constant functions.

In the following we shall need the following simple lemma:

LEMMA 1. *If L is a Markov operator acting on $C(X)$ then* $\|L\| = 1$.

Proof. Taking into account the positivity of L and applying L to the inequalities $-\|f\| \cdot e_0 \leqslant f \leqslant \|f\| \cdot e_0$, we obtain $-\|f\| \cdot e_0 \leqslant L(f) \leqslant \|f\| \cdot e_0$, so that $\|L(f)\| \leqslant \|f\|$, for all $f \in C(X)$. As $\|L(e_0)\| = \|e_0\| = 1$ it follows $\|L\| = 1$. Lemma is proved.

If $(L_n)_{n \geqslant 1}$ is a sequence of Markov operators acting on $C(X)$, let

$$\alpha_n(x) = L_n(d(., x)\,;\, x), \qquad \beta_n(x) = L_n(d^2(., x)\,;\, x), \tag{1}$$

for all $x \in X$ and $n = 1, 2, \ldots$.

Our first Korovkin type theorem is the following:

THEOREM 2. *Let $(L_n)_{n \geqslant 1}$ be a sequence of Markov operators acting on $C(X)$. If $(\alpha_n(x))_{n \geqslant 1}$ converges to zero, uniformly with respect to $x \in X$, then $(L_n(f))_{n \geqslant 1}$ converges uniformly to f, for all $f \in C(X)$.*

Proof. Let $f \in \mathrm{Lip}\,(X)$ and let $K_f \geqslant 0$ be a Lipschitz constant for f, i.e.

$$|f(x) - f(y)| \leqslant K_f \cdot d(x, y),$$

for all $x, y \in X$. This inequality can be rewritten in the form:

$$-K_f \cdot d(., x) \leqslant f(\cdot) - f(x) . e_0 \leqslant K_f \cdot d(., x),$$

for all $x \in X$. Applying to these inequalities the operator L_n and taking into account the positivity of L_n and the notations (1), one obtains:

$$-K_f \cdot \alpha_n(x) \leqslant L_n(f\,;\, x) - f(x) \leqslant K_f \cdot \alpha_n(x)$$

for all $x \in X$, or equivalently,

$$|L_n(f\,;\, x) - f(x)| \leqslant K_f |\alpha_n(x)|, \tag{2}$$

for all $x \in X$. Since, by the hypothesis of the theorem the sequence $(\alpha_n(x))_{n \geqslant 1}$ tends to zero, uniformly for $x \in X$, the inequality (2) implies that $(L_n(f))_{n \geqslant 1}$ tends uniformly to f.

By Theorem 1 the space Lip (X) is dense in $C(X)$ with respect to the uniform norm on $C(X)$ and by Lemma 1, $\|L_n\| = 1$, $n = 1, 2, \ldots$ so that by the

Banach–Steinhaus theorem, the sequence $(L_n(f))_{n\geq 1}$ tends uniformly to f, for all $f \in C(X)$. The theorem is proved.

THEOREM 3. *Let $(L_n)_{n\geq 1}$ be a sequence of Markov operators acting on $C(X)$. If $\beta_n(x)$ is defined by* (1) *and the sequence $(\beta_n(x))_{n\geq 1}$ tends to zero, uniformly with respect to $x \in X$, then the sequence $(L_n(f))_{n\geq 1}$ tends uniformly to f, for all $f \in C(X)$.*

If $f \in$ Lip (X) then, furthermore

$$\| L_n(f) - f \| \leqslant K_f \cdot \sqrt{\| \beta_n \|}, \tag{3}$$

for all $n = 1, 2, \ldots$.

Proof. We have $L_n(e_0) = e_0$ and

$$0 \leqslant L_n((t \cdot f - e_0)^2) = t^2 L_n(f^2) - 2t \cdot L_n(f) + e_0 \quad .$$

for all $t \in R$, implying

$$[L_n(f)]^2 \leqslant L_n(f^2),$$

for all $f \in C(X)$. Applying this inequality to the function $f = d(.,x)$, one obtains:

$$(L_n(d(.,x)\,;x))^2 \leqslant L_n(d^2(.,x)\,;x), \tag{4}$$

for all $x \in X$. Taking into account the notations (1), it follows that the sequence $(\alpha_n(x))_{n\geq 1}$ converges to zero, uniformly for $x \in X$, provided that the sequence $(\beta_n(x))_{n\geq 1}$ converges to zero uniformly for $x \in X$. The first assertion of the theorem follows now from Theorem 2.

The inequality (2), obtained in the proof of Theorem 2, implies

$$\| L_n(f) - f \| \leqslant K_f \cdot \| \alpha_n \|,$$

for all $f \in$ Lip (X). By the inequality (4), $\| \alpha_n \| \leqslant \sqrt{\| \beta_n \|}$, so that

$$\| L_n(f) - f \| \leqslant K_f \cdot \sqrt{\| \beta_n \|},$$

which ends the proof of the theorem.

Now, let H be a real pre-Hilbert space with inner product $\langle .,. \rangle$. For $t \in H$ fixed let the function $e_t : H \to \mathbf{R}$ be defined by $e_t(x) = \langle x, t \rangle$, $x \in H$, and let $e : H \to \mathbf{R}$ be defined by $e(x) = \langle x, x \rangle = \| x \|^2$, $x \in H$.

THEOREM 4. *Let X be a compact subset of the pre-Hilbert space H and let $(L_n)_{n\geq 1}$ be a sequence of Markov operators acting on $C(X)$. If $(L_n(e))_{n\geq 1}$ converges uniformly to e and the sequence $(L_n(e_x\,;x))_{n\geq 1}$ converges to $e(x)$, uniformly for $x \in X$, then the sequence $(L_n(f))_{n\geq 1}$ converges uniformly to f, for all $f \in C(X)$.*

Proof. We have

$$\| t - x \|^2 = e(t) - 2e_x(t) + e(x).$$

Considering x fixed and t variable, applying the operator L_n to this equality and evaluating at the point $t = x$, one obtains:

$$\begin{aligned}\beta_n(x) &= L_n(\| \cdot - x \|^2\,;x) = L_n(e\,;x) - 2L_n(e_x\,;x) + e(x) = \\ &= L_n(e\,;x) - e(x) - 2[L_n(e_x\,;x) - e(x)].\end{aligned} \tag{5}$$

Taking into account the hypotheses of the theorem it follows that the sequence $(\beta_n(x))_{n\geq 1}$ converges to zero uniformly for $x \in X$, and Theorem 4 follows from Theorem 3.

Remark. If $f \in \mathrm{Lip}\,(X)$ then

$$\| L_n(f) - f \| \leqslant K_f \sqrt{\| a_n - 2b_n \|}, \tag{6}$$

where $a_n(x) = L_n(e\,;\,x) - e(x)$ and $b_n(x) = L_n(e_x\,;\,x) - e(x)$, for $x \in X$ and $n = 1, 2, \ldots$.

COROLLARY 1. (Korovkin's theorem). *If $(L_n)_{n\geq 1}$ is a sequence of Markov operators acting on $C[a, b]$ such that $L_n(e_1) \xrightarrow{u} e_1$, $L_n(e_2) \xrightarrow{u} e_2$, where $e_1(x) = x$ and $e_2(x) = x^2$, $x \in [a, b]$, then $(L_n(f))_{n\geq 1}$ converges uniformly to f, for all $f \in C[a, b]$.*

Proof. In Theorem 4 take $H = \mathbf{R}$, $X = [a, b]$ and the inner product be the usual multiplication in $\mathbf{R}$, $\langle x, y\rangle = x \cdot y$. Then $e(x) = x^2 = e_2(x)$, $e_t(x) = t \cdot x = t \cdot e_1(x)$ and $L_n(e_t\,;\,x) = t \cdot L_n(e_1\,;\,x)$. By hypothesis $L_n(e) = L_n(e_2) \xrightarrow{u} e_2 = e$. The corrollary will follow from Theorem 4 if we show that $L_n(e_x\,;\,x) \to x^2$ uniformly for $x \in [a, b]$. By hypothesis $L_n(e_1) \xrightarrow{u} e_1$, so that if $\varepsilon > 0$ is given, there exists $n_\varepsilon \in N$ such that $| L_n(e_1\,;\,x) - x | < \varepsilon/M$ for all $n \geqslant n_\varepsilon$ and all $x \in [a, b]$, where $M = \max\,(|a|, |b|)$. Consequently $| L_n(e_2\,;\,x) - tx | = |t| \cdot | L_n(e_1\,;\,x) - x | < \varepsilon$, for all $n \geqslant n_\varepsilon$ and all x and t in $[a, b]$. In particular for $t = x$, one obtains $|L_n(e_x;\,x) - x^2| < \varepsilon$, for all $n \geqslant n_\varepsilon$ and all $x \in [a, b]$, which shows that the sequence $(L_n(e_x\,;\,x))_{n\geq 1}$ converges to $e(x)$, uniformly for $x \in [a, b]$. The corollary is proved.

If $L_n = B_n$, where B_n denotes the Bernstein polynomial operator defined by

$$B_n(f\,;\,x) = \sum_{k=0}^{n} \binom{n}{k} (1 - x)^{n-k} x^k f\left(\frac{k}{n}\right), \quad x \in [0, 1], f \in C[0, 1],$$

then

$$B_n(e_1\,;\,x) = e_1(x) \text{ and } B_n(e_2\,;\,x) = e_2(x) + \frac{e_1(x) - e_2(x)}{2}.$$

The delimitation (6) gives

$$\| B_n(f) - f \| \leqslant K_f \cdot \frac{1}{2 \cdot \sqrt{n}},$$

for all $f \in \mathrm{Lip}\,[0, 1]$.

Applications. 1°. In the Hilbert space $\mathbf{R}^m$ consider a compact convex set X with nonvoid interior. For $f \in C^1(X)$ (the space of all real-valued continuously differentiable functions on X) and $u \in \mathbf{R}^m$, denote by $\nabla f(u)$ the gradient vector of f at the point u, i.e.

$$\nabla f(u) = \left(\frac{\partial f}{\partial x_1}(u), \ldots, \frac{\partial f}{\partial x_m}(u)\right).$$

LEMMA 2. *If $f \in C^1(X)$ then $f \in \mathrm{Lip}\,(X)$ and* $K_f = \max\limits_{u \in X} \| \nabla f(u) \|$.

Proof. Let $x, y \in X$, $x \neq y$. The mean value theorem implies the existence of a point $u \in X$ (which is an internal point of the segment joining x and y) such that

$$f(x) - f(y) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(u) \cdot (x_i - y_i) = \langle \nabla f(u),\ x - y \rangle.$$

Applying now the Schwarz inequality, one obtains

$$|f(x) - f(y)| = \|\nabla f(u)\| \cdot \|x - y\| \leqslant (\max_{u \in X} \|\nabla f(u)\|) \cdot \|x - y\|.$$

COROLLARY 2. *If $(L_n)_{n \geqslant 1}$ is a sequence of Markov operators acting on $C(X)$, where X is a compact convex subset of $\mathbf{R}^m$ with non-void interior, then*

$$|L_n(f; x) - f(x)| \leqslant \max_{u \in X} \|\nabla f(u)\| \cdot \sqrt{L_n(\| \cdot - x\|^2; x)}, \tag{7}$$

for all $f \in C^1(X)$.

Proof. By Lemma 2, the inequality (7) is a consequence of the inequality (3) (see also (1) for the definition of β_n).

2°. *The Bernstein—Lototsky—Schnabl operator.* If X is a compact space, S is a subspace of $C(X)$ such that $e_0 \in S$ (remind that $e_0(x) = 1$, $x \in X$), L is a Markov operator on $C(X)$ and x is a point in X then a Radon probability measure ν_x on X is called an $L(S)$ — *representing measure* for x if

$$L(f; x) = \int_X f \, d\nu_x,$$

for all $f \in S$.

Suppose from now on that X is a compact convex subset of a pre-Hilbert space H and let $A(X)$ be the space of all real-valued continuous affine functions defined on X. Let $V = (V_n)_{n \geqslant 1}$ be a sequence of Markov operators on $C(X)$ and let $M(V) = \{\nu_{x,n} : n \geqslant 1,\ x \in X\}$ be a set of Radon probability measures on X such that $\nu_{x,n}$ is an $V_n(A(X))$ — representing measure for x, for all $x \in X$ and $n = 1, 2, \ldots$. Suppose further that the family $M(V)$ is such that the functions $E_n : X \to \mathbf{R}$ defined by $E_n(x) = \nu_{x,n}(e)$, $x \in X$, are continuous for all $n = 1, 2, \ldots$. Let $P = (p_{n,j})_{n,j \geqslant 1}$ be a lower triangular stochastic matrix i.e. an infinite matrix such that $p_{n,j} \geqslant 0$ for all $n, j \geqslant 1$,

$$\sum_{j=1}^{n} p_{n,j} = 1 \quad \text{and} \quad p_{n,j} = 0$$

for all $j > n$. If $\rho = (\rho_n)_{n \geqslant 1}$ is a sequence of continuous functions $\rho_n : X \to [0, 1]$, $n = 1, 2, \ldots$, define

$$\nu_{x,n,\rho}^{(V)} = \rho_n(x)\, \nu_{x,n} + (1 - \rho_n(x))\, \varepsilon_x \circ V_n,$$

where ε_t denotes the Dirac measure on X centered at $t \in X$. Let also $\pi_{n,P} : X^n \to X$ be defined by

$$\pi_{n,P}(x_1, x_2, \ldots, x_n) = \sum_{j=1}^{n} p_{n,j} \cdot x_j,$$

for $(x_1, x_2, \ldots, x_n) \in X^n$.

The Bernstein–Lototski–Schnabl operator with respect to $M(V)$, P and ρ is defined by

$$B_n(f; x) = \int_{X^n} f \circ \pi_{n,P} \, d \bigotimes_{1 \le j \le n} \nu_{x,j}^{(V)},$$

for all $x \in X$ and all $f \in C(X)$. It follows that B_n is a Markov operator on $C(X)$ and straightforward calculations (see [5]) show that

$$B_n(e_y; x) = \sum_{j=1}^{n} p_{n,j} \cdot \rho_j(x) \langle x, y \rangle + \sum_{j=1}^{n} p_{n,j} \cdot (1 - \rho_j(x)) \langle x, y \rangle = \langle x, y \rangle = e_y(x), \tag{8}$$

for all $x, y \in X$ and

$$B_n(e; x) = \sum_{j=1}^{n} p_{n,j}^2 [\rho_j(x) \nu_{x,j}(e) + (1 - \rho_j(x)) \cdot V_j(e; x)] + \left(1 - \sum_{j=1}^{n} p_{n,j}^2\right) e(x), \tag{9}$$

for all $x \in X$. Here, the functions $e_y, e : X \to \mathbf{R}$ are defined as above by $e_y(x) = \langle x, y \rangle$ and $e(x) = \langle x, x \rangle$, $x \in X$.

As a consequence of the general convergence theorems one obtains the following result:

THEOREM 5. *If*

$$\lim_{n \to \infty} \sum_{j=1}^{n} p_{n,j}^2 \rho_j(x) \cdot (\nu_{x,j}(e) - V_j(e; x)) + \sum_{j=1}^{n} p_{n,j}^2 (V_j(e; x) - e(x)) = 0, \tag{10}$$

uniformly for $x \in X$, *then* $(B_n(f))_{n \ge 1}$ *converges uniformly to* f, *for all* $f \in C(X)$. *If* $f \in \mathrm{Lip}(X)$ *then furthermore:*

$$\| B_n(f) - f \| \le K_f \left\| \sum_{j=1}^{n} p_{n,j}^2 \rho_j E_j - V_j(e) + \sum_{j=1}^{n} p_{n,j}^2 (V_j(e) - e) \right\|^{1/2}, \tag{11}$$

where $E_j(x) = \nu_{x,j}(e)$, $x \in X$.

Proof. The convergence result follows from Theorem 4. Indeed, by (9),

$$B_n(e; x) - e(x) = \sum_{j=1}^{n} p_{n,j}^2 [\rho_j(x) \nu_{x,j}(e) - V_j(e; x)] + \sum_{j=1}^{n} p_{n,j}^2 (V_j(e; x) - e(x)),$$

and the condition (10) of the theorem implies that $(B_n(e; x))_{n \ge 1}$ converges to $e(x)$, uniformly for $x \in X$. The equality (8) gives for $y = x$, $B_n(e_x; x) = \langle x, x \rangle = e(x)$, for all $x \in X$ and $n = 1, 2, \ldots$. The hypotheses of Theorem 4 are all fulfilled and, consequently, the sequence $(B_n(f))_{n \ge 1}$ converges uniformly to f, for all $f \in C(X)$.

The equalities (5) (for $L_n = B_n$), (8) and (9) give:

$$\beta_n(x) = B_n(e;x) - 2B_n(e_x;x) + e(x) =$$
$$= \sum_{j=1}^{n} p_{n,j}^2[\rho_j(x)\,\nu_{x,j}(e) + (1-\rho_j(x))\cdot V_j(e;x)] +$$
$$+ \left(1 - \sum_{j=1}^{n} p_{n,j}^2\right) e(x) - 2e(x) + e(x) =$$
$$= \sum_{j=1}^{n} p_{n,j}^2[\rho_j(x)\,\nu_{x,j}(e) + (1-\rho_j(x))\,V_j(e;x)] - \sum_{j=1}^{n} p_{n,j}^2\, e(x) =$$
$$= \sum_{j=1}^{n} p_{n,j}^2\,\rho_j(x)\,[E_j(x) - V_j(e;x)] + \sum_{j=1}^{n} p_{n,j}^2[V_j(e;x) - e(x)].$$

It follows that the delimitation (11) is a consequence of the delimitation (3) from Theorem 3.

COROLLARY 3. *If* $\nu_{x,j} = \nu_x$ *for* $j = 1, 2, \ldots$ *and* $\rho_j(x) = 1$, $x \in X$, $j = 1, 2, \ldots$ *then the condition* (10) *from Theorem* 5 *reduces to*

$$\lim_{n\to\infty} \sum_{j=1}^{n} p_{n,j}^2 = 0, \tag{12}$$

and the delimitation (11) *takes the form*.

$$\| B_n(f) - f \| \leqslant K_f \left(\sum_{j=1}^{n} p_{n,j}^2 \| E - e \| \right)^{1/2}, \tag{13}$$

where $E : X \to \mathbf{R}$ *is defined by* $E(x) = \nu_x(e)$, $x \in X$.

Proof. The first assertion of the Corollary follows from the following delimitation for the expression involved in the condition (10), for $\rho_j(x) = 1$ and $\nu_{x,j}(e) = \nu_x(e) = E(x)$:

$$\left| \sum_{j=1}^{n} p_{n,j}^2[E(x) - V_j(e;x)] + \sum_{j=1}^{n} p_{n,j}^2[V_j(e;x) - e(x)] \right| \leqslant$$
$$\leqslant \sum_{j=1}^{n} p_{n,j}^2 \| E - V_j(e) \| + \sum_{j=1}^{n} p_{n,j}^2 \| V_j(e) - e \| \leqslant$$
$$\leqslant (\| E \| + 2\| V_j(e) \| + \| e \|) \sum_{j=1}^{n} p_{n,j}^2 \leqslant (\| E \| + 3) \sum_{j=1}^{n} p_{n,j}^2 .$$

The delimitation (13) follows immediately from (11), taking $\rho_j = 1$ and $E_j = E$.

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