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EXTENSION OF HÖLDER FUNCTIONS AND SOME RELATED PROBLEMS OF BEST APPROXIMATION

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1. Let (X,d) be a metric space and $\alpha \in (0,1]$. A function $f : X \to R$ is called *Hölder of class α on X* if there exists $K \geq 0$ such that

$$|f(x) - f(y)| \leq K\, d^{\alpha}(x,y) \,, \tag{1.1}$$

for all $x,y \in X$.

Put

$$\|f\|_{\alpha,X} = \sup \{|f(x) - f(y)| \,/\, d^{\alpha}(x,y) : x,y \in X,\ x \neq y \}. \tag{1.2}$$

Then $\|f\|_{\alpha,X}$ is the smallest constant $K \geq 0$ for which the inequality (1.1) holds and is called *Hölder norm* of f.

Denote by $\Lambda_{\alpha}(X,d)$ the set of all Hölder functions of class α on X [3]. Ther $\Lambda_{\alpha}(X,d)$ is a vector lattice, that is, it is closed under the operations of addition, multiplication by scalars and formation of supremum and infimum of two of its elements.

For a nonvoid subset Y of X, the Hölder norm $\|f\|_{\alpha,Y}$ and the space $\Lambda_{\alpha}(Y,d)$ are defined similarly.

THEOREM 1. *Let* (X,d) *be a metric space,* $Y \subset X$ *and* $\alpha \in (0,1]$. *If* $f \in \Lambda_\alpha(Y,d)$ *then the functions*

$$(1.3)\quad \begin{aligned} F_1(x) &= \inf \{ f(y) + \|f\|_{\alpha,Y}\, d^\alpha(x,y) : y \in Y \}, \quad x \in X, \\ F_2(x) &= \sup \{ f(y) - \|f\|_{\alpha,Y}\, d\ (x,y) : y \in Y \}, \quad x \in X \end{aligned}$$

are extension of f, i.e.

a) $F_1|_Y = F_2|_Y = f$,

b) $\|F_1\|_{\alpha,X} = \|F_2\|_{\alpha,X} = \|f\|_{\alpha,Y}$.

Theorem 1 follows from Corollary 1.2 in [3].

For $f \in \Lambda_\alpha(Y,d)$ denote by $E_Y(f)$ the set of all extensions of f in $\Lambda_\alpha(X,d)$, i.e.

$$(1.4)\quad E_Y(f) = \{ F \in \Lambda_\alpha(X,d) : F|_Y = f,\ \|F\|_{\alpha,X} = \|f\|_{\alpha,Y} \}.$$

Let C be a convex subset of a vector space V. A subset H of C is called a *face* of C if $\lambda x + (1 - \lambda)y \in H$ for some $\lambda \in (0,1)$ and some $x,y \in C$, implies $x,y \in H$. A one-point face of C is called an *extremal element of* C.

In Theorem 2 below we present some properties of the set $E_Y(f)$.

THEOREM 2. *Let* (X,d) *be a metric space,* Y *a nonvoid subset of* X, $\alpha \in (0,1]$ *and* $f \in \Lambda_\alpha(Y,d)$. *Then*

a) $E_Y(f)$ *is a convex subset of* $\Lambda_\alpha(X,d)$;

b) *For every* $F \in E_Y(f)$, $F_1(x) \ge F(x) \ge F_2(x)$, $x \in X$, *where the functions* F_1 *and* F_2 *are defined by* (1.3);

c) *The functions* F_1 *and* F_2 *defined by* (1,3) *are extremal elements of* $E_Y(f)$.

Proof. a) For $F,G \in E_Y(f)$ and $\lambda \in [0,1]$ we have

$$(\lambda F + (1 - \lambda)G)|_Y = \lambda F|_Y + (1 - \lambda)G|_Y = \lambda f + (1 - \lambda)f = f.$$

Since

$$\|\lambda F + (1 - \lambda)G\|_{\alpha,X} \le \lambda\|F\|_{\alpha,X} + (1 - \lambda)\|G\|_{\alpha,X} = \lambda\|f\|_{\alpha,Y} + (1 - \lambda)\|f\|_{\alpha,Y} = \|f\|_{\alpha,Y}$$

and

$$\|\lambda F + (1 - \lambda)G\|_{\alpha,X} \ge \|(\lambda F + (1 - \lambda)G)|_Y\|_{\alpha,Y} = \|f\|_{\alpha,Y}.$$

it follows that

$$\|\lambda F + (1 - \lambda)G\|_{\alpha,X} = \|f\|_{\alpha,Y}$$

i.e. $\lambda F + (1 - \lambda)G \in E_Y(f)$.

b) Let $F \in E_Y(f)$ and $x \in X$. Then for $y \in Y$ we have

$$F(x) - f(y) = F(x) - F(y) \ge -\|F\|_{\alpha,X}\, d^\alpha(x,y) = -\|f\|_{\alpha,Y}\, d^\alpha(x,y)$$

so that

$$F(x) \ge f(y) - \|f\|_{\alpha,Y}\, d^\alpha(x,y), \text{ for all } y \in Y.$$

Therefore

$$F(x) \ge \sup \{ f(y) - \|f\|_{\alpha,Y}\, d^\alpha(x,y) : y \in Y \} = F_2(x).$$

Similarly,

$$F(x) - f(y) = F(x) - F(y) \le \|F\|_{\alpha,X}\, d^\alpha(x,y) = \|f\|_{\alpha,Y}\, d^\alpha(x,y)$$

implies

$$F(x) \le f(y) + \|f\|_{\alpha,Y}\, d^\alpha(x,y), \text{ for all } y \in Y$$

so that

$$F(x) \le \inf \{ f(y) + \|f\|_{\alpha,Y}\, d^\alpha(x,y) : y \in Y \} = F_1(x).$$

c) If $F, G \in E_Y(f)$ and $\lambda \in (0,1)$ are such that $\lambda F + (1 - \lambda)G = F_1 = \lambda F_1 + (1 - \lambda)F_1$, then $\lambda(F_1 - F) = (1 - \lambda)(G - F_1)$ and since by b), $G - F_1 \le 0$ it follows that $F_1 \le F$. But $F_1 \ge F$ and hence $F = F_1$. Then the relation $\lambda F + (1 - \lambda)G = F_1$ yields also $G = F_1$.

The case of the function F_2 can be treated similarly. ∎

2. For a nonvoid subset Y of metric space (X,d) denote

(2.1) $$Y^{\perp} = \{ f \in \Lambda_\alpha(X,d) \ : \ f|_Y = 0 \}.$$

Obviously, $Y^{\perp}$ is a closed subspace of $\Lambda_\alpha(X,d)$.

A subset S of a normed space $(V,\|\ \|)$ is called *proximinal* if for every $x \in V$ there exists $y_0 \in S$ such that :

(2.2) $$\|x - y_0\| = d(x,S) = \inf \{ \|x - y\| \ : \ y \in S \}.$$

An element $y_0 \in S$ for which the infimum in (2.2) is attained is called an *element of best approximation* of x by elements in S. If for every $x \in V$ there exists a unique element of best approximation of x in S, then the set S is called *Chebyshevian* [6]. Denote by $P_S(x)$ the set of all best approximation elements of x in S.

THEOREM 3. *If Y is a nonvoid subset of a metric space (X,d) then*

a) *The subspace $Y^{\perp}$ is proximinal and*

(2.3) $$d(f,Y^{\perp}) = \|f|_Y\|_{\alpha,Y},$$

for every $f \in \Lambda_\alpha(X,d)$;

b) *Every element $g_0 \in Y^{\perp}$ of best approximation for f has the form $g_0 = f - F$, where $F \in E_Y(f|_Y)$, and, conversely, for every $F \in E_Y(f|_Y)$, $f - F$ is an element of best approximation for f in $Y^{\perp}$*, i.e.

$$P_{Y^{\perp}}(f) = f - E_Y(f|_Y) \ ;$$

c) *The subspace $Y^{\perp}$ is Chebyshevian if and only if for every $f \in \Lambda_\alpha(X,d)$ the function $f|_Y$ has a unique extension in* $\Lambda_\alpha(X,d)$.

Proof. a) Let $f \in \Lambda_\alpha(X,d)$ and $F \in E_Y(f|_Y)$. Then

(2.4) $$\|f - (f - F)\|_{\alpha,X} = \|F\|_{\alpha,X} = \|f|_Y\|_{\alpha,Y}.$$

Since $f - F \in Y^{\perp}$, it follows

$$\inf \{\|f - g\|_{\alpha,X} : g \in Y^{\perp} \} \le \|f\|_{\alpha,Y} .$$

On the other hand

$$\|f|_Y\|_{\alpha,Y} = \sup \{ |(f-g)(x) - (f-g)(y)| \ / \ d^\alpha(x,y) \ : \ x,y \in Y;$$
$$x \ne y \} \le \sup \{ |(f-g)(x) - (f-g)(y)| \ / \ d^\alpha(x,y) \ : \ x,y \in X \ ;$$
$$x \ne y \} = \| f - g \|_{\alpha,X} ,$$

for every $g \in Y^{\perp}$, so that

$$\|f|_Y\|_{\alpha,Y} \le \inf \{ \|f - g\|_{\alpha,X} : \ g \in Y^{\perp} \} .$$

Therefore $\|f|_Y\|_{\alpha,Y} = \|f - (f - F)\|_{\alpha,X} = d(f,Y^{\perp})$, which shows that $Y^{\perp}$ is proximinal and that the formula (2.3) holds.

b) By (2.4), it follows that $f - F \in Y^{\perp}$ is an element of best approximation for f in $Y^{\perp}$, where $F \in E_Y(f|_Y)$. If $g_0 \in Y^{\perp}$ is an element of best approximation for f by elements of $Y^{\perp}$, then

$$\|f - g_0\|_{\alpha,X} = \|f|_Y\|_{\alpha,Y} \ \text{ and } \ (f - g_0)|_Y = f|_Y$$

so that $f - g_0 \in E_Y(f|_Y)$ and $F_0 = f - g_0$ is an extension of $f|_Y$.

c) If $Y^{\perp}$ is Chebyshevian, then every $f \in \Lambda_\alpha(X,d)$ has a unique element of best approximation in $Y^{\perp}$ and by b), the set $E_Y(f|_Y)$ contains only one element. ∎

3. Let now (X,d) be a metric space of finite diameter, i.e. $d(X) = \sup \{ d(x,y) : x,y \in X \} < \infty$. Then every function $f \in \Lambda_\alpha(X,d)$ is bounded, for if $x_0 \in X$ is fixed, then

$$|f(x)| \le |f(x) - f(x_0)| + |f(x_0)| \le |f(x_0)| + \|f\|_{\alpha,X}\, d^\alpha(x,x_0) \le$$
$$\le |f(x_0)| + \|f\|_{\alpha,X}\, [d(X)]^\alpha, \quad \text{for every } x \in X.$$

In this case, we can define the uniform norm on $\Lambda_\alpha(X,d)$,

(3.1) $$\|f\|_{u,X} = \sup \{ |f(x)| \ : \ x \in X \}.$$

Let Y be a subset of metric space of finite diameter (X,d) and let $Y^{\perp}$ be defined by (2.1). For $f \in \Lambda_\alpha(X,d)$ let $G(f)$ denote the set of all best approximation elements, with respect to the Hölder norm, of f by elements in $Y^{\perp}$. Consider the following problem:

Find g_* and g^* in $G(f)$ such that

$$(3.2)\quad \begin{aligned} \|f - g_*\|_{u,X} &= \inf \{ \|f - g\|_{u,X} \ : \ g \in G(f) \} , \\ \|f - g^*\|_{u,X} &= \sup \{ \|f - g\|_{u,X} \ : \ g \in G(f) \} . \end{aligned}$$

By Theorem 3 b), the problem (3.2) is equivalent to the following problem:

Find two extensions F_* and F^* in $E_Y(f|_Y)$ such that

$$(3.3)\quad \begin{aligned} \|F_*\|_{u,X} &= \inf \{ \|F\|_{u,X} \ : \ F \in E_Y(f|_Y) \} , \\ \|F^*\|_{u,X} &= \sup \{ \|F\|_{u,X} \ : \ F \in E_Y(f|_Y) \} . \end{aligned}$$

The next theorem shows that the problem considered above has always a solution.

THEOREM 4. a) *The infimum in* (3.2) *is attained, namely for every function* $g_* \in G(f)$ *of the form* $g_* = f - F_*$ *where* $F_* \in E_Y(f|_Y)$ *is such that* $\|F_*\|_{u,X} = \|f|_Y\|_{u,X}$ *;*

b) *The supremum in* (3.2) *is attained, namely for* $g^* = f - F_1$ *or* $g^* = f - F_2$ *or for both of these functions, where* F_1 *and* F_2 *are defined by* (1.3).

Proof. a) First, observe that there exists $F_* \in E_Y(f|_Y)$ such that $\|F_*\|_{u,X} = \|f|_Y\|_{u,X}$. Indeed, if for $F \in E_Y(f|_Y)$ put

$$(3.4)\quad \begin{aligned} F_*(x) &= \|f|_Y\|_{u,Y} & &\text{if } F(x) > \|f|_Y\|_{u,Y} , \\ &= F(x) & &\text{if } -\|f|_Y\|_{u,Y} \le F(x) \le \|f|_Y\|_{u,Y} , \\ &= -\|f|_Y\|_{u,Y} & &\text{if } F(x) < -\|f|_Y\|_{u,Y} , \end{aligned}$$

then F_* is in $E_Y(f|_Y)$ and $\|F_*\|_{u,X} = \|f|_Y\|_{u,Y}$.

Since

$$\|F\|_{u,X} \ge \sup \{ |F(y)| \ : \ y \in Y \} = \|f|_Y\|_{u,Y} ,$$

for every $F \in E_Y(f|_Y)$, it follows that

$$\inf \{ \|F\|_{u,X} \ : \ F \in E_Y(f|_Y) \} = \|F_*\|_{u,X}$$

where F_* is defined by (3.4).

b) Since

$$F_2(x) \le F(x) \le F_1(x) , \ x \in X ,$$

for every $F \in E_Y(f|_Y)$ (Theorem 2. b)) , it follows that

$$\|F\|_{u,X} \le \max \{ \|F_1\|_{u,X} , \|F_2\|_{u,X} \} . \ \blacksquare$$

Remarks. a) The set of the functions of the form (3.4) is a convex subset of $E_Y(f|_Y)$;

b) The functions F_1, F_2 for which the supremum in (3.3) is attained are extremal elements of the convex set $E_Y(f|_Y)$.

4. Let (X,d) be a metric space of finite diameter. Consider on $\Lambda_\alpha(X,d)$ the norms

$$(4.1)\quad \begin{aligned} \|f\|_s &= \|f\|_{\alpha,X} + \|f\|_{u,X} \\ \|f\|_m &= \max \{ \|f\|_{\alpha,X} \ ; \ \|f\|_{u,X} \} \end{aligned}$$

called the "sum-norm" and the "max-norm" , respectively.

If Y is a subset of X and f is in $\Lambda_\alpha(Y,d)$, then it is natural to ask if f has an extension $F \in \Lambda_\alpha(X,d)$ preserving the norms (4.1). An afirmative answer to this question and some consequences will be objects of the following theorems.

THEOREM 5. *If* (X,d) *is a metric space of finite diameter and* Y *is a nonvoid subset of* X *, then for every* $f \in \Lambda_\alpha(Y,d)$ *there exists* $F \in \Lambda_\alpha(X,d)$ *such that*

(4.2) $\quad F|_Y = f$ *and* $\|F\|_s = \|f\|_s$.

Proof. Let F_1 be defined by (1.3) and let

(4.3)
$$\begin{aligned}\overline{F}_1(x) &= F_1(x) \quad \text{if } F_1(x) \le \|f\|_{u,Y} \\ &= \|f\|_{u,Y} \quad \text{if } F_1(x) > \|f\|_{u,Y}\end{aligned}$$

Then $\overline{F}_1|_Y = f$ and $\overline{F}_1$ preserves both of the Hölder and uniform norms, so that it preserves the "sum-norm", also.

Similarly, the function

(4.4)
$$\begin{aligned}\overline{F}_2(x) &= F_2(x) \quad \text{if } F_2(x) \ge -\|f\|_{u,Y} , \\ &= \|f\|_{u,Y} \quad \text{if } F_2(x) < -\|f\|_{u,Y} ,\end{aligned}$$

where F_2 is defined by (1.3), is also an extension of f, preserving both of the Hölder and uniform norms.

Denote by $E_s(f)$ the set of all extensions of $f \in \Lambda_\alpha(Y,d)$ which preserve the "sum-norm", i.e.

(4.5) $\quad E_s(f) = \{ F \in \Lambda_\alpha(X,d) \ : \ F|_Y = f, \ \|F\|_s = \|f\|_s \}$. ∎

THEOREM 6. *If* (X,d) *is a metric space of finite diameter, Y a subset of X and* $f \in \Lambda_\alpha(Y,d)$ *, then*

(4.6) $\quad$ a) $\|F\|_{\alpha,X} = \|f\|_{\alpha,Y}$ *and* $\|F\|_{u,X} = \|f\|_{u,X}$

for every $F \in E_s(f)$;

b) *If* $\overline{F}_1$ *and* $\overline{F}_2$ *are defined by* (4.3) *and* (4.4) *then*

(4.7) $\quad \overline{F}_1(x) \ge F(x) \ge \overline{F}_2(x) \quad , \ x \in X$,

for every $F \in E_s(f)$;

c) *The set* $E_s(f)$ *is a convex subset of the ball (with respect to "sum-norm") of center 0 and radius* $\|f\|_s$ *in* $\Lambda_\alpha(X,d)$;

d) *The functions* $\overline{F}_1$ *and* $\overline{F}_2$ *are extremal elements of* $E_s(f)$;

e) *If* $\|f\|_s = 1$, *then* f *is an extremal element of the unit ball of* $\Lambda_\alpha(Y,d)$ *(with respect to the "sum-norm") if and only if* $E_s(f)$ *is a face of the unit ball of* $\Lambda_\alpha(X,d)$.

Proof. a) Let $F \in E_s(f)$. Since $F|_Y = f$, it follows that $= \|F\|_{\alpha,X} \ge \|f\|_{\alpha,Y}$ and $\|F\|_{u,X} \ge \|f\|_{u,Y}$. If $\|F\|_{\alpha,X} > \|f\|_{\alpha,Y}$ then $\|f\|_{\alpha,Y} + \|f\|_{u,Y} < \|F\|_{\alpha,X} + \|F\|_{u,X} = \|F\|_s = \|f\|_s = \|f\|_{\alpha,Y} + \|f\|_{u,Y}$, which is impossible. Therefore $\|F\|_{\alpha,X} = \|f\|_{\alpha,Y}$ and $\|F\|_{u,X} = \|F\|_s - \|F\|_{\alpha,X} = \|f\|_s - \|f\|_{\alpha,Y} = \|f\|_{u,Y}$.

The proof of b), c), d) proceeds similarly to the proofs of assertions a), b), c) of Theorem 2.

e) Let f be an extremal element of the unit ball of $\Lambda_\alpha(Y,d)$ and let $\lambda \in (0,1)$. If F_1 , F_2 are elements of the unit ball of $\Lambda_\alpha(X,d)$ such that $\lambda F_1 + (1-\lambda)F_2 \in E_s(f)$, then $\lambda F_1|_Y + (1-\lambda)F_2|_Y = f$. Since f is extremal it follows that $F_1|_Y = F_2|_Y = f$ and $\|F_1\|_s = \|F_2\|_s = 1$, i.e. $F_1, F_2 \in E_s(f)$. We have shown that $E_s(f)$ is a face of the unit ball of $\Lambda_\alpha(X,d)$.

Conversely, suppose that $\|f\|_s = 1$ and f is not an extremal element of the unit ball of $\Lambda_\alpha(Y,d)$. Then there exists f_1, f_2 in $\Lambda_\alpha(Y,d)$, $f_1 \ne f_2$, $\|f_i\|_s = 1$, $i = 1,2$, and $\lambda \in (0,1)$ such that $\lambda f_1 + (1-\lambda)f_2 = f$. If $F_i \in E_s(f_i)$, $i = 1,2$, then $\lambda F_1|_Y + (1-\lambda)F_2|_Y = f$ and $1 = \|\lambda F_1|_Y + (1-\lambda)F_2|_Y\|_s \le \|\lambda F_1 + (1-\lambda)F_2\|_s \le 1$, which show that $\lambda F_1 + (1-\lambda)F_2 \in E_s(f)$. Consequently $E_s(f)$ is not a face of the unit ball of $\Lambda_\alpha(X,d)$. ∎

THEOREM 7. *Let* (X,d) *be a metric space of finite diameter, Y a subset of X and suppose* Λ_α (Y,d) *and* Λ_α (X,d) *endoved with the "max-norm". Then*

a) *For every* $F \in E_s(f)$, $\|F\|_m = \|f\|_m$;

b) *If*

(4.8) $$E_m(f) = \{ F \in \Lambda_\alpha(X,d) : F|_Y = f,\ \|F\|_m = \|f\|_m \},$$

then there exists $\tilde{F}_1$ *and* $\tilde{F}_2$ *in* $E_m(f)$ *such that*

(4.9) $$\tilde{F}_1(x) \ge F(x) \ge \tilde{F}_2(x)\ ,\ x \in X\ ,$$

for every $F \in E_m(f)$;

c) *The set* $E_m(f)$ *is a convex subset of the ball (in the norm* $\|\ \|_m$ *) of center* 0 *and radius* $\|f\|_m$ *in* $\Lambda_\alpha(X,d)$;

d) *The functions* $\tilde{F}_1$ *and* $\tilde{F}_2$ *satisfying* (4.9) *are extremal elements of the set* $E_m(f)$;

e) *The function* f *is an extremal element of the unit ball of* $\Lambda_\alpha(Y,d)$ *if and only if* $E_m(f)$ *is a face of the unit ball of* $\Lambda_\alpha(X,d)$.

Proof. a) If $F \in E_s(f)$ then , by Theorem 6. a) ,

$$\|F\|_{\alpha,X} = \|f\|_{\alpha,Y} \quad \text{and} \quad \|F\|_{u,X} = \|f\|_{u,Y}\ ,$$

so that $\|F\|_m = \|f\|_m$.

b) Let

$$H_1(x) = \inf\ \{ f(y) + \|f\|_m\, d^\alpha(x,y) : y \in Y \},\ x \in X.$$

Then (see [2]) the function H_1 has the properties

$$H_1|_Y = f\ ,\ \|H_1\|_{\alpha,X} = \|f\|_m\ .$$

The function

(4.10) $$\begin{aligned} \tilde{F}_1(x) &= H_1(x) \quad \text{if} \quad H_1(x) \le \|f\|_m\ , \\ &= \|f\|_m \quad \text{if} \quad H_1(x) > \|f\|_m\ , \end{aligned}$$

has the properties

$$\tilde{F}_1|_Y = f\quad ,\quad \|\tilde{F}_1\|_m = \|f\|_m$$

that is $\tilde{F}_1 \in E_m(f)$.

Similarly, by truncating the function

$$H_2(x) = \sup\ \{ f(y) - \|f\|_m\, d^\alpha(x,y) : y \in Y \}\ ,\ x \in X$$

one obtains the function

(4.11) $$\begin{aligned} \tilde{F}_2(x) &= H_2(x) \quad \text{if} \quad H_2(x) \ge -\|f\|_m\ , \\ &= -\|f\|_m \quad \text{if} \quad H_2(x) < -\|f\|_m\ , \end{aligned}$$

which is an extension of f with respect to the "max-norm", i.e. $\tilde{F}_2 \in E_m(f)$.

The inequalities (4.9) can be obtained reasoning like in the proof of assertion b) in Theorem 2.

The proofs of c) and d) are similar to the proofs of assertion a) and c) of Theorem 2 and the proof of e) is similar to the proof of e) in Theorem 6. ■

From Theorem 6 and 7 obtains the following corolary:

COROLLARY 1. *If* (X,d) *is a metric space of finite diameter,* Y *is a subset of* X *and* $f \in \Lambda_\alpha(Y,d)$, *then*

a) $E_s(f) \subset E_m(f)$;

b) $\tilde{F}_1(x) \ge \bar{F}_1(x) \ge \bar{F}_2(x) \ge \tilde{F}_2(x)\ ,\ x \in X\ ,$

where the functions $\tilde{F}_1, \tilde{F}_2, \bar{F}_1, \bar{F}_2$ *are defined by* (4.3), (4.4), (4.10) *and* (4.11) , *respectively.*

5. In the following we shall give a procedure to find the global extrema of a function $f \in \Lambda_\alpha(X,d)$ by using the extensions of the restriction of f to some finite subset of X.

Let (X,d) be a compact metric space and let f be in $\Lambda_\alpha(X,d)$. If Y is a subset of X and $q \ge \|f|_Y\|_{\alpha,Y}$ (here $f|_Y$ denotes the restriction of f to Y), then the functions

(5.1) $$\begin{aligned} U_q(x) &= \inf\ \{ f(y) + q\, d^\alpha(x,y) : y \in Y \},\ x \in X\ , \\ u_q(x) &= \sup\ \{ f(y) - q\, d^\alpha(x,y) : y \in Y \},\ x \in X\ , \end{aligned}$$

are extensions of $f|_Y$ which belongs to $\Lambda_\alpha(X,d)$ and have the Hölder norms at most q (see [2]). If U and u denotes the functions defined by (5.1) for $q = \|f\|_{\alpha,X}$ then

(5.2) $\quad u(x) \le f(x) \le U(x)\ ,\ x \in X\ ,$

and for $q > \|f\|_{\alpha,X}$ we have

(5.3) $\quad u_q(x) \le u(x) \le f(x) \le U(x) \le U_q(x)\ ,\ x \in X.$

A *maximum* (respectively a *minimum*) point of a function $f : X \to R$ is a point $x^* \in X$ such that

(5.4) $\quad f(x^*) \ge f(x) \quad$ (respectively $f(x^-) \le f(x)$)

for all $x \in X$.

For a bounded real function f on X put

(5.5) $$M_f = \sup\{ f(x) : x \in X \}\ ,\quad m_f = \{ f(x) : x \in X \}\ ,$$
$$E_f = \{ x \in X : f(x) = M_f \}\ ,\ e_f = \{ x \in X : f(x) = m_f\}.$$

Let now (X,d) be a compact metric and let f be a function in $\Lambda_\alpha(X,d)$.

We define now inductively two sequences $(x_n)_{n\ge0}$ and $(M_n)_{n\ge0}$ of points in X and of real numbers, respectively, as follows:

Let $q \ge \|f\|_{\alpha,X}$ be fixed and let x_0 be a fixed point in X. Let $U^0(x) = f(x_0) + q\, d^\alpha(x,x_0)\ ,\ x \in X$, the greatest extension of f obtained from (5.1) for $Y = \{x_0\}$ and let

$$M_0 = \sup\ \{ U^0(x) : x \in X \}\ .$$

Let $x_1 \in X$ be a point with $U^0(x_1) = M_0$.

Suppose now that for a natural number $n \ge 1$, the points $x_0, x_1, \dots, x_{n-1}$ and the numbers $M_0, M_1, \dots, M_{n-1}$ were defined. Let U^{n-1} be the greatest extension of $f|_Y$ obtained from (5.1) for $Y = \{ x_0, x_1, \dots, x_{n-1} \}$. Put

$$M_n = \sup\ \{ U^{n-1}(x) : x \in X \}\ ,$$

and let x_n be a point in X such that $U^{n-1}(x_n) = M_n$.

The properties of the so defined sequences $(x_n)_{n\ge0}$ and

$(M_n)_{n\ge0}$ are described in the following theorem:

Theorem 8. *Let (X,d) be a compact metric space and let $f \in \Lambda_\alpha(X,d)$. For a fixed $q \ge \|f\|_{\alpha,X}$ let the sequences $(x_n)_{n\ge0}$ and $(M_n)_{n\ge0}$ be defined as above. Then*

a) $\lim_{n\to\infty} M_n = M_f$;

b) $\lim_{n\to\infty} [\ \inf\ \{ d(x,x_n) : x \in E_f \}\ ] = 0$;

c) *The sequence $(f(x_n))_{n\ge0}$ has the number M_f as a limit point.*

Proof. Since $U^n \le U^{n-1}$ for $n = 1,2,\dots$ it follows that the sequence $(M_n)_{n\ge0}$ is nonincreasing . By (5.3), $M_n = U^{n-1}(x_n) \ge$ $\ge f(x_n) \ge \min\ \{ f(x) : x \in X \}$ so that the sequence $(M_n)_{n\ge0}$ is also bounded . Therefore there exists $M = \lim_{n\to\infty} M_n$. By (5.3), $f(x) \le U^n(x) \le M_n$, for all $x \in N$ so that

(5.6) $\quad f(x) \le M\ ,$ for all $x \in X$.

The metric space X being compact , the sequence $(x_n)_{n\ge0}$ contains a subsequence $(x_{n_k})_{k\ge0}$ converging to a point $z \in X$. Since the function f is continuous it follows that

(5.7) $\quad f(x_{n_k}) \longrightarrow f(z)\ ,\ k \longrightarrow \infty\ .$

But, for $k \ge 1$,

$$|U^{n_k-1}(z) - M_{n_k-1}| = |U^{n_k-1}(z) - U^{n_k-1}(x_{n_k})| \le q\, d^\alpha(z,x_{n_k}) \to 0$$

for $k \to \infty$, and $M_{n_k-1} \to M$ for $k \to \infty$, so that

(5.8) $\quad U^{n_k-1}(z) \longrightarrow M\ ,\ $ for $k \longrightarrow \infty$.

By the relation

$$|U^{n_k}(z) - f(x_{n_k})| = |U^{n_k}(z) - U^{n_k}(x_{n_k})| \le q\, d^\alpha(z,x_{n_k}) \longrightarrow 0,$$

$k \longrightarrow \infty$, and by (5.7) it follows that

(5.9) $U^{n_k}(z) \longrightarrow f(z)$, $k \longrightarrow \infty$.

Therefore , if in the inequalities

$$f(z) \le U^{n_k - 1}(z) \le U^{n_{k-1}}(z) , \quad k \ge 1 ,$$

we let $k \longrightarrow \infty$ one obtains $f(z) \le M \le f(z)$, so that $f(z) = M$. Taking into account (5.6) it follows that

$$M = f(z) = \max \{ f(x) : x \in X \}.$$

To prove b) , observe that if contrary, then there exist $\varepsilon > 0$ and an infinite subset J of N such that

(5.10) $\inf \{ d(x,x_j) : x \in E_f \} \ge \varepsilon$,

for all $j \in J$. The space X being compact there exists a subsequence $(x_{j_k})_{k\ge0}$ of $(x_j)_{j\in J}$ converging to a point $y \in X$. But then, repeating the above arguments will follows that $y \in E_f$, which contradicts (5.10).

The affirmation c) follows from (5.7). ■

Remarks. 1) In the clase $X = [a,b]$ and $\alpha = 1$ a similar result is proved in [7].

2) If the extensions U^n are replaced by the extensions u^n and $m_n = \inf \{ u^n(x) : x \in X \}$, $u^n(x_{n+1}) = m_n$, then one obtains a procedure to find the minimum m_f of a function $f \in \Lambda_\alpha(X,d)$.

Example. Let $X = [0,1]$, $d(x,y) = |x - y|$ and

$$f(x) = x \cdot \sin (1/x) , \quad \text{if } x \in (0,1] ,$$
$$= 0 , \quad \text{if } x = 0 .$$

It is known that $f \in \Lambda_\alpha(X,d)$ if and only if $\alpha \in (0,1/2]$ (see [8], Problem 153) and in this case

$$\|f\|_{\alpha,X} \le [1 + 2 \ln (1 + 2\pi) + 2\pi]^{1/2} < 4 .$$

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ON VECTOR TOPOLOGIES ON FINITE-DIMENSIONAL VECTOR SPACES

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1. Introduction.

This article grew out from some basic ideas presented by I.Muntean in his book [1]. We wrote it thinking that the question of the independence of the defining axioms of a vector topology is still an interesting one.

Let X be a vector space on K (K being $\mathbb{R}$ or $\mathbb{C}$, endowed with the Euclidian topology). Let O be the origin of X.

A topology τ in X is named vector topology if it satisfies:

TV1) The adition "+" : $X \times X \to X$, $+(x,y) = x + y$ is continuous.

TV2) The multiplication "$\cdot$" : $K \times X \to X$, $\cdot(\alpha,x) = \alpha x$ is continuous.

Generally, the axioms TV1) and TV2) are independent.

2. Independence of the axioms.

If X is a vector space, $X \neq \{0\}$ with the discrete topology, TV1) is verified without TV2) being verified.

In 1988, V.Anisiu gave an exemple of a vector topology wich verifies TV2) without TV1) being satisfied. The vector space he considered was infinite dimensional. We give now an exemple of