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CONTENTS

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ON A PROBLEM OF EXTREMUM

Costică Mustăţa

Let $[a,b]$ be an interval of the real axis and let $D : a = x_0 < x_1 < x_2 < \ldots < x_n = b$ be a division of this real interval. Let

$V = \{ y_k : k = 0,1,\ldots,n \} \subset \mathbb{R}$ and let $M > 0$ be such that $M > |y_{k+1} - y_k| / (x_{k+1} - x_k)$, $k = 0,1,\ldots,n-1$.

A function $f : [a,b] \to \mathbb{R}$ is called Lipschitz on $[a,b]$ if there exists a number $K \geq 0$ such that:

$$(1) \qquad |f(x) - f(y)| \leq K|x - y| \quad ,$$

for all $x,y \in [a,b]$. We shall denote by K_f the smallest of the numbers K for which the relation (1) holds and we shall call it the *Lipschitz norm* of the function f. Obviously that K_f is given by:

$$K_f = \sup\{ |f(x) - f(y)| / |x - y| \; ; \; x,y \in [a,b] \; , \; x \neq y \} .$$

Denote by $Lip[a,b]$ the set of all real-valued Lipschitz functions defined on $[a,b]$ and let

$$(2) \quad \mathcal{F}(D,V,M) = \{ f \in Lip\,[a,b] : f(x_k) = y_k \; , \; k = \overline{0,n} \; , \; K_f \leq M \}.$$

The function whose graph is the polygonal line joining the points (x_k, y_k), $k = 0,1,\ldots,n$ belongs to $\mathcal{F}(D,V,M)$ so that

$$\phi \neq \mathcal{F}(D,V,M) \subset Lip\,[a,b] \subset C\,[a,b] \; ,$$

where, as usually, C [a,b] denotes the Banach space of all continuous real-valued functions defined on [a,b], equiped with the uniform norm:

(3) $\|f\| = \sup \{ |f(x)| : x \in [a,b] \}$, $f \in C[a,b]$.

As a subset of the Banach space C[a,b] the set $\mathcal{F}(D,V,M)$ has the following properties:

THEOREM 1. a) *The set* $\mathcal{F}(D,V,M)$ *is a convex subset of* C [a,b];

b) *The functions* F_i *and* F_s *given by*

(4)
$$F_i(x) = \max \{ f(x_k) - M|x - x_k| , k = 0,1,\ldots,n \} ,$$
$$F_s(x) = \min \{ f(x_k) + M|x - x_k| , k = 0,1,\ldots,n \} ,$$

for $x \in [a,b]$, *are extremal points of* $\mathcal{F}(D,V,M)$;

c) *The set* $\mathcal{F}(D,V,M)$ *is compact with respect to the uniform topology of the space* C [a,b].

Proof. a) Let $f_1, f_2 \in \mathcal{F}(D,V,M)$, $\lambda \in [0,1]$ and $f = \lambda f_1 + (1 - \lambda)f_2$. Then, obviously, $f(x_k) = y_k$, $k = 0,1,\ldots,n$ and

$$|f(x) - f(y)| \le \lambda|f_1(x) - f_1(y)| + (1 - \lambda)\cdot|f_2(x) - f_2(y)| \le$$
$$\le (\lambda K_{f_1} + (1 - \lambda)K_{f_2})\cdot|x - y| \le$$
$$\le (\lambda M + (1 - \lambda)M) \; |x - y| = M \cdot |x - y| ,$$

for all $x,y \in [a,b]$, impling $K_f \le M$. It follows that $f \in \mathcal{F}(D,V,M)$.

b) By a theorem of McShane [2], the functions F_i and F_s defined by (4) are in $\mathcal{F}(D,V,M)$ and furthermore

(5) $$F_i(x) \le f(x) \le F_s(x) \quad , \quad x \in [a,b] ,$$

for all $f \in \mathcal{F}(D,V,M)$. To prove the second inequality in (5), suppose, on the contrary, that there exists a function $f \in \mathcal{F}(D,V,M)$ on a point $c \in [a,b]$ such that $f(c) > F_s(c)$. As $F_i(x_k) = f(x_k) = F_s(x_k) = y_k$, $k = 0,1,\ldots,n$, it follows that

there exists $k_0 \in \{0,1,\ldots,n\}$ such that $c \in (x_{k_0}, x_{k_0+1})$. But then

$$\frac{f(c) - f(x_{ko})}{c - x_{ko}} > \frac{F_s(c) - F_s(x_{ko})}{c - x_{ko}} = M$$

or

$$\frac{f(x_{ko+1}) - f(c)}{x_{ko+1} - c} < \frac{F_s(x_{ko+1}) - F_s(c)}{x_{ko+1} - c} = - M,$$

according as c belongs to the interval

$$\left(x_k , \frac{x_{ko+1} + x_{ko}}{2} + \frac{y_{ko+1} - y_{ko}}{2M} \right] \quad \text{or}$$

$$\left[\frac{x_{ko+1} + x_{ko}}{2} + \frac{y_{ko+1} - y_{ko}}{2M} , x_{ko+1} \right) ,$$

respectively. In both of the cases it follows $K_f > M$, contradicting the hypothesis $f \in \mathcal{F}(D,V,M)$. The first inequality in (5), $F_i(x) \le f(x)$, for all $x \in [a,b]$, can be proved similary.

To prove that F_s is an extreme point of the convex set $\mathcal{F}(D,V,M)$ suppose that $F_s = \lambda f_1 + (1 - \lambda)f_2$ for two functions $f_1, f_2 \in \mathcal{F}(D,V,M)$ and a number $\lambda \in (0,1)$. We have to show that $f_1 = f_2 = F_s$, but this follows immediately from the inequalities (5).

c) By the Arzelà - Ascoli theorem (see e.g. [5]) it is sufficient to show that $\mathcal{F}(D,V,M)$ is a closed, uniformly bounded and equicontinuous subset of C [a,b]. By the definition of $\mathcal{F}(D,V,M)$ it is obvious that if (f_n) is a sequence in $\mathcal{F}(D,V,M)$ converging to $f \in C[a,b]$ then f is in $\mathcal{F}(D,V,M)$ too, and by (5)

$$\|f\| \le \max \{\|F_i\|,\|F_s\|\}$$

showing that $\mathcal{F}(D,V,M)$ is a closed and uniformly bounded subset of C [a,b].

Now, for $\varepsilon > 0$ let $\delta = \varepsilon/(M + 1)$. Then

$$|f(x) - f(y)| \le M|x - y| < M \frac{\varepsilon}{M + 1} < \varepsilon ,$$

for all $x,y \in [a,b]$ with $|x - y| < \delta$ and all f in $\mathcal{F}(D,V,M)$ proving the equicontinuity of the set $\mathcal{F}(D,V,M)$ and, by the above quated result of Arzelà - Ascoli, also its compactness. ■

Exemple. In the paper [3] there are given several solutions to the following problem: let $f : [0,2] \to \mathbb{R}$ be a continuous function derivable on $(0,2)$ and such that $|f'(x)| \le 1$, for all $x \in (0,2)$ and $f(0) = f(2) = 1$. Show that $1 < \int_0^2 f(x)dx < 3$.

The hypotesis of the problem show that f belongs to a class of the type $\mathcal{F}(D,V,M)$, namely for $D = \{0,2\}$, $V = \{1,1\}$ and $M = 1$.

In this case the exremal functions F_i and F_s are not derivable in the point $x = 1$, explaining why the inequalities in the conclusion of the problem are strict.

Consider now for $p \in N$ the functional $I_p : \mathcal{F}(D,V,M) \to \mathbb{R}$, defined by:

$$I_p(f) = \int_a^b |f(x)|^p \, dx . \tag{6}$$

One asks to find the minimal and the maximal values of this functional. The solution of this problem is given by:

THEOREM 2. a) *If the numbers* $\alpha_k = \frac{y_{k+1} + y_k}{2} - \frac{M}{2}(x_{k+1} - x_k)$ *are non-negative for all* $k = 0,1,\ldots,n - 1$ *then*

$$\max I_p(f) = \int_a^b (F_s(x))^p \, dx , \quad \textit{and}$$
(7)
$$\min I_p(f) = \int_a^b (F_i(x))^p \, dx ;$$

b) *If the numbers* $\beta_k = \frac{y_{k+1} + y_k}{2} + \frac{M}{2}(x_{k+1} - x_k)$ *are non - positive for all* $k = 0,1,\ldots,n-1$, *then*

$$\max I_p(f) = \int_a^b |F_i(x)|^p \, dx , \quad \textit{and}$$
(8)
$$\min I_p(f) = \int_a^b |F_s(x)|^p \, dx ;$$

c) *If the numbers* y_k *are non - negative for all* $k = 0,1,\ldots,n$ *then*

$$\max I_p(f) = \int_a^b (F_s(x))^p \, dx ,$$
(9)
$$\min I_p(f) = \int_a^b (\max\{F_i(x),0\})^p \, dx ;$$

d) *If the numbers* y_k *are non - positive for all* $k = 0,1,\ldots,n$ *then*

$$\max I_p(f) = \int_a^b |F_i(x)|^p \, dx , \quad \textit{and}$$
(10)
$$\min I_p(f) = \int_a^b (\max\{|F_s(x)|,0\})^p \, dx .$$

Proof. a) The numbers α_k, $k = 0,1,\ldots,n - 1$ are the relative minima of the function F_i on the interval $[a,b]$. If $\alpha_k \ge 0$ for $k = 0,1,\ldots,n - 1$, then

$$0 \le F_i(x) \le f(x) \le F_s(x) , \quad x \in [a,b] ,$$

implying the inequalities

$$0 \le (F_i(x))^p \le (f(x))^p \le (F_s(x))^p , \quad x \in [a,b],$$

which by integration over $[a,b]$ yield a).

b) The numbers β_k are the relative maxima of the function F_s on $[a,b]$. If $\beta_k \le 0$ for all $k = 0,1,\ldots,n - 1$, then $F_i(x) \le f(x) \le F_s(x) \le 0$, $x \in [a,b]$, implying:

$$- F_i(x) \ge - f(x) \ge - F_s(x) \ge 0 , \quad x \in [a,b].$$

Rising to the power p and integrating over $[a,b]$ one obtains b).

c) If the numbers y_k , $k = 0,1,\ldots,n$ are all non - negative then the inequalities

$$\max \{F_i(x),0\} \le |f(x)| \le F_s(x) \ , \ x \in [a,b] \ ,$$

hold, which rised to the power p and integrated over [a,b] give (9).

d) The proof is similar to that in the case c). ∎

Remark 1. The set $\mathcal{F}(D,V,M)$ being compact every continuous functional defined on $\mathcal{F}(D,V,M)$ attains its extrema.

Remark 2. All the integrals appearing in the calculation of the extrema of the functional I_p (the formulae (7) - (10)) can be easily calculated, taking into account the fact that the functions F_i and F_s are segmentary linear functions and have very simple expression. For instance, in the case a) :

$$\max I_p(f) = \sum_{k=0}^{n-1} \int_{x_k}^{x_{k+1}} (F_s(x))^p \, dx = \sum_{k=0}^{n-1} \left(\int_{x_k}^{x_{M_k}} (F_s(x))^p \, dx + \int_{x_{M_k}}^{x_{k+1}} (F_s(x))^p \, dx \right) = \sum_{k=0}^{n-1} \int_{x_k}^{x_{M_k}} [M(x - x_k) + y_k]^p \cdot dx + \sum_{k=0}^{n-1} \int_{x_{M_k}}^{x_{k+1}} [- M(x - x_{k+1}) + y_{k+1}]^p \, dx \ ,$$

where $x_{M_k} = \dfrac{x_k + x_{k+1}}{2} + \dfrac{y_{k+1} - y_k}{2M}$ is the point of relative maximum of the function F_s on the interval $[x_k, x_{k+1}]$, $k = 0,1,\ldots,n - 1$.

In the case $D = \{0,2\}$, $V = \{1,1\}$ and $M = 1$, $p \in N$ one obtains the following result:

The inequalities

$$m_p \le I_p(f) \le M_p \ ,$$

hold for every $f \in \mathcal{F}(D,V,M)$, where

$$m_p = \int_0^2 |F_i(x)|^p \, dx = \int_0^1 (- x + 1)^p \, dx + \int_1^2 (x - 1)^p \, dx = 2/(p + 1) \ ,$$

$$M_p = \int_0^2 |F_s(x)|^p \, dx = \int_0^1 (x + 1)^p \, dx + \int_1^2 (- x + 3)^p \, dx = 2(2^{p+1} - 1)/(p + 1) \ .$$

For $p = 1$ we find

$$1 \le \int_0^2 f(x) dx \le 3 \ ,$$

for all $f \in \mathcal{F}(D,V,M)$, i.e a non - sharp version of the inequality proved in [3].

Considering the L_p - norm of a function $f \in \mathcal{F}(D,V,M)$ one gets

$$\sqrt[p]{\frac{2}{p + 1}} \le \|f\| \le \sqrt[p]{2 \, \frac{2^{p+1} - 1}{p + 1}}$$

which for $p \to \infty$ yields the uniform bounds of the set $\mathcal{F}(D,V,M)$: $1 \le \|f\| \le 2$ for every $f \in \mathcal{F}(D,V,M)$.

R E F E R E N C E S

1. ARONSSON, G., Extension of functions satisfying Lipschitz conditions, Arkiv för Mathematik 6 (1967) Nr.28, 551 - 561.

2. Mc SHANE, E.J., Extension of range of functions, Bull. Amer. Math. Soc. 40 (1934), 837 - 842.

3. MOCANU, P., Variaţiuni pe o temă de concurs, Seminarul "Didactica Matematicii", 1985 - 1986, 123 - 128.

4. MUSTĂŢA, C., On the extension problem with prescribed norm, Seminar of Functional Analysis and Numerical Methods, Preprint Nr.4 (1981), 93 - 99..

5. TRENOGUINE, V., Analyse fonctionnelle, Edition Mir, Moscow 1985.

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