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COMMON SELECTIONS FOR THE METRIC PROJECTIONS

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1. Let X be a normed space and M a closed subspace of X. The subspace M is called *proximinal (Chebyshev)* if for every $x \in X$, the set of the elements of best approximation for x in M, given by

(1) $$P_M(x) = \{y \in M : \| x - y \| = d(x, M)\},$$

where

(2) $$d(x, M) = \inf \{\| x - y \| : y \in M\},$$

is nonvoid (respectively a one-point set).

The quantity $d(x, M)$ is called the *distance* from x to M.

If M is a proximinal subspace of X, then the operator $P_M : X \to 2^M$ is called the *metric projection* on M, and the set

(3) $$\ker P_M = \{x \in X : 0 \in P_M(x)\} = \{x \in X : \| x \| = d(x, M)\},$$

is called the *kernel of the metric projection* P_M.

DEFINITION 1. A function $p : X \to M$ is called a *selection* for the metric projection P_M, if $p(x) \in P_M(x)$, for all $x \in X$.

The existence of continuous (and eventualy linear) selections and characterizations of continuous or linear selections for P_M have been studied in [2], for arbitrary normed spaces X.

The finding of continuous or linear metric selections in concrete normed spaces is a problem specific to each considered case. Two such concrete cases were considered in [3] and [9],

This paper will be concerned with the following natural problem: if $\| \; \|_1$ and $\| \; \|_2$ are two norms on a linear space X and M is a subspace of X which is proximinal with respect to each of these norms, find a common selection for the metric projections P^1_M and P^2_M; i.e. an application $p : X \to M$ such that $p(x) \in P^1_M(x) \cap P^2_M(x)$, for all $x \in X$.

The following characterization result of common linear selections for two metric projections is an immediate consequence of Theorem 2.2 in [2].

THEOREM A. *Let M be a subspace of the linear space X which is proximinal with respect to each of the norms $\| \; \|_1$, $\| \; \|_2$ on X. Then the following assertions are equivalent:*

1° *P^1_M and P^2_M admit a common linear selection;*

2° *The set* $\ker P^1_M \cap \ker P^2_M$ *contains a closed subspace N such that $X = M \oplus N$ (algebraic direct sum);*

3° *The set* ker $P^1_M \cap$ ker P^2_M *contains a closed subspace* N *such that* $X = M + N$ *(algebraic and topological direct sum).*

In the following we shall iluustrate Theorem A in a concrete setting.

First, we shall define a linear and continuous selection for the extension operator which preserves both the Lipschitz and uniform norms and then, using a Phelps' type result ([8], Theorem 3) we shall define a common selection for the operators of metric projection in the Lipschitz and in the uniform norms.

2. Let $a, b, c, d \in R$ be such that $c < a < b < d$ and let $X = [c, d]$ $Y = [a, b]$, $x_0 \in [a, b]$ fixed and $d(x, y) = |x - y|$.

A function $f : Y \to R$ is called Lipschitz (on Y) if there exists $K \geqslant 0$ such that

$$|f(x) - f(y)| \leqslant K \cdot d(x, y), \tag{4}$$

for all $x, y \in Y$. The smallest constant K for which (4) holds is

$$\|f\|_L = \sup \{|f(x) - f(y)|/d(x, y) : x, y \in Y, x \neq y\}, \tag{5}$$

and is called the *Lipschitz norm* of the function $f \in \mathrm{Lip}_0\, Y$, where

$$\mathrm{Lip}_0 Y = \{f ; f : [a, b] \to R, f \text{ is Lipschitz on } Y, f(x_0) = 0\}, \tag{6}$$

is the Banach space of all real valued Lipschitz functions defined on Y and vanishing at the fixed point $x_0 \in Y$.

The Banach space $\mathrm{Lip}_0\, X$ is defined in a similar way (the fixed point x_0 is the same as for $\mathrm{Lip}_0\, Y$)

Since Y is a compact subset of R one can also define the uniform norm on $\mathrm{Lip}_0\, Y$ by

$$\|f\|_u = \max \{|f(y)| : y \in Y\}, \tag{7}$$

for $f \in \mathrm{Lip}_0\, Y$. The uniform norm on $\mathrm{Lip}_0\, X$ is defined similarly.

It holds :

THEOREM 1. (a) *For every* $f \in \mathrm{Lip}_0\, Y$ *there exists* $F \in \mathrm{Lip}_0\, X$ *such that*

$$F|_Y = f \text{ and } \|F\|_L = \|f\|_L ; \tag{8}$$

(b) *For every* $f \in \mathrm{Lip}_0\, Y$ *there exists* $\bar{F} \in \mathrm{Lip}_0\, X$ *such that*

$$\bar{F}|_Y = f \text{ and } \|\bar{F}\|_u = \|f\|_u. \tag{9}$$

Proof. The assertion (a) is a particular case of a theorem of Mc Shane [4] (see also [1], [6]).

Denoting by

$$E_L(f) = \{F \in \mathrm{Lip}_0 X : F|_Y = f \text{ and } \|F\|_L = \|f\|_L\{, \tag{10}$$

the non-void set of all extensions of the function f, preserving the Lipschitz norm, let

$$T : E_L(f) \to E_L(f),$$

be the truncation operator defined, for $F \in E_L(f)$, by $T(F) = \bar{F}$, where

(11) $$\bar{F}(x) = \|f\|_u \quad \text{if} \quad F(x) > \|f\|_u,$$

$$= F(x) \quad \text{if} \quad -\|f\|_u \leqslant F(x) \leqslant \|f\|_u,$$

$$= -\|f\|_u \text{ if } \quad F(x) < -\|f\|_u.$$

Obviously, $T(F) = \bar{F} \in E_L(f)$ and $\|\bar{F}\|_u = \|f\|_u$, proving the assertion (b) of the Theorem.

Denote by

(12) $$E_u(f) = \{\bar{F} \in \mathrm{Lip}_0\, X : \bar{F}|_Y = f \text{ and } \|\bar{F}\|_u = \|f\|_u\},$$

the set of all extensions of the function $f \in \mathrm{Lip}_0\, Y$ which preserve the uniform norm of f (this set is non-void by the assertion (b) of the above theorem).

Since the truncation $\bar{F}$ of the extension F of a function $f \in \mathrm{Lip}_0\, Y$ is in $E_L(f) \cap E_u(f)$, it follows that

(13) $$E_L(f) \cap E_u(f) \neq \emptyset,$$

for every $f \in \mathrm{Lip}_0\, Y$ and it holds the inclusion

(14) $$T(E_L(f)) \subset E_u(f),$$

(the example given at the end of this paper shows that the inclusion can be strict).

A function $e_L : \mathrm{Lip}_0\, Y \to \mathrm{Lip}_0\, X$ is called a selection associated to the extension operator

$$E_L : \mathrm{Lip}_0\, Y \to 2^{\mathrm{Lip}_0 X}$$

if $e_L(f) \in E_L(f)$, for every $f \in \mathrm{Lip}_0 Y$. A selection e_u associated to the extension operator E_u is defined in a similar way.

Now we shall consider the following problem : there exists a common linear and continuous selection associated to the operators E_L and E_u?

The answer is given by the following theorem :

THEOREM 2. *The function $e : \mathrm{Lip}_0 Y \to \mathrm{Lip}_0 X$, given by the equality*

(15) $$e(f) = \frac{1}{2}(F_1 + F_2), \quad f \in \mathrm{Lip}_0 Y,$$

where

$$(16)\qquad \begin{aligned} F_1(x) &= \max\ \{f(y) - \|f\|_L \cdot |x-y| : y \in Y\},\\ F_2(x) &= \min\{f(y) + \|f\|_L \cdot |x-y| : y \in Y\}, \end{aligned}$$

is a common linear and continuous selection for the operators E_L and E_u.

Proof. For $f \in \mathrm{Lip}_0\, Y$, the functions given by (16) are extensions of the function f, preserving the Lipschitz norm (see [1], [4], [8] for the properties of the functions F_1, F_2 and of the set $E_L(f)$).

For $x \in X$ we find

$$(17)\qquad \begin{aligned} e(f)(x) &= f(a) \text{ for } x \in [c, a),\\ &= f(x) \text{ for } x \in [a,\ b] = Y,\\ &= f(b) \text{ for } x \in (b,\ d]. \end{aligned}$$

Obviously that $\|e(f)\|_L = \|f\|_L$ so that $e(f) \in E_L(f)$. Furthermore, $\|e(f)\|_u = \|f\|_u$ so that $e(f)$ belongs to the set $E_u(f)$, too.

In [9] Theorem 4 and Corollary 5, it was proved that e is a linear selection, continuous with respect to the topology generated by the Lipschitz norm.

We shall show that e is continuous with respect to the topology generated by the uniform norm, too. To this end, let $\varepsilon > 0$ and let $0 < \delta < \varepsilon$. If $f,\ g \in \mathrm{Lip}_0 Y$ are such that $\|f - g\|_u < \delta$, then $\|e(f) - e(g)\|_u = \|f - g\|_u < \delta < \varepsilon$, proving the continuity of e with respect to the uniform norm. Theorem is proved.

There is a close relation between the selections associated to the extensions operators E_L and E_u and those associated to the operators of metric projection (in the Lipschitz norm and respectively in the uniform norm) on the annihilator of the set Y in $\mathrm{Lip}_0 X$, i.e. on the subspace.

$$(18)\qquad Y^\perp = \{G \in \mathrm{Lip}_0 X :\ G|_Y = 0\}.$$

Let $\overset{L}{P}_{Y^\perp}$, $\overset{u}{P}_{Y^\perp} : \mathrm{Lip}_0 X \to 2^{Y^\perp}$ denote the operators of metric projection on $Y^\perp$ (in the Lipschitz respectively in the uniform norm) and let

$$(19)\qquad \begin{aligned} d_L(F, Y^\perp) &= \inf\ \{\|F - G\|_L : G \in Y^\perp\},\\ d_u(F, Y^\perp) &= \inf\{\|F - G\|_u : G \in Y^\perp\}, \end{aligned}$$

be the distances from an element $F \in \mathrm{Lip}_0 X$ to the subspace $Y^\perp$ with respect to the Lipschitz and the uniform norm, respectively.

An element $G_1 \in Y^\perp$ such that $\|F - G_1\|_L = d_L(F,\ Y^\perp)$ is called an L *— nearest point* to F in $Y^\perp$ and en element $G_2 \in Y^\perp$ for which $\|F - G_2\|_u = d_u(F, Y^\perp)$ is called a u *— nearest point* to F in $Y^\perp$.

THEOREM 3. (a) *The equalities*

$$d_L(F, Y^\perp) = \| F|_Y \|_L \,; \quad d_u(F, Y^\perp) = \| F|_Y \|_u \tag{20}$$

hold, for every $F \in \mathrm{Lip}_0 \ X$;

(b) *A function* $G_1 \in Y^\perp$ *is an L-nearest point to* F *in* $Y^\perp$ *if and only if* $G_1 = F - H_1$, *for a function* $H_1 \in E_L(F|_Y)$;

(c) *A function* $G_2 \in Y^\perp$ *is a u-nearest point to* F *in* $Y^\perp$ *if and only iff* $G_2 = F - H_2$, *for a function* $H_2 \in E_u(F|_Y)$.

Proof. The first equality in (20) was proved in [5], Theorem 2 and Lemma 1. To prove the second one, observe that

$$\| F|_Y \|_u = \| F|_Y - G|_Y \|_u \leqslant \| F - G \|_u,$$

for every $G \in Y^\perp$ and, taking the infimum with respect to $G \in Y^\perp$, one obtains the inequality

$$\| F|_Y \|_u \leqslant d_u(F, Y^\perp).$$

On the other hand

$$d_u(F, Y^\perp) \leqslant \| F - (F - H) \|_u = \| H \|_u = \| F|_Y \|_u,$$

for every $H \in E_u(F|_Y)$, implying $\| F|_Y \|_u \geqslant d_u(F, Y^\perp)$.

The fact that every L-nearest point G_1 to F in $Y^\perp$ has the form $G_1 = F - H_1$, for an $H_1 \in E_L(F|_Y)$ was proved in [6], Lemma 1.

The assertion (c) can be proved in a similar way.

From Theorem 3 it follows that

$$\overset{L}{P}_{Y^\perp}(F) = F - E_L(F|_Y)\,; \quad \overset{u}{P}_{Y^\perp}(F) = F - E_u(F|_Y), \tag{21}$$

and, taking into account relations (13) and (14), we find

$$\overset{u}{P}_{Y^\perp}(F) \cap \overset{L}{P}_{Y^\perp}(F) \supset F - T(E_L(F|_Y)), \tag{22}$$

for every $F \in \mathrm{Lip}_0 X$.

The following Corollary holds:

COROLLARY 1. (a) *For every* $F \in \mathrm{Lip}_0 \ X$ *there exists a function* $G_0 \in Y^\perp$ *which is simultaneously an L-nearest point and a u-nearest point for* F *in* $Y^\perp$;

(b) *If* $F \in \mathrm{Lip}_0 \ X$ *is such that* $\| F|_Y \|_L = \| F|_Y \|_u$, *then*

$$d_L(F, Y^\perp) = d_u(F, Y^\perp)\,; \tag{23}$$

(c) *The function* $p : \mathrm{Lip}_0 X \to Y^{\perp}$ *given by*

$$p(F) = F - e(F|_Y) \tag{24}$$

is a common linear and continuous selection of the metric projection operators $\overset{L}{P}_{Y^{\perp}}$ *and* $\overset{u}{P}_{Y^{\perp}}$.

(d) *The subspace* $W = \{e(F|_Y) : F \in \mathrm{Lip}_0 X\}$ *is the algebraic and topological complement of* $Y^{\perp}$ *in* $\mathrm{Lip}_0 X$.

Furthermore, the assertions (c) *and* (d) *are equivalent.*

Proof. The assertions (a) and (b) are immediate consequences of Theorem 3. It is obvious that the selection p, given by (24) is linear and continuous (both in the Lipschitz and in the uniform norms). The fact that W is the complement of $Y^{\perp}$ (with respect to the Lipschitz norm) was proved in [9], Corollary 7, but it follows also from Theorem 2.2 in [2]. The same theorem implies the equivalence of the assertions (c) and (d). Every function $F \in \mathrm{Lip}_0\ X$ can be uniquely written in the form $F = G + H$, with $G \in Y^{\perp}$ and $H \in W$, where the functions G and H can be explicitly given by

$$\begin{aligned} G(x) &= 0 && \text{for } x \in [a, b], \\ &= F(x) - F(a) && \text{for } c \leqslant x < a, \\ &= F(x) - F(b) && \text{for } b < x \leqslant d, \end{aligned}$$

and

$$\begin{aligned} H(x) &= F(x) && \text{for } x \in [a, b], \\ &= F(a) && \text{for } c \leqslant x < a, \\ &= F(b) && \text{for } b < x \leqslant d, \end{aligned}$$

Remarks. 1° The assertions (c) and (d) from Corollary 1 are illustrations to the Theorem 2.2 in [2] (assertions (1), (2) and (4)).

2° The kernels of the applications $\overset{L}{P}_{Y^{\perp}}$ and $\overset{u}{P}_{Y^{\perp}}$ are given by $\ker\ \overset{L}{P}_{Y^{\perp}} = \{F \in \mathrm{Lip}_0 X : \|F\|_L = \|F|_Y\|_L\}$, respectively $\ker\ \overset{u}{P}_{L^{\perp}} = \{F \in \mathrm{Lip}_0 X\ : \|F\|_u = \|F|_Y\|_u\}$, and W is a closed subspace of $\mathrm{Lip}_0 X$ contained in $\ker\ \overset{L}{P}_{Y^{\perp}} \cap \ker\ \overset{u}{P}_{Y^{\perp}}$.

3° As $Y^{\perp}$ and W are closed subspaces of $\mathrm{Lip}_0\ X$, which is a Banach space with respect to both of Lipschitz and uniform norms, it follows that their algebraic sum is also topological (a consequence of the open mapping theorem)

Example. Let $Y = [a, b] = [1, 3]$, $X = [-2{,}5]$, $x_0 = 2$.

The function

$$f(x) = -x + 2 \text{ for } x \in [1, 2],$$
$$= 2x - 4 \quad \text{for } x \in (2, 3]$$

is in $\mathrm{Lip}_0 Y$ and $\|f\|_L = \|f\|_u = 2$

The functions (16) from Theorem 2 are

$$F_1(x) = 2x - 1,\ x \in [-2, 1] \text{ and } \quad F_2(x) = -2x + 3,\ x \in [-2, 1]$$
$$= f(x),\ x \in (1, 3] \qquad\qquad = f(x),\ x \in (1, 3]$$
$$= -2x + 8,\ x \in (3, 5] \qquad\qquad = 2x - 4,\ x \in (3, 5]$$

Every function $F \in E_L(f)$ verifies the inequalities

$$F_1(x) \leqslant F(x) \leqslant F_2(x), \quad x \in [-2, 5].$$

implying that

$$T(F_1) \leqslant T(F) \leqslant T(F_2).$$

We have

$$\begin{aligned} T(F_2)(x) = \bar{F}_2(x) &= 2, & x &\in [-2, 1/2] \\ &= -2x + 3, & x &\ (1/2, 1) \\ &= f(x), & x &\in [1, 3] \\ &= 2, & x &\in (3, 5]. \end{aligned}$$

Let

$$\begin{aligned} H(x) &= 2, & x &\in [-2, 3/4] \\ &= -4x + 5, & x &\in (3/4, 1] \\ &= \bar{F}_2(x), & x &\in (1, 5]. \end{aligned}$$

Then $\|H\|_L = 4$ and $\|H\|_u = 2$ so that $H \notin T(E_L(f))$ but $H \in E_u(f)$. This example shows that the inclusion (11) can be strict.

Also

$$Y^\perp = \{G \in \mathrm{Lip}_0[-2,5] : G|_{[1,3]} = 0\}.$$

If $F \in \mathrm{Lip}_0[-2, 5]$ is such that $F|_{[1,3]} = f$ then $\|F|_Y\|_L = \|F|_Y\|_L$ and, consequently,

$$d_L(F, Y^\perp) = d_u(F, Y^\perp) = 2,$$

showing that condition (23) from Corollary 1 is fulfilled.

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