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Dedicated to Professor Iulian Coroian on his 60th anniversary

On the derivative-interpolating spline functions of even degree

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The aim of the present note is to show that the derivative-interpolating spline functions, considered in some recent papers ([2],[6]) are in fact primitives, chosen in an appropriate way, of interpolating natural spline functions ([3]).

This fact allows to derive some properties of derivative-interpolating spline functions of even order from the corresponding properties of interpolating natural spline functions.

Let $m, n \in \mathbb{N}$, $m \leq n$, and $[a, b]$ an interval contained in $\mathbb{R}$. Let also

$$\Delta_n := a < x_1 < x_2 < \cdots < x_n < b \tag{1}$$

be a fixed partition of the interval $[a, b]$.

Definition 1 *A function $s : [a, b] \to \mathbb{R}$ verifying the conditions*
1⁰ $s \in C^{2m-2}[a, b]$,
2⁰ $s \in \mathcal{P}_{m-1}$ on (a, x_1) and (x_n, b),
3⁰ $s \in \mathcal{P}_{2m-1}$ on (x_i, x_{i+1}), $i = 1, 2, ..., n-1$,
where $\mathcal{P}_k$ stands for the set of polynomials of degree at most k $(k \in \mathbb{N})$, is called an interpolating natural spline function (associated to the partition Δ_n).

Denoting by $\mathcal{S}_{2m-1}(\Delta_n)$ the set of all interpolating natural spline functions, one sees that $\mathcal{S}_{2m-1}(\Delta_n)$ is an n-dimensional subspace of the linear space $C^{2m-2}[a, b]$.

If $s \in \mathcal{S}_{2m-1}(\Delta_n)$ then s is of the form

$$s(x) = \sum_{i=0}^{m-1} B_i x^i + \sum_{k=1}^{n} b_k (x - x_k)_+^{2m-1}, \tag{2}$$

where

(3) $$\sum_{k=1}^{n} b_k x_k^j = 0,\ j = 0, 1, ..., m-1$$

(see [3]).

If $Y = (y_1, y_2, ..., y_n)$ is a fixed vector in $\mathbb{R}^n$ then there exists exactly one function $s_Y \in \mathcal{S}_{2m-1}(\Delta_n)$ verifying the equalities:

(4) $$s_Y(x_i) = y_i, \quad i = 1, 2, ..., n$$

see ([3]).

Let

(5) $$H_2^m[a,b] \ :=\left\{f \in C^{m-1}[a,b] : f^{(m-1)} \text{ is absolutely continuous and } f^{(m)} \in L_2[a,b]\right\}$$

and

(6) $$H_{2,Y}^m[a,b] := \{f \in H_2^m[a,b] : f(x_i) = y_i, \quad i = 1, 2, ..., n\}.$$

Then

(7) $$H_{2,Y}^m[a,b] \cap \mathcal{S}_{2m-1}(\Delta_n) = \{s_Y\}$$

and the functional $J : H_{2,Y}^m[a,b] \to \mathbb{R}_+$ given by

(8) $$J(f) = \int_a^b \left[f^{(m)}(x)\right]^2 dx, \quad f \in H_{2,Y}^m[a,b]$$

has the property

(9) $$\min\left\{J(f) : f \in H_{2,Y}^m[a,b]\right\} = J(s_Y)$$

i.e. the minimum of the L_2-norms of the derivatives of order m of the functions in $H_{2,Y}^m[a,b]$ is attained at the interpolating natural spline function s_Y ("*the minimal norm property*").

Also, for each $f \in H_{2,Y}^m[a,b]$ the inequality

(10) $$\left\|f^{(m)} - s_Y^{(m)}\right\|_2 \le \left\|f^{(m)} - s\right\|_2$$

holds for any $s \in \mathcal{S}_{2m-1}(\Delta_n)$ ("*the best approximation property*") (see [3]).

Now, we shall introduce the derivative-interpolating spline functions of even order $2m$, having properties similar to (9) and (10). (see [2]).

Let $m, n \in \mathbb{N}, m \le n+1$, and Δ_n the partition (1) of the interval $[a,b]$.

Definition 2 *([2]). A function $S : [a,b] \to \mathbb{R}$ is called a natural spline function of order $2m$ if it verifies the conditions:*

1^0 $S \in C^{2m-1}[a,b]$,

2^0 $S \in \mathcal{P}_m$ on (a,x_1) and (x_n,b),

3^0 $S \in \mathcal{P}_{2m}$ on (x_i,x_{i+1}), $i = 1,2,...,n-1$.

The set of all interpolating natural spline functions of order $2m$ will be denoted by $\mathcal{S}_{2m}(\Delta_n)$. It follows that $\mathcal{S}_{2m}(\Delta_n)$ is an $(n+1)$-dimensional subspace of $C^{2m-1}[a,b]$. (see [6]).

If $\overline{Y} = (y_\alpha, y_1, ..., y_n)$ is a fixed vector in $\mathbb{R}^{n+1}$ then there exists only one function $S_{\overline{Y}} \in \mathcal{S}_{2m}(\Delta_n)$ verifying the conditions

$$
\begin{aligned}
S_{\overline{Y}}(\alpha) &= y_\alpha \\
S'_{\overline{Y}}(x_i) &= y_i, \quad i = 1,2,...,n,
\end{aligned} \tag{11}
$$

where x_i, $i \doteq \overline{1,n}$ are the nodes of the partition Δ_n given by (??).

The function $S_{\overline{Y}} \in \mathcal{S}_{2m}(\Delta_n)$ verifying the condition (??) is called the *derivative-interpolating spline of even order* $2m$ associated to the vector $\overline{Y}$ and to the partition Δ_n.

Any function $S \in \mathcal{S}_{2m}(\Delta_n)$ admits the representation

$$
S(x) = \sum_{i=0}^{m} A_i x^i + \sum_{k=1}^{n} a_k (x - x_k)_+^{2m}, \tag{12}
$$

$$
\sum_{k=0}^{n} a_k x_k^j = 0, \quad j = 0,1,...,m-1 \tag{13}
$$

see ([2] or [6]).

Let

$$
H_2^{m+1}[a,b] \; : \; = \left\{ f \in C^m[a,b], f^{(m)} \text{ is absolutely continuous and } f^{(m+1)} \in L_2[a,b] \right\} \tag{14}
$$

and

$$
H_{2,\overline{Y}}^{m+1} \; : \; = \left\{ g \in H_2^{m+1}[a,b] : g(\alpha) = y_\alpha \text{ and } g'(x_i) = y_i, \; i = 1,2,...,n \right\}. \tag{15}
$$

Then (see [2])

$$H_{2,\overline{Y}}^{m+1}[a,b] \cap \mathcal{S}_{2m}(\Delta_n) = \{S_{\overline{Y}}\} \tag{16}$$

and the functional $J_\alpha : H_{2,\overline{Y}}^{m+1}[a,b] \to \mathbb{R}_+$ defined by

$$J_\alpha(g) = \int_a^b \left[g^{(m+1)}(x)\right]^2 dx \tag{17}$$

attains its minimum at the function $S_{\overline{Y}}$:

$$\min\left\{J_\alpha(g) : g \in H_{2,\overline{Y}}^{m+1}[a,b]\right\} = J_\alpha(S_{\overline{Y}}) \tag{18}$$

Also, the inequality

$$\left\|g^{(m+1)} - S_{\overline{Y}}^{(m+1)}\right\|_2 \le \left\|g^{m+1} - S^{(m+1)}\right\|_2, \tag{19}$$

holds for any $S \in \mathcal{S}_{2m}(\Delta_m)$.

The relations (18) and (19) (called "the minimal norm property" and "the best approximation property", respectively) are proved in [2], following a way similar to that used to prove the corresponding properties for interpolating natural spline functions (see [6], Theorems 3 and 4).

We mention that the derivative-interpolating spline functions of order $2m$ have been successfully used for the numerical solution of boundary value problems (Cauchy problems) for differential equations with modified argument ([6]). Spline functions of degree 5 (particular cases of p-derivative-interpolating spline functions for $p = 2$ and $m = 2$) were used in [8] to solve a singularly perturbed bilocal problem.

In the next we shall show that the functions used in [8] are spline functions obtained by integrating the interpolation natural cubic spline functions.

Lemma 3 *Let $s \in \mathcal{S}_{2m-1}(\Delta_n)$, $\alpha \in [a,b]$ fixed and*

$$\hat{I}(s) := \left\{\int_\alpha^x s(t)\,dt + C : C \in \mathbb{R}\right\}. \tag{20}$$

Then every $S \in \hat{I}(s)$ belongs to $\mathcal{S}_{2m}(\Delta_n)$.

Proof. By (2)

$$s(x) = \sum_{i=0}^{m-1} B_i x_i + \sum_{k=1}^{n} b_k (x - x_k)_+^{2m-1},$$

with

$$\sum_{k=1}^{n} b_k x_k^j = 0, \quad j = 0, 1, 2, ..., m-1.$$

Consequently

$$\begin{aligned} S(x) &= \int_\alpha^x s(t)\,dt + C_0 = C_0 + \sum_{i=0}^{m-1} \tfrac{B_i}{i+1} x^{i+1} + \sum_{i=1}^{n} \tfrac{b_k}{2m} (x - x_k)_+^{2m} = \\ &= \sum_{i=0}^{m} A_i x^i + \sum_{k=1}^{n} a_k (x - x_k)_+^{2m} \end{aligned}$$

where $A_0 = C_1$, $C_1 = C_0 - \sum_{1=0}^{m-1} \frac{B_i}{i+1}\alpha^{i+1} - \sum_{k=1}^{n} \frac{b_k}{2m} (\alpha - x_k)_+^{2m}$, $A_i = \frac{B_{i-1}}{i}$, $i = 1, 2, ..., m$; $a_k = \frac{b_k}{2m}$, $k = 1, 2, ..., n$ and $\sum_{k=1}^{n} a_k x_k^j = 0$, $j = 0, 1, ..., m-1$. Taking into account (12) and (13) it follows $S \in \mathcal{S}_{2m}(\Delta_n)$. □

Lemma 4 *Let $f \in H_2^m[a,b]$ and*

(21) $$\hat{I}(f) : \left\{ \int_\alpha^x f(t)\,dt + c : c \in \mathbb{R} \right\}.$$

Then $g \in \hat{I}(f)$ if and only if $g \in H_2^{m+1}[a,b]$.

Proof. Obviously that $\hat{I}(f) \subset C^m[a,b]$, and if $g \in \hat{I}(f)$ then $g^{(m)} = f^{(m-1)}$ (absolutely continuous on $[a,b]$) and $g^{(m+1)} = f^{(m)} \in L_2[a,b]$, showing that $g \in H_2^{m+1}[a,b]$.

If $g \in H_2^{m+1}[a,b]$ then $g' \in H_2^m[a,b]$ so that

$$g(x) = \int_\alpha^x g'(t)\,dt + g(\alpha)$$

i.e. $g \in \hat{I}(g')$. □

Lemma 5 *Let* $Y = (y_1, y_2, ..., y_n) \in \mathbb{R}^n$ *and* $\overline{Y} = (y_\alpha, Y) = (y_\alpha, y_1, y_2, ..., y_n) \in \mathbb{R}^{n+1}$. *Then the operator*

$$I_\alpha : H^m_{2,Y}[a,b] \to H^{m+1}_{2,Y}[a,b]$$

defined by

(22) $$I_\alpha(f)(x) = \int_\alpha^x f(t)\,dt + y_\alpha, \quad x \in [a,b]$$

is bijective.

Proof. Obviously that

$$\begin{aligned} I'_\alpha(f)(x_i) &= f(x_i) = y_i, \quad i = 1,2,...,n \\ I_\alpha(f)(\alpha) &= y_\alpha \end{aligned}$$

showing that $I_\alpha(f) \in H^{m+1}_{2,\overline{Y}}[a,b]$, for every $f \in H^m_{2,Y}[a,b]$.

If $f_1, f_2 \in H^{m+1}_{2,Y}[a,b]$ and $I_\alpha(f_1) = I_\alpha(f_2)$ then

$$\int_\alpha^x f_1(t)\,dt = \int_\alpha^x f_2(t)\,dt$$

for all $x \in [a,b]$, implying $f_1(t) = f_2(t)$ for all $t \in [a,b]$, i.e.

I_α is injective.

Let $g \in H^{m+1}_{2,\overline{Y}}[a,b]$. Then $g' \in H^m_{2,Y}[a,b]$, and $I_\alpha(f) = g$, for $f = g'$, showing that I_α is surjective, too. □

Lemma 6 $I_\alpha(s_Y) = S_{\overline{Y}}$

Proof. By Lemmas 1 and 2.

$$I_\alpha(s_Y) \in \mathcal{S}_{2m}(\Delta_n) \cap H_2^{m+1}(\Delta_n) = \{S_{\overline{Y}}\}$$

so that

$$I_\alpha(s_Y) = S_{\overline{Y}}. \qquad \square$$

Lemma 7 $J_\alpha = J \circ I_\alpha^{-1}$

Proof. For $g \in H^{m+1}_{2,\overline{Y}}[a,b]$ we have

$$\begin{aligned} J_\alpha(g) &= \int_a^b \left[g^{(m+1)}(x)\right]^2 dx = \int_a^b \left[(g')^{(m)}(x)\right]^2 dx = \\ &= \int_a^b \left(\left[I_\alpha^{-1}(g)(x)\right]^{(m)}\right)^2 dx = J\left(I_\alpha^{-1}(g)\right) = \left(J \circ I_\alpha^{-1}\right)(g) \quad \square. \end{aligned}$$

Theorem 8 *a) If the functional $J : H^m_{2,Y}[a,b] \to \mathbb{R}_+$ attains its minimal value at the spline function $s_Y \in H^m_{2,Y}[a,b] \cap \mathcal{S}_{2m-1}(\Delta_n)$ then the functional $J_\alpha : H^{m+1}_{2,\overline{Y}}[a,b] \to \mathbb{R}_+$ attains its minimal value at $S_{\overline{Y}} \in H^{m+1}_{2,\overline{Y}}[a,b] \cap \mathcal{S}_{2m}(\Delta_n)$;*

b) If $f \in H^m_{2,Y}[a,b]$ and

$$\left\|f^{(m)} - s_Y^{(m)}\right\|_2 \leq \left\|f^{(m)} - s^{(m)}\right\|_2,$$

for any $s \in \mathcal{S}_{2m-1}(\Delta_n)$ then

$$\left\|I_\alpha^{(m+1)}(f) - S_{\overline{Y}}^{(m+1)}\right\|_2 \leq \left\|I_\alpha^{(m+1)} - S^{(m+1)}\right\|_2,$$

for any $S \in \mathcal{S}_{2m}(\Delta_n)$.

Proof. a) For $g \in H^{m+1}_{2,\overline{Y}}[a,b]$ we have

$$\left\|g^{(m+1)}\right\|_2^2 - \left\|S_{\overline{Y}}^{(m+1)}\right\|_2^2 = \left\|(g')^{(m)}\right\|_2^2 - \left\|s_Y^{(m)}\right\|_2^2 \geq 0,$$

because $g' \in H^m_{2,Y}$. Also $\left\|f^{(m)}\right\|_2^2 \geq \left\|s_Y^{(m)}\right\|_2^2$ for any $f \in H^m_{2,Y}$ (see (??)). Therefore

$$\min\left\{J_k(g) : g \in H^{m+1}_{2,\overline{Y}}\right\} = I_a(S_{\overline{Y}}).$$

b)

$$\begin{aligned} \left\|I_\alpha^{(m+1)}(f) - S_{\overline{Y}}^{(m+1)}\right\|_2 &= \left\|(I'_\alpha)^{(m)}(f) - s_Y^{(m)}\right\|_2 = \left\|f^{(m)} - s_Y^{(m)}\right\|_2 \leq \\ &\leq \left\|f^{(m)} - s^{(m)}\right\|_2, \end{aligned}$$

for any $s \in \mathcal{S}_{2m-1}(\Delta_n)$. Since, by Lemma 1,

$$S(x) = \int_\alpha^x s(t)\,dt + C \in \mathcal{S}_{2m}(\Delta_n)$$

it follows $s^{(m)} = S^{(m+1)}$ and $f^{(m)} = (I_\alpha(f))^{(m+1)}$ so that

$$\left\| I_\alpha^{(m+1)}(f) - s_Y^{(m+1)} \right\|_2 \leq \left\| (I_\alpha(f))^{(m+1)} - S^{(m+1)} \right\|_2$$

for any $S \in \mathcal{S}_{2m}(\Delta_n)$. ■.

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