Extension of bilinear functionals and best approximation in 2-normed spaces

EXTENSION OF LIPSCHITZ FUNCTIONS AND BEST APPROXIMATION

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The aim of this paper is to present various extension results for Lipschitz functions and to put in evidence their relevance for some best approximation problems in spaces of Lipschitz functions.

1. Extension theorems for Lipschitz functions

Let (X, d) be a metric space having at least two distinct points. *A* function $f : X \to \mathbb{R}$ is called *Lipschitz* if there exists a number $L \geq 0$ such that

$$|f(x) - f(y)| \leq Ld(x, y) \ . \tag{1.1}$$

for all $x, y \in X$.

The smallest number $L \geq 0$ for which (1.1) holds is called the *Lipschitz norm* of the function f and it is denoted by $\|f\|$. It can be calculated by the formula

$$\|f\| = \sup \left\{ \frac{|f(x) - f(y)|}{d(x, y)} : x, y \in X,\ x \neq y \right\} . \tag{1.2}$$

Denote by Lip X the set of all real-valued Lipschitz functions on X. With respect to the pointwise operations of addition and multiplication by scalars Lip X is a vector space and (1.2) is a seminorm on Lip X (constant functions have Lipschitz norm zero).

In order that the formula (1.2) define a proper norm we fix a point $x_0 \in X$ (if X is a vector space, then one usually takes $x_0 = 0$) and consider the space

$$\mathrm{Lip}_0\ X = \{f \in \mathrm{Lip}\ X : f(x_0) = 0\} \ . \tag{1.3}$$

In this case (1.2) is a norm on $\mathrm{Lip}_0 X$ and $(\mathrm{Lip}_0 X, \|\|)$ is a Banach space, even a conjugate one. If the metric space X is bounded then the product of two Lipschitz functions is a Lipschitz function too and

$$\|f \cdot g\| \leq \|f\| \, \|g\| \ . \tag{1.4}$$

It follows that, in this case, $\mathrm{Lip}_0 X$ is a Banach algebra too. The properties of Banach spaces and Banach algebras of Lipschitz functions have been intensively studied in the papers of J. A. Johnson [57, 58, 59], K. de Leeuw [67], W. E. Mayer-Wolf [77], D. R. Sherbert [130], N. Weaver [144, 145, 146, 147].

As in the cases of spaces of continuous function or in the case of normed spaces an essential tool in developing the theory of spaces of Lipschitz functions is an extension theorem for Lipschitz functions. In the case of continuous functions we have Tietze's extension theorem in its scalar or vector versions (see [34] and [33]) and in the case of normed spaces we have Hahn-Banach extension theorem.

Theorem 1.1 ([78])**.** *Let (X, d) be a metric space and Y a subset of Y. Then any real Lipschitz function on Y admits at last one extension to X with the same Lipschitz constant.*

If $f \in \mathrm{Lip}\ Y$ with Lipschitz constant $L \geq 0$ then two such extension are given by the formulae

$$F_1(x) = \sup \{f(y) - L \cdot d(x, y) : y \in Y\} , \quad x \in X , \tag{1.5}$$

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and

$$F_2(x) = \inf\{f(y) + L \cdot d(x,y) : y \in Y\}, \quad x \in X. \tag{1.6}$$

Any other extension F with Lipschitz constant L satisfies the inequalities

$$F_1(x) \le F(x) \le F_2(x). \tag{1.7}$$

In particular there exists at least one extension F of f verifying

$$\|F\| = \|f\|, \tag{1.8}$$

where, in (1.8), $\|f\|$ is the Lipschitz norm of f in Lip Y *and $\|F\|$ is the Lipschitz norm of F in* Lip X.

The problem of the extension of vector-valued functions is more complicated and delicate. The norm preserving extension is not always possible and sometimes is possible only by increasing the Lipschitz constant.

A metric space (X,d) is said to have the *binary intersection property* if any collection of mutually intersecting closed balls in X has nonvoid intersection. This property was first introduced by L. Nachbin [97] in connection with the vector version of Hahn-Banach extension property for operators. A Banach space Y is said to have the *Hahn-Banach extension property* if for any normed space X and any subspace Z of X every continuous linear operator $A \in L(Z,Y)$ admits a norm-preserving extension $B \in L(X,Y)$. *L.* Nachbin [97] proved that Y has the Hahn-Banach extension property if and only if it has the binary intersection property and in its turn, this happens exactly when Y is isometrically isomorphic to a space $C(T)$, with T an extremally disconnected compact space (see J. L. Kelley [63] for the real case and M. Hasumi [48] for the complex one).

Remark that in the real case the fact that the compact T is extremally disconnected is equivalent to the fact that $C(T)$ is a complete lattice.

The binary intersection property is also relevant in the case of the extension of Lipschitz functions, but we need a property considered by M. Kirszbraun [64] (see also F. Valentine [140]). A pair $(X,d_1),(Y,d_2)$ of metric spaces is said to have *property* (K) provided

$$\bigcap_{i\in I} B_X(x_i,r_i) \neq \emptyset \tag{1.9}$$

implies

$$\bigcap_{i\in I} B_Y(y_i,r_i) \neq \emptyset, \tag{1.10}$$

for all families of closed balls $\{B_X(x_i,r_i) : i \in I\}$ and $\{B_Y(y_i,r_i) : i \in I\}$ in X and Y, respectively, such that

$$d_2(y_i,y_j) \le d_1(x_i,x_j) \quad (\forall i,j \in I). \tag{1.11}$$

One can consider also the problem of the extension of Lipschitz or Lipschitz-Hölder maps from subsets of X to Y to the whole X.

A map $f : Z \to Y$ form a subset Z of X is called a *Lipschitz-Hölder map* of order α if

$$d_2(f(x_1), f(x_2)) \le L d_1(x_1,x_2)^{\alpha}, \quad x_1,x_2 \in Z. \tag{1.12}$$

If $\alpha = 1$ and $L < 1$ then f is called a *contraction* from Z to Y. If $\alpha = 1$ and $L = 1$ then f is called *nonexpansive*. It is called an *isometry* provided

$$d_2(f(x_1), f(x_2)) = d_1(x_1,x_2), \quad x_1,x_2 \in Z. \tag{1.13}$$

A pair X,Y of metric spaces is said to have the extension property for Lip_α-*maps* (*nonexpansive maps*, *isometries*) provided any Lip_α *−map* (nonexpansive maps, isometry) f from a subset Z of X

to Y admits a Lip_α-extension with the same Lipschitz constant (an extension which is nonexpansive, an isometry).

Supposing $0 < \alpha \le 1$ then, replacing the metric d_1 by $L \cdot d_1^\alpha$, one can suppose that f is always nonexpansive, i.e. the extension problem for Lipschitz-Hölder maps of order α with $0 < \alpha \le 1$, reduces to the problem of the extension of nonexpansive maps.

Theorem 1.2 (M. Kirszbraun [64]). *The pair of metric spaces (X, d_1), (Y, d_2) has the extension property for nonexpansive maps if and only if the pair (X, Y) has property (K).*

To characterize metric spaces Y with the extension property for nonexpansive maps for every other metric space X we need a further notion. A metric space (Y, d_2) is called *metrically convex* if $x, y \in Y$ and $0 < \lambda < 1$ imply the existence of a point $z \in Y$ with the property $d_2(x, z) = \lambda d_2(x, y)$ and $d_2(y, z) = (1 - \lambda) d_2(x, y)$. Obviously that any normed linear space is metrically convex.

Theorem 1.3 (M. Kirszbraun [64], see also [148]). *A metric space Y has the extension property for nonexpansive maps for every metric space X if and only if Y is metrically convex and has the binary intersection property. In this case (X, Y) has the Lip_α extension property for every metric space X and $0 < \alpha \le 1$.*

Theorem 1.4 ([148, 79, 126]). *If H is a Hilbert space then the pair (H, H) has the extension property for nonexpansive maps and Lip_α-maps, $0 < \alpha \le 1$.*

Theorem 1.5 ([148]). *If H is a Hilbert space then (H, H) has the isometric extension property if and only if H is finite dimensional.*

The following results show that, in some sense, the Hilbert space setting is the most general in the class of Banach spaces for which the contraction extension property holds.

Theorem 1.6 ([128]). *If X is a strictly convex Banach space then the pair (X, X) has the extension property for nonexpansive maps if and only if X is a Hilbert space.*

Theorem 1.7 ([128]). *If X, Y are Banach spaces, with Y strictly convex and of dimension at least 2, then (X, Y) has the extension property for nonexpansive maps if and only if X and Y are Hilbert spaces.*

It can be shown that the finite dimensional space l_n^p, $1 < p < \infty$, does not have the property (K) for $n > 1$ and $p \ne 2$. (see [148], p.50).

Other extension results for Banach valued Lipschitz functions can be found in [3, 4, 5, 39, 69, 127, 128, 141]. A good account of all these problems is given in the book [148].

As in the case of norm preserving extension theorem for linear maps (Hahn-Banach theorem) it is natural to study the existence of Lipschitz extensions preserving also other properties of the function. Results of this kind are known for convex and starshaped functions.

Theorem 1.8 ([20]). *Let X be a normed space and Y a convex subset of X. Then any convex function $f \in \mathrm{Lip}\, Y$ admits a norm-preserving convex extension $F \in \mathrm{Lip}\, X$.*

The maximal extension F_2 given by (1.6) is convex and there exists also a minimal convex norm preserving extension $\overline{F} \in \mathrm{Lip}\, X$ such that

$$\overline{F}(x) \le F(x) \le F_2(x), \quad \forall x \in X, \tag{1.14}$$

for any other convex norm preserving extension F of f.

A proof of the fact that the maximal extension F_2 of a convex Lipschitz function is convex appear also in [50] (see also [52]).

A result, similar to that proved Theorem 1.8, but for starshaped Lipschitz functions defined on starshaped subsets of Banach spaces, was proved by C. Mustăţa [87].

W. Rzymowski [122] has found the following condition in order that a function admit a convex Lipschitz extension.

Theorem 1.9. *Let $\Omega \subset \mathbb{R}^n$ be open non-empty convex and let $f : \mathrm{bd}(\Omega) \to \mathbb{R}$. The function f admits a convex extension $F : \mathbb{R}^n \to \mathbb{R}$ satisfying the Lipschitz condition with constant L if and only if the following condition is fulfilled*

$$f(z) - \frac{f(x) + f(y)}{2} \le L \left\| z - \frac{x+y}{2} \right\| , \tag{1.14a}$$

for all $x, y, z \in \mathrm{bd}(\Omega)$.

Some extension results for Lipschitz functions on p-normed spaces were proved by W. Ruess [121]. A function $\|\cdot\| : X \to \mathbb{R}$ defined on a real vector space X is called a p-norm on X, $0 < p \le 1$, provided it verifies the axioms
p1) $\|x\| \ge 0$, $\|x\| = 0 \iff x = 0$
p2) $\|x + y\|^p \le \|x\|^p + \|y\|^p$
p3) $\|\lambda x\| = |\lambda| \|x\|$,
for all $x, y \in X$ and $\lambda \in \mathbb{R}$.
W. Ruess [121] considered as a dual for X the cone

$$\begin{aligned} C_X^p = \{h : X \to \mathbb{R}_+ : h(x+y) \le h(x) + h(y) \\ \text{and } h(\lambda x) = |\lambda|^p h(x), \text{ for all } x, y \in X \text{ and } \lambda \in \mathbb{R}\} . \end{aligned} \tag{1.15}$$

in $\mathrm{Lip}_0\, X$.
In this case one can prove the following extension result.

Theorem 1.10 ([121])**.** *If Y is a linear subspace of the p - normed space $(X, \|\cdot\|)$ and $h \in C_Y^p$ then the function $H : X \to \mathbb{R}_+$ given by*

$$H(x) = \inf \{h(y) + \|h\| \cdot \|x - y\| : y \in Y\} \tag{1.16}$$

is a norm preserving extension of h in C_X^p, i.e.

$$H \in C_X^p, \quad H|_Y = h \text{ and } \|H\| = \|h\| . \tag{1.17}$$

where $\|H\|$ and $\|h\|$ stand for the Lipschitz norms (with respect to the p-norm $\|\cdot\|$) of H and h, respectively.

These extension results can be used as a base for developing a duality theory with Lipschitz functions instead of continuous linear functional. This has been done in [123, 124, 125].

For instance K. Schnatz [124, 125] considers a metric linear space X with a translation invariant metric d and takes as dual space to X the space o-$\mathrm{Lip}_0\, X$ formed of all odd Lipschitz functions on X, called the non-linear dual space of X, and shows that known results in the linear case as Alaoglu-Bourbaki, Krein-Milman a.o. theorems, hold in this nonlinear case, too. Similar ideas appear in [123], but working with Lipschitz functions on a Banach space.

Another way to obtain a norm on a space of Lipschitz functions is to consider the vector space of bounded Lipschitz functions, denoted by $\mathrm{BLip}\, X$ (X a metric space) and equip it with the norm

$$\|f\|_s = \|f\|_L + \|f\|_\infty , \tag{1.18}$$

where $\|f\|_L$ stands for the Lipschitz norm (1.2) and $\|f\|_\infty$ for the uniform norm. Again $\mathrm{BLip}\, X$ is a Banach space with respect to the norm (1.18) (see [57]), and the following extension result holds true.

Theorem 1.11 ([93])**.** *Let (X, d) be a metric space and Y a subset of X. Then any $f \in \mathrm{BLip}\, Y$ admits a norm preserving extension $F \in \mathrm{BLip}\, X$, i.e. satisfying*

$$F|_Y = f \text{ and } \|F\|_s = \|f\|_s . \tag{1.19}$$

Two such extensions are given by

$$\overline{F}_1(x) = \begin{cases} F_1(x) & \text{if} \quad F_1(x) \le \|f\|_\infty \\ \|f\|_\infty & \text{if} \quad F_1(x) > \|f\|_\infty \end{cases} \tag{1.20}$$

and

$$\overline{F}_2(x) = \begin{cases} F_2(x) & \text{if} \quad F_2(x) \le -\|f\|_\infty \\ -\|f\|_\infty & \text{if} \quad F_2(x) < -\|f\|_\infty \end{cases} \tag{1.21}$$

where F_1, F_2 are the extremal norm preserving extensions of f given by (1.5) and (1.6) respectively.

Remark 1.12. The extensions $\overline{F}_1$, $\overline{F}_2$ given by (1.20) and (1.21), preserve both the Lipschitz and uniform norms of the function f, implying $\|\overline{F}_1\|_m = \|f\|_m = \|\overline{F}_2\|_m$, where $\|\cdot\|_m$ denotes the norm on $\mathrm{BLip}\, X$, equivalent to (1.18), given by

$$\|g\|_m = \max\{\|g\|_L, \|g\|_\infty\}, \quad g \in \mathrm{BLip}\, X. \tag{1.22}$$

The paper [93] contains also similar extension results for Hölder-Lipschitz functions of order $\alpha, 0 < \alpha < 1$. Furthermore, based on this extension property, one gives an algorithm for finding the global minimum of a function $F \in \mathrm{Lip}_\alpha X$ for a compact metric space (X, d). One starts with the restriction of F to a subset Y of X and one uses the norm preserving extensions of $F|_Y$ to X (a similar procedure was used in [131] in a particular case).

When (X, d) is a compact metric space, then, by the Stone-Weierstrass theorem, $\mathrm{Lip}\, X$ is dense in $C(X)$ with respect to the uniform norm. Using this fact, some convergence results for sequences of Markov operators (linear positive operators on X which preserve the constant functions) were proved in [1].

An extension theorem for multivalued functions which are Lipschitz with respect to the Hansdorff-Pompeiu metric was proved by A. Bressan and A. Cortesi [11]. This extension does not preserve the Lipschitz constant, and the authors give an example of a multivalued Lipschitz map which does not admit extensions with the same Lipschitz constant.

Theorem 1.13 ([11])**.** *Let H be a Hilbert space, Ω a subset of H and $f : \Omega \to \mathcal{C}(\mathbb{R}^m)$ a set-valued map taking values in the family $\mathcal{C}(\mathbb{R}^m)$ of all nonempty compact convex subsets of $\mathbb{R}^m$.*

If f is Lipschitz with respect to the Hansdorff-Pompeiu metric with constant L, then it admits an extension $F : H \to \mathcal{C}(\mathbb{R}^m)$ which is Lipschitz with Lipschitz constant $Lm\sqrt{28/3}$.

Extension theorems for Lipschitz fuzzy-valued functions were proved by N. Furukama [42].

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2. Applications to best approximation in spaces of Lipschitz functions

For a real normed space X a subset Y of X and an element $x \in X$ put

$$\begin{aligned} d(x, Y) &= \inf\{\|x - y\| : y \in Y\} \\ P_Y(x) &= \{y \in Y : \|x - y\| = d(x, Y)\}. \end{aligned}$$

The elements (if any) of the set $P_Y(x)$ are called *nearest points* to x in Y (or *elements of best approximation*). The set Y is called *proximinal* if $P_Y(x) \ne \emptyset$ for all $x \in X$, a *uniqueness set* if card $P_Y(x) \le 1$ for all $x \in X$ and *Chebyshevian* if card $P_Y(x) = 1$ for all $x \in X$.

If Y is a subspace of X let

$$Y^\perp = \{x^* \in X^* : x^*|_Y = 0\} \tag{2.1}$$

be the annihilator space of X in X^*. R. R. Phelps [106] proved that the subspace $Y^\perp$ is always proximinal and that it is Chebyshevian if and only if every $y^* \in Y^*$ has a unique norm preserving extension $x^* \in X^*$.

It can be shown that similar results hold in the case of Lipschitz functions.

Let (X,d) be a metric space, x_0 a fixed point in X and Y a subset of X containing x_0. Denote by $\mathrm{Lip}_0 X$ ($\mathrm{Lip}_0 Y$) the spaces of all Lipschitz functions on X (respectively on Y) vanishing at $x_0 \in Y$. Normed by (1.2) they are Banach spaces.

Put

$$Y^\perp = \{F \in \mathrm{Lip}_0 X : F|_Y = 0\} \ , \tag{2.2}$$

and, for $f \in \mathrm{Lip}_0 Y$, let

$$E(f) = \{F \in \mathrm{Lip}_0 X : F \text{ is a norm preserving extension of } f\} \ , \tag{2.3}$$

i.e.

$$F \in E(f) \iff F|_Y = f \quad \text{and} \quad \|F\|_X = \|f\|_Y \, .$$

By the extension theorem (Theorem 1.1) the set $E(f)$ is non-empty for any $f \in \mathrm{Lip}_0 Y$.

Theorem 2.1 ([83]). *Let (X,d), x_0, Y be as above and $Y^\perp$ be defined by (2.2)*

1° *The space $Y^\perp$ is always proximinal in $\mathrm{Lip}_0 X$ and for every $F \in \mathrm{Lip}_0 X$*

$$d\big(F, Y^\perp\big) = \|F|_Y\| \ , \tag{2.4}$$

and

$$P_{Y^\perp}(F) = F - E(F|_Y) \ . \tag{2.5}$$

2° *The space $Y^\perp$ is Chebyshevian in $\mathrm{Lip}_0 X$ if and only if every $f \in \mathrm{Lip}_0 Y$ has a unique norm preserving extension $F \in \mathrm{Lip}_0 X$.*

Proof. For $F \in \mathrm{Lip}_0 X$ and arbitrary $G \in Y^\perp$ we have

$$\begin{aligned} \|F|_Y\| &= \sup\left\{\frac{|F(x) - F(y)|}{d(x,y)} : x,y \in Y, x \neq y\right\} \\ &= \sup\left\{\frac{|(F-G)(x) - (F-G)(y)|}{d(x,y)} : x,y \in Y; x \neq y\right\} \leq \\ &\leq \sup\left\{\frac{|(F-G)(x) - (F-G)(y)|}{d(x,y)} : x,y \in X; x \neq y\right\} = \|F-G\| \ , \end{aligned}$$

implying

$$\|F|_Y\| \leq \inf\left\{\|F-G\| : G \in Y^\perp\right\} = d\big(F, Y^\perp\big).$$

By Theorem 1.1 there exists $G \in \mathrm{Lip}_0 X$ such that $G|_Y = F|_Y$ and $\|G\| = \|F|_Y\|$. It follows $F - G \in Y^\perp$ and

$$d\big(F, Y^\perp\big) \leq \|F - (F-G)\| = \|G\| = \|F|_Y\| \ ,$$

showing that (2.4) holds.

Also $G \in Y^\perp$ is a nearest point to F in $Y^\perp$ iff

$$\|F - G\| = d\big(F, Y^\perp\big) = \|F|_Y\| \ .$$

Since $(F-G)|_Y = F|_Y$ we have $F - G \in E(F|_Y)$ which is equivalent to $G \in F - E(F|_Y)$. Therefore

$$G \in P_{Y^\perp}(F) \iff G \in F - E(F|_Y) \ ,$$

proving the formula (2.5).

The second assertion of the theorem is an immediate consequence of this formula. □

By Theorem 1.1, any norm preserving extension of $F|_Y$ is contained between the extremal extensions F_1, F_2 given by (1.5) and (1.6), respectively, it follows that $E(F|_Y)$ is a singleton if and only if

$$\sup\{F|_Y(y) - \|F|_Y\|\, d(x,y) : y \in Y\} = \\ \inf\{F|_Y(y) + \|F|_Y\|\, d(x,y) : y \in Y\}\ ,$$

for all $x \in X$.

Since

$$\inf(F|_Y)(Y) + \|F|_Y\|\, d(x,y) \le \sup(F|_Y)(Y) - \|F|_Y\|\, d(x,Y)\ ,$$

we have

$$d(x,Y) \le \frac{\sup(F|_Y)(Y) - \inf(F|_Y)(Y)}{2\,\|F|_Y\|}\ , \tag{2.6}$$

for every $x \in X$ and every $F \in \mathrm{Lip}_0 X \setminus Y^{\perp}$.

Using this inequality one can give conditions on Y in order that its annihilator $Y^{\perp}$ be Chebyshevian.

Proposition 2.2. *Let (X,d) be a metric space and Y a subset of X containing the distinguished point x_0.*

1° *If $\overline{Y} = X$, then $Y^{\perp}$ is Chebyshevian in $\mathrm{Lip}_0 X$.*

2° *If $Y^{\perp}$ is Chebyshevian and Y contains at least one accumulation point, then $\overline{Y} = X$.*

Similar results hold in the case of the extension of convex Lipschitz functions.

By Theorem 1.8 every convex function $f \in \mathrm{Lip}\,Y$ admits a convex norm preserving extension $F \in \mathrm{Lip}\,X$. The minimal extension F_1 given by (1.5) is convex and there exists a maximal extension $\overline{F}$ too.

Now, if $0 \in Y$ is the fixed point, then put

$$K_Y = \{f \in \mathrm{Lip}_0 Y : f \text{ is convex on } Y\}\ .$$

It follows that K_Y is a convex cone and let

$$X_c = K_X - K_X$$

be the linear space generated by the cone

$$K_X = \{F \in \mathrm{Lip}_0 X : F \text{ is convex on } X\}.$$

Let also

$$Y_c = K_Y - K_Y\ ,$$

and

$$Y_c^{\perp} = \{F \in X_c : F|_Y = 0\}.$$

Theorem 2.3 ([20])**.** 1° *If $F \in K_X$, then*

$$\|F|_Y\| = d\big(F, Y_c^{\perp}\big)$$

2° *The space $Y_c^{\perp}$ is K_X-proximinal and, for $F \in K_X$, a function $G \in Y_c^{\perp}$ is a nearest point to F in $Y_c^{\perp}$ if and only if $G = F - H$ where H is a convex norm preserving extension of $F|_Y$.*

3° *The space $Y_c^{\perp}$ is K_X-Chebyshevian if and only if every $f \in K_Y$ has a unique convex norm preserving extension to X.*

Similar results hold for starshaped Lipschitz functions (see [90]).

References

[1] D. Andrica, C. Mustăţa, *An abstract Korovkin type theorem and applications,* Studia Univ. Babes-Bolyai, Ser. Math. **34** (1989), 41-44.
[2] R. F. Arens, J. Eells Jr., *On embedding uniform and topological spaces,* Pacific J. Math. **6** (1956), 397-405.
[3] Zvi Artstein, *Extensions of Lipschitz selections and an application to differential inclusions,* Nonlinear Anal. **16** (1991), 701-704.
[4] G. Aronsson, *Extension of Lipschitz functions satisfying a Lipschitz condition,* Arkiv Math. **6** (1976), 551-561.
[5] N. Aronszajn, P. Panichpakdi, *Extension of uniformly continuous and hyperconvex metric spaces,* Pacific J. Math **6** (1956), 405-439.
[6] S. Banach, *Wstep do teorii funcji rzeczwistych*(Polish),[Introduction to the theory of real functions], Warszawa/Wroclaw 1951.
[7] I. Beg, M. Iqbal, *Extension of linear 2-operators,* Math. Montesnigri **2** (1993), 1-10.
[8] C. Bessaga, A. Pelczynski, *Selected Topics in Infinite Dimensional Topology,* PWN Warszawa, 1975.
[9] J. M. Borwein, M. Fabian, *Characterizations of Banach spaces via convex and other locally Lipschitz functions,* Acta Math. Vietnam. **22** (1997), 53-69.
[10] W. W. Breckner, *Hölder continuity of certain generalized convex functions,* Optimization **28** (1994), 201-209.
[11] A. Bressan, A. Cortesi, *Lipschitz extensions of convex-valued maps,* Atti. Acad. Naz. Lincei, Rendinconti Classe Sci. Fis. Mat. Natur. (8) **80** (1986), 530-532.
[12] H. Brezis, *Prolongement d'applications lipschitziennes et des semi-groupes de contractions,* Séminaire Choquet: Initiation à l'analyse, 9 e anné, no. 19 (1969/70).
[13] B. M. Brown, D. Elliot, D. F. Paget, *Lipschitz constant for the Bernstein polynomials of Lipschitz continuous functions,* J. Approx. Theory **49** (1982), 196-199.
[14] T. Caputti, *A note on the extension of Lipschitz functions,* Rev. Un. Mat. Argentina **31** (1984), 122-129.
[15] E. W. Cheney, D. E. Wulbert, *The existence and unicity of best approximation,* Math. Scand. **24** (1969), 113-140, Corrigendum: Math. Scand. **27** (1970), 245.
[16] S. Cobzaş, *On the Lipschitz properties of continuous convex functions,* Mathematica **21** (1979), 123-125.
[17] S. Cobzaş, *Lipschitz properties of convex functions,* Seminar on Mathematical Analysis, Babes-Bolyai University, Faculty of Mathematics, Research Seminaries, Preprint No. 7, Cluj-Npoca 1985, 77-84.
[18] S. Cobzaş, *Extreme points in Banach spaces of Lipschitz functions,* Mathematica **31** (1989), 25-33.
[19] S. Cobzaş, I. Muntean, *Continuous and locally Lipschitz convex functions,* Mathematica **18** (1976), 41-51.
[20] S. Cobzaş, C. Mustăţa, *Norm preserving extension of convex Lipschitz functions,* J. Approx. Theory **24** (1978), 555-564.
[21] S. Cobzaş, C. Mustăţa, *Selections associated to the metric projection,* Rev. Anal. Numér. Théor. Approx. **24** (1995), 45-52.
[22] S. Cobzaş, C. Mustăţa, *Extension of bilinear operators and best approximation in 2-normed spaces,* Rev. Anal. Numér. Théor. Approx. **25** (1996), 63-75
[23] S. Cobzaş, C. Mustăţa, *Extension of bilinear operators and best approximation in 2-normed spaces,* Proc. 6th Workshop of the DGOR-Working Group-Multicriteria and Decision Theory, Halle 1996, A. Göpfert, J. Seeländer, Chr. Tammer, Eds. Deutsche Hochschulschriften vol. 2398, H¨ansel-Hohenhausen Frankfurt 1997, pp. 19-29.
[24] S. Cobzaş, C. Mustăţa, *Extension of bilinear functionals and best approximation in 2-normed spaces,* Studia Univ. Babes-Bolyai, Ser. Math. (in print).
[25] J. Czipser, L. Gehér, *Extension of functions satisfying a Lipschitz condition,* Acta Math. Sci. Acad. Sci. Hungar. **6** (1955), 213-220.
[26] L. Danzer, B. Grünbaum, V. Klee, *Helly's theorem and its relatives,* in *Convexity,* Proc. Symp. Pure Appl. Math. vol. 7, Amer. Math. Soc, Providence RI 1963, pp. 101-180.
[27] V. F. Demyanov, *Point derivations for Lipschitz functions and Clarke's generalized derivative,* Appl. Math. (Warsaw), **24** (1997), 465-474.
[28] F. Deutsch, *Linear selections for the metric projection,* J. Func. Anal. **49** (1982), 269-292.
[29] F. Deutsch, *A survey of the metric selections,* Contemporary Mathematics vol. 18, Amer. Math. Soc., Providence RI 1983, pp. 49-71.
[30] F. Deutsch, Wu Li, Sung-Ho Park, *Tietze extensions and continuous selections for the metric projection,* J. Approx. Theory **64** (1991), 56-68.
[31] F. Deutsch, Wu Li, Sizwe Mabizela, *Helly extensions and best approximation,* in Parametric Optimization and Related Topics, Approx. Optim., 3, Lang, Frankfurt am Main, 1993, pp. 107-120.
[32] M. Dorfner, *The extension of Lipschitz continuous operators,* Istit. Lombardo Accad. Sci. Lett. Rend. A **129** (1995), 47-59.

[33] J. Dugundji, *An extension of Tietze's theorem,* Pacific J. Math. **1** (1951), 353-367.
[34] R. Engelking, *General Topology,* PWN Warszawa 1977.
[35] H. Fakhouri, *Séléctions linéaires associées au théorème de Hahn-Banach*, J. Func. Anal. **11** (1972), 436-452.
[36] J. D. Farmer, *Extreme points of the unit ball of the space of Lipschitz functions,* Proc. Amer. Math. Soc. **121** (1994), 807-813.
[37] D. de Figueiredo, L. Karlovitz, *On the projection into convex sets and the extension of contractions*, in Proc.Conf. on Projections and Related Topics, Clemson Unic. 1967.
[38] D. de Figueiredo, L. Karlovitz, *On the extension of contractions on normed spaces*, in Proc. Symp. Pure Math. vol.18, part.1, Amer.Math.Soc., RI 1970.
[39] T.M. Flett, *Extension of Lipschitz functions*, J. London Math. Soc. **7** (1974), 604-608.
[40] C. Franchetti, *Lipschitz maps on the unit ball of normed spaces*, Confer. Sem. Mat. Univ. Bari vol. 202 (1985).
[41] C. Franchetti, *Lipschitz maps and the geometry of the unit ball in normed spaces*, Arch. Math. **46** (1986), 76-84.
[42] N. Furukama, *Convexity and local Lipschitz continuity of fuzzy-valued mappings*, Fuzzy Sets and Systems, **93** (1998), 113-119.
[43] S. Gähler, *Linear 2-normierte Räume,* Math. Nachr. **28** (1965), 1-45.
[44] B. Grünbaum, *A generalization of theorems of Kirszbraun and Minty,* Proc. Amer. Math. Soc. **13** (1962), 812-814.
[45] B. Grünbaum, *On a theorem of Kirszbraun,* Bull. Res. Council Israel **7F** (1958), 129-132.
[46] B. Grünbaum, E. Zarantonello, *On the extension of unniformly continuous mappings,* Michigan Math. J. **15** (1968), 65-78.
[47] P. Harmand, D. Werner, W. Werner, M-*ideals in Banach Spaces and Banach Algebras,* Lect. Notes Math. vol. 1547 Springer-Verlag, Berlin 1993.
[48] M. Hasumi, *The extension property of complex Banach spaces,* Tohoku Math. J. **10** (1958), 135-142.
[49] T. Hayden, J. Wells, *On the extension of Lipschitz-Hölder maps of order* β, J. Math. Anal. Appl. **33** (1971), 627-640.
[50] J.-B. Hiriart-Urruty, *Extension of Lipschitz functions,* J. Math. Anal. Appl. **77** (1980), 539-554.
[51] J.-B. Hiriart-Urruty, *Extension of Lipschitz integrands and minimization of nonconvex integral functionals . Applications to the optimal resource problem in discrete time,* Prob. Math. Stat. **3** (1982), 19-36.
[52] J.-B. Hiriart-Urruty, *Enveloppe k-lipschitzienne d'une fonction,* Rev. Math. Spéciales **106** (1995-1996), 785-793.
[53] R. B. Holmes, *A Course on Optimization and Best Approximation,* Lect. Notes Math. vol. 257, Springer-Verlag, Berlin 1972.
[54] R. B. Holmes, *Geometric Functional Analysis and its Applications,* Springer-Verlag, Berlin 1975.
[55] R. B. Holmes, B. Scranton, J.D. Ward, *Approximation from the space of compact operators and other* M*-ideals,* Duke Math. J. **42** (1975(, 259-269.
[56] R. Jensen, *Uniqueness of Lipschitz extensions: minimizing the sup norm of the gradient,* Arch. Rat. Mech. Anal. **123** (1993), 51-74.
[57] J. A. Johnson, *Banach spaces of Lipschitz functions and vector-valued Lipschitz functions,* Trans. Amer. Math. Soc. **148** (1970), 147-171.
[58] J. A. Johnson, *Lipschitz spaces,* Pacific J. Math. **51** (1974), 177-186.
[59] J. A. Johnson, *A note on Banach spaces of Lipschitz functions*, Pacific J. Math. **58** (1975), 475-482.
[60] W. B. Johnson, J. Lindenstrauss and G. Schechtman, *Extensions of Lipschitz maps into Banach spaces*, Israel J. Math. **54** (1986), 129-138.
[61] V. M. Kadets, *Lipschitz mappings on metric spaces*, Matematika, Izvestija VUZOV (1985) no.1, 30-34, (in Russian).
[62] H. Kamowitz, S. Scheinberg, *Some properties of endomorphisms of Lipschitz algebras*, Studia Math. **96** (1990), 61-67.
[63] J. L. Kelley, *Banach spaces with the extension property*, Trans. Amer. Math. Soc. **72** (1952), 323-326.
[64] M. Kirszbraun, *Über die zusammenziehenden und Lipschitzschen Transformationen*, Fund. Math. **22** (1934), 77-108.
[65] S.V. Konyagin, *On the level sets of Lipschitz functions*, Tatra Mount. Math. Publ. **2** (1993), 51-59.
[66] A. Langenbach, *Uber lokale eigenschaften von Lipschitz-Abbildungen*, Math. Nachr. **194** (1998), 127-137.
[67] K. de Leeuw, *Banach spaces of Lipschitz functions,* Studia Math. **21** (1961), 55-66.
[68] R. Levi, M. D. Rice, *The approximation of uniformly continuous mappings by Lipschitz and Hölder mappings,* in General Topology and its Relations to Modern Analysis and Algebra V, Prague 1981, 455-461.
[69] J. Lindensstrauss, *Non-linear projections in Banach spaces,* Michigan Math. J. **11** (1964), 263-287.
[70] J. Lindenstrauss, *Extension of compact operators,* Memoirs Amer. Math. Soc. vol. 48 (1964).
[71] S. Mabizela, *On bounded 2-linear functionals,* Math. Japonica **35** (1990), 51-55.

[72] S. Mabizela, *The relationship between Lipschitz extensions, best approximations, and continuous selections,* Quaestiones Math. **14** (1991), 261-268.
[73] L. Marco, J. A. Murillo, *Locally Lipschitz and convex functions,* Mathematica **38** (1996), 121-131.
[74] J. Matoušek, *Extension of Lipschitz mappings on metric trees,* Comment. Math. Univ. Carol. **31** (1990), 99-101.
[75] J. Matoušek, *Note on bi-Lipschitz embeddings into normed spaces,* Comment. Math. Univ. Carol. **33** (1992), 51-55.
[76] J. Matoušek, *On Lipschitz mappings onto square,* in: The Mathematics of P. Erdös, vol. II, Springer-Verlag, Berlin 1997
[77] W. E. Mayer-Wolf, *Isometries between Banach spaces of Lipschitz functions,* Israel J. Math. **38** (1981), 58-74.
[78] J. A. McShane, *Extension of range of functions,* Bull. Amer. Math. Soc. **40** (1934), 837-842.
[79] E. Mickle, *On the extension of a transformation,* Bull. Amer. Math. Soc. **55** (1949), 160-164.
[80] V. A. Milman, *Extension of functions preserving the modulus of continuity,* Mat. Zametki **61** (1997), 236-245 (in Russian).
[81] V. A. Milman, *Lipschitz extensions of linearly bounded functions,* Mat. Sbornik **189** (1998), 67-92 (in Russian).
[82] G. J. Minty, *On the extension of Lipschitz-Hölder continuous, and monotone functions,* Bull. Amer. Math. Soc. **76** (1970), 334-339.
[83] C. Mustăţa, *On some Chebyshevian subspaces of the normed space of Lipschitz functions,* Rev. Anal. Numer. Teor. Aprox. **2** (1973), 81-87 (in Romanian).
[84] C. Mustăţa, *On the unicity of the extension of continuous p-seminorms,* Rev. Anal. Numer. Teor. Aprox. Vol.2 Fasc.2 (1973), 173-177 (Romanian).
[85] C. Mustăţa, *M-ideals in metric spaces,* Babes-Bolyai University Reserch Seminaries, Preprint No. 7 (1988), 65-74.
[86] C. Mustăţa, *On the best approximation in metric spaces,* Rev. Anal. Numér. Théor. Approx. **4** (1975), 45-50.
[87] C. Mustăţa, *Norm preserving extension of starshaped Lipschitz functions,* Rev. Anal. Numér. Théor. Approx. **19** (1977), 183-187.
[88] C. Mustăţa, *Best approximation and unique extension of Lipschitz functions,* J. Approx. Theory **19** (1977), 222-230.
[89] C. Mustăţa, *A characterization of Chebyshevian subspaces of $Y^{\perp}$-type,* Rev. Anal. Numér. Théor. Approx. **6** (1977), 51-56.
[90] C. Mustăţa, *The extension of starshaped Lipschitz functions,* Rev. Anal. Numér. Théor. Approx. **9** (1980), 93-99.
[91] C.Mustăţa, *On the extension of Hölder functions*, Babes-Bolyai University Reserch Seminaries, Seminar on Functional Analysis and Numerical Methods, Preprint No. 7 (1985), 71-86.
[92] C. Mustăţa, *On the unicity of the extension of odd Lipschitz functions,* Babes-Bolyai University, Seminar on Optimization Theory, Report No. 8 (1987), 75-80.
[93] C. Mustăţa, *Extension of Hölder functions and some related problems of best approximation,* Babes-Bolyai University, Faculty of Mathematics Research Seminaries, Preprint no. 7 (1991), 71-86.
[94] C. Mustăţa, *Selections associated to McShane's extension theorem for Lipschitz functions,* Rev. Anal. Numér. Théor. Approx. **21** (1992, 135-145.
[95] C. Mustăţa, *On the metric projection and quotient mapping,* Rev. Anal. Numér. Théor. Approx. **24** (1995, 191-199.
[96] L. Nachbin, *On the Hahn-Banach theorem,* Anais Acad. Brasil. Ci. **21** (1949), 151-154.
[97] L. Nachbin, *A theorem of Hahn-Banach type for linear transformations,* Trans. Amer. Math. Soc. **68** (1950), 28-46.
[98] L. Nachbin, *Some problems in extending and lifting continuous linear transformations,* in Proc. International Symp. Linear Spaces, Jerusalem 1960, Jerusalem Acad. Press and Pergamon Press, Jerusalem and London 1961, pp. 340-350.
[99] Nguyen Van Khue, Nguyen To Xhu, *Extending locally Lipschitz maps with values in an infinite dimensional nuclear Fréchet space,* Bull. Acad. Polon. Sci. Ser. Math. **29** (1981), 609-616.
[100] Nguyen Van Khue, Nguyen To Xhu, *Lipschitz extensions and Lipschitz retractions in metric spaces,* Colloq. Math. **45** (1981), 245-250.
[101] E. Oja, *On the uniqueness of the norm preserving extension of the linear functional in the Hahn-Banach theorem,* Proc. Acad. Sci. Estonian SSR **33** (1984), 424-433.
[102] D. O'Keeffe, *The tangent stars of a set, and extensions of functions in Lipschitz classes,* Proc. Roy. Irish Acad. Sect. A, **97** (1997), 5-13.
[103] G. Pantelidis, *Approximationstheorie für metrische lineare Räume,* Math. Ann. **184** (1969), 30-48.
[104] Sung-Ho Park, *Quotient mapping, Helly-extensions, Hahn-Banach extension, Tietze extension, Lipschitz extension and best approximation,* J. Korean Math. Soc. **29** (1992), 239-250.
[105] J. Partanen, J. Vaisala, *Extension of bi-Lipschitz maps on compact polyhedra,* Math. Scand. **72** (1993), 235-264.
[106] R. R. Phelps, *Uniqueness of Hahn-Banach extension and unique best approximation,* Trans. Amer. Math. Soc. **95** (1960), 238-255.

[107] B. H. Pourciau, *Analysis and optimization of Lipschitz continuous mappings,* J. Optim. Theory Appl. **22** (1977), 311-351.
[108] B. H. Pourciau, *Hadamard's theorem for locally Lipschitzian mappings,* J. Math. Anal. Appl. **85** (1982), 279-285.
[109] B. H. Pourciau, *Univalence and degree for Lipschitz continuous maps*, Arch. Rat. Mech. Anal. **81** (1983), 289-299.
[110] B. H. Pourciau, *Global properties of proper Lipschitzian maps, SIAM J. Math. Anal.* **14** *(1983), 796-799.*
[111] K. Przeslawski, *Lipschitz retracts, selections, and extensions,* Michigan Math. J. **42** (1995), 555-571.
[112] K. Przeslawski, *Lipschitz continuous selections. I. Linear selections,* J. Convex Anal. **5** (1998), 249-267.
[113] S. Rolewicz, *Metric Linear Spaces,* PWN Warszawa 1972.
[114] S. Rolewicz, *On Lipschitz projection–a geometrical approach,* Ann. Univ. Maria Sklodowska-Curie **38** (1984), 135-138.
[115] S. Rolewicz, *On extremal points of the unit ball in Banach spaces of Lipschitz continuous functions,* J. Austral. Math. Soc. Ser. A **41** (1986), 95-98.
[116] S. Rolewicz, *Generalized Asplund inequalities on Lipschitz functions,* Arch. Math. **61** (1993), 484-488.
[117] S. Rolewicz, *On optimal observability of Lipschitz systems,* in Selected Topics in Oper. Res. and Math. Economics, Lect. Notes in Economics and Math. Systems vol 226, pp. 151-158, Springer-Verlag, Berlin 1993.
[118] S. Rolewicz, *On an extension of Mazur's theorem on Lipschitz functions,* Arch. Math. **63** (1994), 535-540.
[119] S. Rolewicz, *Duality and convex analysis in the absence of linear structure*, Math. Japon. **44** (1996), 165-182.
[120] A. K. Roy, *Extreme points and linear isometries of the Banach spaces of Lipschitz functions*, Canad. J. Math. **20** (1968), 1150-1164.
[121] W. Ruess, *Ein Dualkegel für p-konvexe topologische lineare Räume*, Geselschaft für Mathematik und Datenverarbeitung Nr. 60 (1973).
[122] W. Rzymowski, *Convex extension preserving Lipschitz constants*, J. Math. Anal. Appl. **212** (1997), 30-37.
[123] I. Sawashima, *Methods of duals in nonlinear analysis-Lipschitz duals of Banach spaces and some applications*, Lect. Notes Econ. Math. Syst. vol. 419, Springer-Verlag 1975, pp.247-249.
[124] K. Schnatz, *Approximationstheorie in metrischen Vektorräumen*, Dissertation. Frankfurt am Main, 1985.
[125] K. Schnatz, *Nonlinear duality and best approximation in metric linear spaces*, J. Approx.Theory **49** (1987), 201-218.
[126] I. Schoenberg, *On a theorem of Kirszbraun and Valentine*, Amer. Math. Monthly **44** (1953), 620-622.
[127] S. O. Schönbeck, *Extension of nonlinear contractions,* Bull. Amer. Math. Soc. **72** (1966), 99-101.
[128] S. O. Schönbeck, *On the extension of Lipschitz maps,* Ark Math. **7** (1967-1969), 201-209.
[129] D. R. Sherbert, *Banach algebras of Lipschitz functions,* Pacific J. Math. **13** (1963), 1387-1399.
[130] D. R. Sherbert, *The structure of ideals and point derivations in Banach algebras of Lipschitz functions*, Trans. Amer. Math. Soc., **111** (1964), 240-272.
[131] D. R. Shubert, *A sequential method seeking the global maximum of function*, SIAM J. Numer. Anal., **9** (1972), 379-388.
[132] I. Singer, *Best Approximation in Normed Linear Spaces by Elements of Linear Subspaces*, Ed. Academiei and Springer Verlag, Bucharest-Berlin 1970.
[133] I. Singer, *Extension with larger norm and separation with double support in normed linear spaces*, Bull. Austral. Math. Soc. **21** (1980), 93-106.
[134] R. Smarzewski, *Extreme points of unit balls in Lipschitz functions spaces*, Proc. Amer. Math. Soc. **125** (1997), 1391-1397.
[135] Ch. Stegall, *Spaces of Lipschitz functions on Banach spaces*, In Funct. Anal. (Essen 1991), 265-278. M. Dekker, New York 1994.
[136] V. Trenoguine, *Analyse fonctionnelle*, Edition Mir, Moscou 1985.
[137] S. Ustunel, *Extension of Lipschitz functions on Wiener space,* in Stochastic Anal. and Appl. (Powys 1995), 465-470, World Sci. 1996
[138] J. Vaisala, *Bi-Lipschitz and quasi-symmetric extension properties,* Ann. Acad. Sci. Fenn. SerI Math. 11 (1986), 239-274.
[139] J. Vaisala, *Banach spaces and bi-Lipschitz maps*, Studia Math. 103 (1992), 291-294.
[140] F. Valentine, *A Lipschitz condition preserving extension of a vector function*, Amer. J. Math. 67 (1945), 83-93.
[141] F. Valentine, *On the extension of a vector function so as to preserve a Lipschitz condition*, Bull. Amer. Math. Soc. 49 (1943), 100-108.
[142] F. Valentine, *Contractions in non-euclidean spaces*, Bull. Amer. Math. Soc. 50 (1944), 710-713.
[143] L. Waelbroeck, *Closed ideals of Lipschitz functions, in Function Algebras* (F. T. Birtel ed.), pp. 322-325, Scott Foresman 1966,
[144] N. Weaver, *Lattices of Lipschitz functions*, Pacific J. Math. **164** (1994), 179-193.
[145] N. Weaver, *Isometries of noncompact Lipschitz spaces* , Canad. Math. Bull. **38** (1995), 242-249.

[146] N. Weaver, *Subalgebras of little Lipschitz algebras* , Pacific J. Math. **173** (1996), 283-293.
[147] N. Weaver, *Lipschitz algebras and derivations of von Neumann algebras*, J. Func. Anal. **139** (1996), 261-300.
[148] J. H. Wells, L. R. Williams, *Embeddings and Extensions in Analysis*, Springer-Verlag, Berlin 1975.
[149] L. Williams, J. Wells, T. Hayden, *On the extension of Lipschitz-Hölder maps on L^p spaces*, Studia Math. 39 (1971), 29-38.