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Dedicated to Maria S. Pop on her 60th anniversary

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Costică MUSTĂŢA

Let X be a nonvoid set and $d : X \times X \to [0, \infty)$ a function satisfying the following conditions:

(i) $d(x,y) = 0 \Longleftrightarrow x = y$,

(ii) $d(x,y) \leq d(x,z) + d(z,y)$,

for all $x, y, z \in X$.We call d a *quasi-metric* on $X$and the pair (X, d), a *quasi-metric space.* Remark that the main difference with respect to a metric is the symmetry condition, $d(x,y) = d(y,x)$,which is not satisfied by a quasi-metric.

The conjugate of a quasi-metric d,denoted by d^{-1} is defined by

$$d^{-1}(x,y) = d(y,x) \tag{1}$$

for all $x, y \in X$. Obviously, that the mapping $d^s : X \times X \to [0, \infty)$ defined by

$$d^s(x,y) = \max\left\{d(x,y),\ d^{-1}(x,y)\right\},\ x,y \in X \tag{2}$$

is a metric on X, i.e. d^s satisfies the conditions (i), (ii) and the symmetry condition:

$$d^s(x,y) = d^s(y,x), \quad x, y \in X. \tag{iii}$$

A function $f : X \to \mathbb{R}$,defined on a quasi-metric space (X, d) is called *semi-Lipschitz* provided there exists a number $K \geq 0$ such that

$$f(x) - f(y) \leq K\, d(x,y), \tag{3}$$

for all $x, y \in X$. A function $f : X \to \mathbb{R}$ is called $\leq_d$- *increasing* if

$$d(x,y) = 0 \Longrightarrow f(x) - f(y) \leq 0 \tag{3a}$$

for all $x, y \in X$.

The definition of $\leq_d$ - increasing function $f : X \to \mathbb{R}$ is consistent for T_0 - separated quasi-metric space (X, d) (see [6]). In this note the quasi-metric space $(X.d)$ is T_1 - separated (see the condition (i) and (ii)).

Since $d(x,y) = 0 \Longleftrightarrow x = y$, it follows that $f(x) \leq f(y)$ for any function $f : X \to \mathbb{R}$ i.e. any real - valued function on a quasi - metric space X is $\leq_d$ - increasing.

Theorem 1 *Let $f : X \to \mathbb{R}$ be such that*

$$\|f\|_d = \sup \left\{ \frac{(f(x) - f(y)) \vee 0}{d(x,y)} : x,\ y \in X,\ d(x,y) > 0 \right\} < \infty \tag{4}$$

Then f satisfies the inequality

$$f(x) - f(y) \leq \|f\|_d \cdot d(x,y),\ \forall x, y \in X \tag{5}$$

and $\|f\|_d$ is the smallest constant for which the inequality (3) holds.

P roof. nce f is $\leq_d$ - increasing (see (3a)) it follows that $f(x) - f(y) > 0$ implies $d(x,y) > 0$. But then

$$\frac{f(x) - f(y)}{d(x,y)} > 0 \text{ and } \|f\|_d = \sup_{d(x,y)>0} \frac{(f(x) - f(y)) \vee 0}{d(x,y)} \geq \frac{f(x) - f(y}{d(x,y)}$$

implying

$$f(x) - f(y) \leq \|f\|_d \cdot d(x,y).$$

If $f(x) - f(y) \leq 0$ then

$$\frac{(f(x) - f(y)) \vee 0}{d(x,y)} = 0$$

implying $f(x) - f(y) \leq \|f\|_d \cdot d(x,y)$.

Let now $K \geq 0$ be such that

$$f(x) - f(y) \leq K \cdot d(x,y)$$

for all $x, y \in X$. Then f is $\leq_d$ - increasing and

$$\frac{(f(x) - f(y)) \vee 0}{d(x,y)} = \frac{f(x) - f(y)}{d(x,y)} \leq K \quad \text{if} \quad f(x) - f(y) > 0$$

and

$$\frac{(f(x)-f(y))\vee 0}{d(x,y)}=0\leq K \quad \text{if} \quad f(x)-f(y)\leq 0.$$

Consequently, $\|f\|_d \leq K$. ■

Denoting by $SLip\ X$ the set of all real - valued semi - Lipschitz functions defined on a quasi - metric space (X,d) we have

$$SLipX=\left\{f:X\to\mathbb{R},\ \sup_{d(x,y)>0}\frac{(f(x)-f(y))\vee 0}{d(x,y)}<\infty\right\}. \tag{6}$$

Let $Y\subset X$, $Y\neq\emptyset$, where (X,d) is a quasi - metric space. It follows that (Y,d) is a quasi - metric space, too, and let's denote by $SLip\ Y$ the set of all semi - Lipschitz functions on Y.

The following extension problem arises naturally: for $f\in SLip\ Y$ find $F\in SLip\ X$ such that

$$F|_Y=f \text{ and } \|F\|_d=\|f\|_d. \tag{7}$$

The answer is affirmative. In [5] it was shown that the functions

$$F(x)=\inf_{y\in Y}\left[f(y)+\|f\|_d\cdot d(x,y)\right],\ x\in X, \tag{8}$$

$$G(x)=\sup_{y\in Y}\left[f(y)-\|f\|_d\cdot d^{-1}(x,y)\right],\ x\in X \tag{9}$$

satisfy the equalities

$$F|_Y=G|_Y=f \quad \text{and} \quad \|F\|_d=\|G\|_d=\|f\|_d.$$

In other words, for any $f\in SLip\ Y$ the set

$$E_Y^d(f):=\{H\in SLipX: H|_Y=f \quad \text{and} \quad \|H\|_d=\|f\|_d\} \tag{10}$$

of all extensions of f which preserve the smallest Lipschitz constant is non-void.

Concerning the unicity of the extension ($card\ E_Y^d(f)=1$) one can prove:

Theorem 2 *Let (X,d) be a quasi - metric space, $Y\subset X$ and $f\in SLipY$. Then*

a) For every $H\in E_Y^d(f)$ the following inequalities hold:

$$G(x)\leq H(x)\leq F(x),\quad x\in X \tag{11}$$

where the functions F, G *are defined by (8), (9);*

b) card $E_Y^d(f) = 1$ *if and only if*

$$\sup_{y\in Y}\left[f(y) - \|f\|_d d^{-1}(x,y)\right] = \inf_{y\in Y}\left[f(y) + \|f\|_d d(x,y)\right] \tag{12}$$

for all $x \in X$.

P roof. t $H \in E_Y^d(f)$. Then we have for every $x \in X$ and $y \in Y$:

$$H(x) - H(y) \le \|f\|_d d(x,y)$$
$$H(y) - H(x) \le \|f\|_d d(y,x) = \|f\|_d \cdot d^{-1}(x,y).$$

The first inequality implies

$$H(x) \le H(y) + \|f\|_d \cdot d(x,y) = f(y) + \|f\|_d d(x,y)$$

and, taking the infimum with respect to $y \in Y$, we have

$$H(x) \le F(x), \quad x \in X.$$

Similarly, we get

$$H(x) \ge H(y) - \|f\|_d d^{-1}(x,y) = f(y) - \|f\|_d d^{-1}(x,y).$$

Taking the supremum with respect to $y \in Y$ one obtains

$$H(x) \ge G(x), \quad x \in X.$$

The assertion b) is a direct consequence of the inequalities (11). ■

Remark. 1^0.If the function $f : X \to \mathbb{R}$ is constant on X then $\|f\|_d = 0$, and the equality (12) holds.

Consider on $\mathbb{R}$de quasi-metric

$$d(x,y) = \begin{cases} x - y, & x \ge y \\ 0, & x < y \end{cases}$$

and let $Y = [0,1]$and $f(y) = 2y$, $y \in Y$.Then $\|f\|_d = 2$ and the extremal extensions F, G are

$$F(x) = \begin{cases} 2 & x < 0 \\ 2x & x \ge 0 \end{cases} \quad \text{and} \quad G(x) = \begin{cases} 2x & x \le 1 \\ 0 & x > 1 \end{cases}$$

which are distinct.

2^0. By Theorem 2, if $f \in SLipY$ has a unique extension then the equality (12) holds and, since

$$\inf_{y\in Y}\left[f(y) + \|f\|_d \cdot d(x,y)\right] \geq \inf_{y\in Y} f(y) + \|f\|_d \cdot d(x,Y),$$
$$\sup_{y\in Y}\left[f(y) - \|f\|_d \cdot d^{-1}(x,y)\right] \leq \sup_{y\in Y} f(y) - \|f\|_d \cdot d^{-1}(x,Y)$$

where

$$d(x,Y) = \inf\{d(x,y) : y \in Y\}$$

and

$$d^{-1}(x,Y) = \inf\{d(y,x) : y \in Y\}$$

we obtain the inequality

$$(13) \qquad d(x,Y) + d^{-1}(x,Y) \leq \frac{1}{\|f\|_d}\left(\sup_{y\in Y} f(y) - \inf_{y\in Y} f(y)\right).$$

Theorem 3 *Let (X,d) be a quasi - metric space and $Y \subset X$, $Y \neq X$, containing at least one cluster point. If each function $f \in SLipY$ has a unique extension then $\overline{Y} = X$.*

P roof. t $y_0 \in Y$ be a cluster point of the set Y and let $y_n \in Y\setminus\{y_0\}$, $n = 1, 2...$, be such that $\lim_{n\to\infty} d(y_n, y_0) = 0$. ■

Claim: There exists $x_0 \in X$ such that $d(x_0, y_0) > 0$ and $d(x_0, y_n) > 0$, $n = 1, 2, \ldots$.

Indeed, if contrary, then for every $x \in X$, $d(x, y_0) = 0$ or $d(x, y_n) = 0$ for all $n \in \mathbb{N}$.In the first case $x = y_0 \in Y$ and in the second $x = y_n \in Y$.It follows $Y = X$,a contradiction.

Consider the function $f : X \to \mathbb{R}$ defined by

$$f(x) = d(x, y_0) - d(x_0, y_0),\ x \in X.$$

We have

$$f(y_0) = d(y_0, y_0) - d(x_0, y_0) = -d(x_0, y_0) < 0$$
$$f(y_n) = d(y_n, y_0) - d(x_0, y_0) > -d(x_0, y_0)$$

for all $n = 1, 2, \ldots$. Define the sequence of functions $\varphi_n : f(X) \to [0,1]$ by

$$\varphi_n(t) = \begin{cases} 1 & \text{if } \ t < f(y_0) \\ \dfrac{f(y_n) - t}{f(y_n) - f(y_0)} & t \in [f(y_0), f(y_n)] \\ 0 & t > f(y_n) \end{cases}$$

The function $\Psi_n = \varphi_n \circ f : X \to \mathbb{R}$, $n = 1, 2, \ldots$, satisfy

$$\|\Psi_n\|_d \geq \frac{(\varphi_n(f(y_0)) - \varphi_n(f(y_n))) \vee 0}{d(y_0, y_n)} = \frac{1}{d(y_0, y_n)} \to \infty,$$

for $n \to \infty$.

By the inequality (12)

$$d(x, Y) + d^{-1}(x, Y) \leq \frac{1 - 0}{\|\Psi_n\|_d} \to 0$$

for $n \to \infty$,showing that $Y$is dense in X,with respect to the quasi - metric d and with respect to d^{-1},as well. ■

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