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EXTENSION OF BOUNDED STARSHAPED SEMI-LIPSCHITZ FUNCTIONS ON QUASI-METRIC LINEAR SPACES

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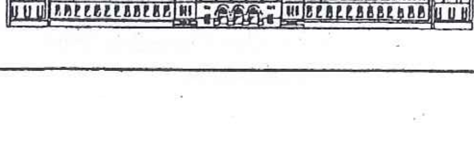

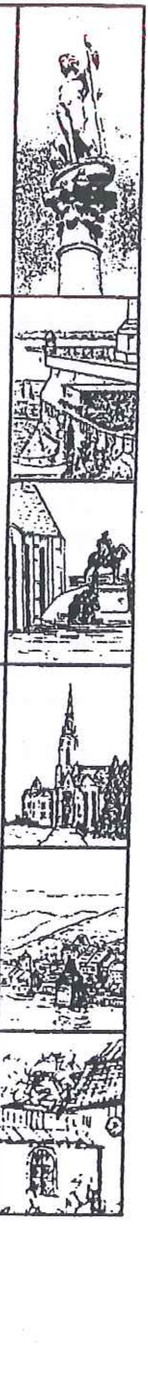

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Extension of bounded starshaped semi-Lipschitz functions on quasi-metric linear spaces

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1 Introduction

Let X be a nonvoid set. A function $d : X \times X \to [0, \infty)$ satisfying the conditions:

(i) $d(x,y) = 0 \Longleftrightarrow x = y,$

(ii) $d(x,y) \leq d(x,z) + d(z,y),$

for all $x, y, z \in X$ is called a *quasi-metric* on X and the pair (X, d) is called quasi-metric space. The essential difference with respect to a metric on X is that a quasi-metric does not satisfy the symmetry condition $d(x,y) = d(y,x)$.

If X is a linear space and d a quasi-metric on X then the pair (X, d) is called a *quasi-metric linear space.*

Let $\theta \in X$ be the null element of the linear space X. A subset Y of X is called *starshaped* (with respect to θ) if it satisfies the condition:

$$(1) \qquad \forall y \in Y \qquad \forall \alpha \in [0,1] : \alpha y \in Y.$$

If Y is a starshaped subset of the linear space X then a function $f : Y \to \mathbb{R}$ is called a *starshaped function* provided:

$$(2) \qquad \forall y \in Y \qquad \forall \alpha \in [0,1] : f(\alpha y) \leq \alpha f(y).$$

Obviously that the condition (1) implies $\theta \in Y$ and the condition (2) implies $f(\theta) \leq 0$. In what follows *we shall consider only starshaped functions on Y which vanish at θ*, i.e. $f(\theta) = 0$.

If (X, d) is a quasi-metric linear space and $Y \subset X$ is a starshaped set, the quasi-metric d is called starshaped on Y if

(3) $$\forall x,y\in Y,\quad \forall\alpha\in[0,1]:\quad d(\alpha x,\alpha y)\leq \alpha d(x,y).$$

Let (X,d) be a quasi-metric space and $Y\subset X,\ Y\neq\emptyset$. A function $f:Y\to\mathbb{R}$ is called *semi-Lipschitz* if it satisfies the condition

(4) $$\exists K_Y\geq 0\ :\ f(x)-f(y)\leq K_Y\cdot d(x,y),$$

for all $x,y\in Y$.

A number $K_Y\geq 0$ for which (4) holds is called a *semi-Lipschitz constant* for f (on Y).

One sees that

(5) $$\|f\|_Y=\sup\left\{\frac{(f(x)-f(y))\vee 0}{d(x,y)}:x,y\in Y,\ d(x,y)>0\right\}$$

is the smallest semi-Lipschitz constant for the function f on the set Y (see [6] and [4, Th.1]).

Let

(6) $$SLipY:=\{f:Y\to\mathbb{R},\ f\text{ is semi-Lipschitz}\}$$

the set of all real-valued semi-Lipschitz functions defined on the quasi-metric space (Y,d), $Y\subseteq X$.

If (X,d) is a quasi-metric linear space and Y is a subset of X containing θ then the set

(7) $$SLip_0Y:=\{f|f\in SLipY\text{ and }f(\theta)=0\}$$

is a semilinear space and the functional $\|\cdot\|_Y:SLip_0Y\to\mathbb{R}_+$ defined by

(8) $$\|f\|_Y:\sup\left\{\frac{(f(x)-f(y))\vee 0}{d(x,y)}:x,y\in Y,\ d(x,y)\neq 0\right\}$$

is a quasi-norm on $SLip_0X$ (see [4] and [6]).

A semilinear space satisfies axioms similar to those defining a linear space, excepting the existence of the opposite element (the inverse with respect to +) and that the multiplication is defined only for positive scalars. (see [6]).

2 Extensions

The following problem is treated in [4]:

Let (X,d) be a quasi-metric space, $Y \subset X$ and $f \in SLip\, Y$. One asks to find a function $F \in SLipX$ such that

$$F|_Y = f \text{ and } \|F\|_X = \|f\|_Y . \tag{9}$$

One shows ([4, Th.2]) that for every $f \in SLipY$ there exists $F \in SLipX$ satisfying (9). This result is similar to a result of Mc Shane from 1934 (see [2]) asserting that every real valued Lipschitz function defined on a subset of a metric space X admits an extension to the whole space with the same Lipschitz constant.

In the present paper we study the extension problem for bounded starshaped semi-Lipschitz functions defined on sharshaped subsets of quasi-metric linear spaces:

For a quasi-metric linear space (X,d) with the quasi-metric d starshaped, a starshaped subset Y of X and a bounded semi-Lipschitz starshaped function $f \in SLip_0Y$ find a bounded starshaped function $F \in SLip_0X$ such that

$$F|_Y = f,\ \|F\|_X = \|f\|_Y \text{ and } \|F\|_\infty = \|f\|_\infty , \tag{10}$$

where $\|\cdot\|_\infty$ denotes the sup-norm.

Observe that if Y is a subspace of X and $f \in SLip_0Y$ is starshaped then it is possible that f be unbounded on Y.

More exactly we have:

Lemma 1 *Let (X,d) be a quasi-metric linear space, Y a subspace of X and $f \in SLip_0Y$ be starshaped. If there exists $x_0 \in Y$ such that $f(x_0) > 0$ then f is unbounded on Y.*

Proof. For every $x \in Y$, $x \neq 0$, the function $h : (0,\infty) \to \mathbb{R}$ defined by $h(t) = f(tx)/t$ is non-increasing. Indeed, if $0 < t_1 < t_2$ then

$$h(t_1) = \frac{f(t_1 x)}{t_1} = \frac{f\left(t_2^{-1} \cdot t_1 (t_2 x)\right)}{t_1} \le \frac{t_1}{t_2}\frac{f(t_2 x)}{t_1} = \frac{f(t_2 x)}{t_2} = h(t_2).$$

In particular, for $t > 1$ and $x_0 \in Y$ with $f(x_0) > 0$ we have

$$0 < f(x_0) \le \frac{f(tx_0)}{t}$$

so that

$$f(tx_0) \ge t f(x_0)$$

for every $t > 1$, which shows that f is unbounded on Y. ∎

The following theorem answers positively the question on the extension of bounded starshaped semi-Lipschitz functions.

Theorem 2 *Let (X,d) be a quasi-metric linear space and Y a starshaped subset of X. Suppose that the quasi-metric d is starshaped on Y.*

Let $f \in SLip_0 Y$ be bounded and starshaped. In order to exist a bounded starshaped function $F \in SLip_0 X$ such that

$$F|_Y = f, \quad \|F\|_X = \|f\|_Y \quad \text{and} \quad \|F\|_\infty = \|f\|_\infty$$

it is necessary and sufficient that $f(y) \leq 0$ for all $y \in Y$.

Proof. Sufficiency suppose that $f(y) \leq 0$ for all $y \in Y$. The function

(11) $$H(x) = \inf_{y \in Y} \left[f(y) + \|f\|_Y d(x,y) \right], \quad x \in X$$

is starshaped on X and satisfies the conditions

$$H|_Y = f \quad \text{and} \quad \|H\|_X = \|f\|_Y$$

(see [4,Th.2] and [5, Th.8]).

Indeed, let $z \in Y$ and $x \in X$. For any $y \in Y$ we have

$$\begin{aligned} f(y) + \|f\|_Y d(x,y) &= f(z) + \|f\|_Y d(x,y) - (f(z) - f(y)) \geq \\ &\geq f(z) + \|f\|_Y d(x,y) - \|f\|_Y d(z,y) = \\ &= f(z) - \|f\|_Y (d(z,y) - d(x,y)). \end{aligned}$$

The inequality $d(z,y) - d(x,y) \leq d(z,x)$ implies

$$f(y) + \|f\|_Y d(x,y) \geq f(z) - \|f\|_Y \cdot d(z,x)$$

showing that for every $x \in X$ the set $\{f(y) + \|f\|_Y d(x,y) : y \in Y\}$ is bounded from above by $f(z) - \|f\|_Y d(z,x)$, and the infimum (11) is finite.

We *show now that $H(y) = f(y)$ for all $y \in Y$.*

Let $y \in Y$. Then

$$H(y) \leq f(y) + \|f\|_Y d(y,y) = f(y).$$

For any $v \in Y$ we have

$$f(y) - f(v) \leq \|f\|_Y \cdot d(y,v)$$

so that

$$f(v) + \|f\|_Y \cdot d(y,v) \geq f(y)$$

and

$$H(y) = \inf \{ f(v) + \|f\|_Y d(y,v) : v \in Y \} \geq f(y).$$

It follows $H(y) = f(y)$.

We prove that $\|H\|_Y = \|f\|_Y$.

Since $H|_Y = f$, the definitions of $\|H\|_Y$ and $\|f\|_Y$ yield $\|H\|_Y \geq \|f\|_Y$.

Let $x_1, x_2 \in X$ and $\varepsilon > 0$. Choosing $y \in Y$ such that

$$H(x_1) \geq f(y) + \|f\|_Y d(x_1, y) - \varepsilon$$

we obtain

$$\begin{aligned} H(x_2) - H(x_1) &\leq f(y) + \|f\|_Y d(x_2, y) - (f(y) + \|f\|_Y \cdot d(x_1, y) - \varepsilon) \\ &= \|f\|_Y [d(x_2, y) - d(x_1, y)] + \varepsilon \\ &\leq \|f\|_Y \cdot d(x_2, x_1) + \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary it follows

$$H(x_2) - H(x_1) \leq \|f\|_Y \cdot d(x_2, x_1)$$

for any $x_1, x_2 \in X$ and $\|H\|_Y \leq \|f\|_Y$. Then $\|H\|_X = \|f\|_Y$.

We shall show that H is also starshaped on X. To this end let $x \in X$, $z \in Y$ and $\alpha \in [0,1]$. We have

$$\begin{aligned} H(\alpha x) &= f(\alpha z) + \|f\|_Y d(\alpha x, \alpha z) \leq \\ &\leq \alpha f(z) + \alpha \|f\|_Y d(x, z) = \\ &= \alpha [f(z) + \|f\|_Y \cdot d(x, z)]. \end{aligned}$$

Taking the infimum with respect to $z \in Y$ we get

$$H(\alpha x) \leq \alpha H(x)$$

for all $x \in X$ and all $\alpha \in [0,1]$, showing that the function H defined by (11) is a starshaped extension of f.

Consider the function

$$F(x) = \begin{cases} H(x) & \text{if } H(x) \leq 0 \\ 0 & \text{if } H(x) > 0 \end{cases}. \tag{12}$$

Since $F|_Y = H|_Y = f$, $\|H\|_X = \|f\|_Y$ and $\|H\|_X \geq \|F\|_X$, it follows that $\|F\|_X = \|f\|_Y$. Therefore F is an extension of f with the same semi-Lipschitz constant.

Let $x \in X$. If $H(x) \leq 0$ then

$$H(\alpha x) \leq \alpha H(x) \leq 0$$

for any $\alpha \in [0,1]$ so that

$$F(\alpha x) = H(\alpha x) \leq \alpha H(x) = \alpha F(x).$$

If $H(x) > 0$ then $F(x) = 0$. Let $\alpha \in (0,1)$. If $H(\alpha x) > 0$then

$$F(\alpha x) = 0 = \alpha F(x).$$

If $H(\alpha x) \leq 0$ then $F(\alpha x) = H(\alpha x)$ and

$$F(\alpha x) = H(\alpha x) \leq 0 = \alpha F(x).$$

It follows that the function F defined by (12) is an extension of f which is starshaped and has the same semi-Lipschitz constant as f.

Since

$$\sup\{|F(x)| : x \in X\} \geq \sup\{|F(y)| : y \in Y\} = \|f\|_\infty$$

it follows $\|F\|_\infty \geq \|f\|_\infty$.

Let $x \in X$. For any $y \in Y$ we have

$$f(y) \leq f(y) + \|f\|_Y \cdot d(x,y)$$

which implies

$$\inf_{y \in Y} f(y) \leq \inf_{y \in Y} [f(y) + \|f\|_Y d(x,y)] = H(x).$$

If $H(x) \leq 0$ then

$$\inf_{y \in Y} f(y) \leq H(x) = F(x).$$

Since $f(y) \leq 0$ for all $y \in Y$ we have $\|f\|_\infty = -\inf f(Y)$ so that

$$-\|f\|_\infty \leq F(x) \Longleftrightarrow \|f\|_\infty \geq -F(x).$$

If $H(x) > 0$ then

$$\|f\|_\infty \geq 0 = F(x).$$

Using the fact that $F(x) \leq 0$, for all $x \in X$. and the above inequalities we obtain

$$\|f\|_\infty \geq \sup_{x \in X} (-F(x)) = \|F\|_\infty.$$

It follows that $\|F\|_\infty = \|f\|_\infty$.

Necessity. Suppose there exists $y \in Y$ such that $f(y) > 0$. By Lemma 1 f has no bounded starshaped extensions to X. ■

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