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Extensions of convex semi-Lipschitz functions on quasi-metric linear spaces

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Abstract. In this paper one shows that a convex semi-Lipschitz functions defined on a convex subset of a quasi-metric linear space X admits an extension to the whole space X, preserving both the convexity and the semi-Lipschitz constant. A similar result is proved for starshaped functions.

1 Introduction

Let X be a set. A function $d : X \times X \to [0, \infty)$ is called a quasi-metric if

(i) $d(x, y) = d(y, x) = 0 \iff x = y$,

(ii) $d(x, y) \le d(x, z) + d(z, y)$,

for all $x.y.z \in X$. If d is a quasi-metric on X then the pair (X, d) is called a *quasi-metric space*. If X is further a linear space and d is a quasi-metric on X then the pair (X, d) is called a *quasi-metric linear space*.

The function $d^{-1} : X \times X \to [0, \infty)$ defined by equality

$$d^{-1}(x, y) = d(y, x), \quad x, y \in X, \tag{1}$$

is called the *conjugate* of the quasi-metric d [1].

Definition 1.1 *Let (X,d) be a quasi-metric linear space and Y a convex subset of X (i.e. $(\forall)\ u_1,u_2 \in Y,\ (\forall)\ \alpha \in [0,1],\ \alpha u_1 + (1-\alpha)\,u_2 \in Y$). The quasi-metric d is called convex on Y if it satisfies the inequality*

$$d(\alpha x_1 + (1-\alpha)\,x_2,\ \alpha y_1 + (1-\alpha)\,y_2) \le \alpha d(x_1,y_1) + (1-\alpha)\,d(x_2,y_2), \tag{2}$$

for all $x_1,x_2,y_1,y_2 \in Y$ and all $\alpha \in [0,1]$.

Definition 1.2 [4]. *Let (X,d) be a quasi-metric linear space. A function $f : X \to \mathbb{R}$ is called semi-Lipschitz if there exists $K \ge 0$ such that*

$$f(x) - f(y) \le K \cdot d(x,y), \tag{3}$$

for all $x,y \in X$.

A number $K \ge 0$ for which (3) holds is called a *semi-Lipschitz constant* for f.

For $Y \subset X$ let

$$\|f\|_Y = \sup\left\{ \frac{(f(x) - f(y)) \vee 0}{d(x,y)} : x,y \in Y,\ d(x,y) > 0 \right\}. \tag{4}$$

The $\|f\|_Y$ is the smallest semi-Lipschitz constant of the function f on Y [3, Th.1].

For $Y \subset X$ let

$$S\,LipY = \{f : Y \to \mathbb{R} : \|f\|_Y < \infty\} \tag{5}$$

be the set of all real-valued semi-Lipschitz functions defined on the quasi-metric space (Y,d).

Definition 1.3 *A function $F \in S\,Lip\,X$ is called an extension of a function $f \in S\,LipY$ if*

(i) $F|_Y = f$,

(ii) $\|F\|_X = \|f\|_Y$.

For $f \in S\,LipY$ one denotes by

$$E_Y(f) = \{F \in S\,LipX : F|_Y = f \text{ and } \|F\|_X = \|f\|_Y\} \tag{6}$$

the set of all extensions of the function f.

By Theorem 2 in [4] it follows that

$$E_Y(f) \neq \emptyset,\ (\forall)\, f \in S\,LipY. \tag{7}$$

The following problem arises naturally: which other properties of the function f (beside the semi-Lipschitz constant) are preserved by at least one of its extensions?

The aim of this paper is to show that two such properties are convexity and starshapedness.

First we prove:

Theorem 1.1 *Let (X,d) be a quasi-metric linear space and Y a convex subset of X. Suppose that the quasi-metric d is convex on Y, in the sense of Definition 1.*

a) If $f \in S\,LipY$ is convex on Y then there exists a convex

$$F \in E_Y(f).$$

b) If $f \in S\,LipY$ is concave (i.e. $-f$ is convex) on Y then there exists a concave

$$G \in E_Y(f).$$

Proof. a) Let $f \in S\,LipY$ be convex on the convex set Y. Consider the function $F : X \to \mathbb{R}$ defined by

$$F(x) = \inf_{y\in Y} \{f(y) + \|f\|_Y \cdot d(x,y)\},\ x \in X. \tag{8}$$

Then F is well defined and $F \in E_Y(f)$ (Theorem 2 in [4]). Indeed, let $z \in Y$ and $x \in X$. For any $y \in Y$ we have

$$\begin{aligned} f(y) + \|f\|_Y\, d(x,y) &= f(z) + \|f\|_Y\, d(x,y) - (f(z) - f(y)) \\ &\geq f(z) + \|f\|_Y\, d(x,y) - \|f\|_Y\, d(z,y) \\ &= f(z) - \|f\|_Y\,(d(z,y) - d(x,y)). \end{aligned}$$

The inequality $d(z,y) - d(x,y) \leq d(z,x) = d^{-1}(x,z)$ implies

$$f(y) + \|f\|_Y\, d(x,y) \geq f(z) - \|f\|_Y \cdot d^{-1}(x,z)$$

showing that for every $x \in X$ the set $\{f(y) + \|f\|_Y d(x,y) : y \in Y\}$ is bounded from above by $f(z) - \|f\|_Y d^{-1}(x,z)$,and the infimum (8) is finite.

We *show now that* $F(y) = f(y)$ *for all* $y \in Y$.

Let $y \in Y$. Then

$$F(y) \le f(y) + \|f\|_Y d(y,y) = f(y).$$

For any $v \in Y$ we have

$$f(y) - f(v) \le \|f\|_Y \cdot d(y,v)$$

so that

$$f(v) + \|f\|_Y \cdot d(y,v) \ge f(y)$$

and

$$F(y) = \inf\{f(v) + \|f\|_Y d(y,v) : v \in Y\} \ge f(y).$$

It follows $F(y) = f(y)$.

We prove that $\|F\|_Y = \|f\|_Y$.

Since $F|_Y = f$, the definitions of $\|F\|_Y$ and $\|f\|_Y$ yield $\|F\|_Y \ge \|f\|_Y$.

Let $x_1, x_2 \in X$ and $\varepsilon > 0$. Choosing $y \in Y$ such that

$$F(x_1) \ge f(y) + \|f\|_Y d(x_1,y) - \varepsilon$$

we obtain

$$\begin{aligned} & F(x_2) - F(x_1) \\ \le\ & f(y) + \|f\|_Y d(x_2,y) - (f(y) + \|f\|_Y \cdot d(x_1,y) - \varepsilon) \\ =\ & \|f\|_Y [d(x_2,y) - d(x_1,y)] + \varepsilon \\ \le\ & \|f\|_Y \cdot d(x_2,x_1) + \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary it follows

$$F(x_2) - F(x_1) \le \|f\|_Y \cdot d(x_2,x_1)$$

for any $x_1, x_2 \in X$ and $\|F\|_Y \le \|f\|_Y$. Then $\|F\|_X = \|f\|_Y$.

Since Y is convex and the quasi-metric d is convex (in the sense of Definition 1) we have

$$\begin{aligned} & F(\alpha x_1 + (1-\alpha) x_2) \\ \leq\ & f(\alpha y_1 + (1-\alpha) y_2) + \\ & + \|f\|_Y \cdot d(\alpha x_1 + (1-\alpha) x_2, \alpha y_1 + (1-\alpha) y_2) \\ \leq\ & \alpha f(y_1) + (1-\alpha) f(y_2) + \\ & + \|f\|_Y [\alpha d(x_1, y_1) + (1-\alpha) d(x_2, y_2)] \\ =\ & \alpha [f(y_1) + \|f\|_Y d(x_1, y_1)] + \\ & + (1-\alpha) [f(y_2) + (1-\alpha) d(x_2, y_2)], \end{aligned}$$

for all $x_1, x_2 \in X$, $y_1, y_2 \in Y$ and $\alpha \in [0,1]$.

Taking the infimum with respect to $y_1, y_2 \in Y$ we obtain

$$F(\alpha x_1 + (1-\alpha) x_2) \leq \alpha F(x_1) + (1-\alpha) F(x_2) \tag{9}$$

for all $x_1, x_2 \in X$ and all $\alpha \in [0,1]$, showing that the function F in $E_Y(f)$, defined by (8), is convex.

b) If $f \in S\,LipY$ is concave on Y, let $G : X \to \mathbb{R}$ be defined by

$$G(x) = \sup_{y \in Y} \{f(y) - \|f\|_Y \cdot d(y, x)\}, \ x \in X \tag{10}$$

Then G is well defined and $G \in E_Y(f)$ [4, Theorem 2].

For any $x_1, x_2 \in X$, $y_1, y_2 \in Y$ and $\alpha \in [0,1]$ we have

$$\begin{aligned} & G(\alpha x_1 + (1-\alpha) x_2) \\ \geq\ & f(\alpha y_1 + (1-\alpha) y_2) - \\ & - \|f\|_Y d(\alpha y_1 + (1-\alpha) y_2, \alpha x_1 + (1-\alpha) x_2) \\ \geq\ & \alpha f(y_1) + (1-\alpha) f(y_2) - \\ & - \|f\|_Y [\alpha \cdot d(y_1, x_1) + (1-\alpha) d(y_2, x_2)] \\ =\ & \alpha [f(y_1) - \|f\|_Y d(y_1, x_1)] + \\ & + (1-\alpha) [f(y_2) - \|f\|_Y d(y_2, x_2)] \end{aligned}$$

Taking the supremum with respect to $y_1, y_2 \in Y$ we get

$$G(\alpha x_1 + (1-\alpha) x_2) \geq \alpha G(x_1) + (1-\alpha) G(x_2) \tag{11}$$

for all $x_1, x_2 \in X$ and $\alpha \in [0,1]$, showing that the function G from $E_Y(f)$, defined by (10), is concave. ■

Definition 1.4 *Let X be a real linear space and $\theta \in Z \subset X$ where θ denotes the null element of X. The set Z is called starshaped if*

$$(\forall)\,\alpha,\ \alpha \in [0,1],\ (\forall)\, z \in Z,\ \alpha z \in Z. \tag{12}$$

Obviously that any convex subset of X which contains θ is starshaped, and the converse is not true in general.

A function $f : Z \to \mathbb{R}$,where Z is a starshaped subset of a linear space X, is called starshaped if

$$f(\alpha z) \leq \alpha f(z). \tag{12a}$$

for all $z \in Z$ and $\alpha \in [0,1]$. A convex function $f : Y \to \mathbb{R}$,defined on a convex subset Y of X containing θ, and such that $f(\theta) \leq 0$ is starshaped but there are starshaped functions on Y which are not convex.

Definition 1.5 *Let (X,d) be a quasi-metric linear space. The quasi-metric d is called starshaped on X if the inequality*

$$d(\alpha x, \alpha y) \leq \alpha d(x,y) \tag{13}$$

holds for all $x, y \in X$ and $\alpha \in [0,1]$.

Remark 1.1 *If (X,d) is a quasi-metric linear space with convex quasi-metric, then d is starshaped because the inequality*

$$d(\alpha x + (1-\alpha)\theta),\ \alpha y + (1-\alpha)\theta \leq \alpha d(x,y) + (1-\alpha) d(\theta,\theta)$$

yields

$$d(\alpha x, \alpha y) \leq \alpha d(x,y)$$

for all $x, y \in X$ and $\alpha \in [0,1]$.

Now we shall prove the extension result for starshaped semi-Lipschitz functions.

Theorem 1.2 *Let (X,d) be a quasi-metric linear space with starshaped quasi-metric d and let Z be a starshaped subset of X.*

Then every starshaped function $\varphi \in S\,LipZ$ admits at least one starshaped extension $\Phi \in E_Z(\varphi)$.

Proof. Let $\varphi \in S\,LipZ$ be starshaped on the starshaped set $Z \subset X$. The function

$$(14) \qquad \Phi(x) = \inf_{z \in Z} \{\varphi(z) + \|\varphi\|_Z\, d(x,z)\}, \; x \in X,$$

belongs to $E_Z(\varphi)$ (Th 2 in [4]).

We shall show that Φ is also starshaped on X. To this end let $x \in X, z \in Z$ and $\alpha \in [0,1]$. We have

$$\begin{aligned} \Phi(\alpha x) &= \varphi(\alpha z) + \|\varphi\|_Z\, d(\alpha x, \alpha z) \\ &\leq \alpha\varphi(z) + \alpha\|\varphi\|_Z\, d(x,z) \\ &= \alpha[\varphi(z) + \|\varphi\|_Z \cdot d(x,z)]. \end{aligned}$$

Taking the infimum with respect to $z \in Z$ we get

$$\Phi(\alpha x) \leq \alpha\Phi(x)$$

for all $x \in X$ and all $\alpha \in [0,1]$, showing that the function Φ defined by (14) is a starshaped extension of φ. ∎

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