

THE CONVERGENCE OF MEAN VALUE ITERATION FOR A FAMILY OF MAPS

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We consider a mean value iteration for a family of functions, which corresponds to the Mann iteration with $\lim_{n \rightarrow \infty} \alpha_n \neq 0$. We prove convergence results for this iteration when applied to strongly pseudocontractive or strongly accretive maps.

1. Introduction

Let X be a real Banach space. The map $J : X \rightarrow 2^{X^*}$ given by

$$Jx := \{f \in X^* : \langle x, f \rangle = \|x\|^2, \|f\| = \|x\|\}, \quad \forall x \in X, \quad (1.1)$$

is called *the normalized duality mapping*. Let $y \in X$ and $j(y) \in J(y)$; note that $\langle \cdot, j(y) \rangle$ is a Lipschitzian map.

Remark 1.1. The above J satisfies

$$\langle x, j(y) \rangle \leq \|x\| \|y\|, \quad \forall x \in X, \forall j(y) \in J(y). \quad (1.2)$$

Definition 1.2. Let B be a nonempty subset of X . The map $T : B \rightarrow B$ is strongly pseudocontractive if there exist $k \in (0, 1)$ and $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y) \rangle \leq k \|x - y\|^2, \quad \forall x, y \in B. \quad (1.3)$$

A map $S : B \rightarrow B$ is called strongly accretive if there exist $k \in (0, 1)$ and $j(x - y) \in J(x - y)$ such that

$$\langle Sx - Sy, j(x - y) \rangle \geq k \|x - y\|^2, \quad \forall x, y \in B. \quad (1.4)$$

In (1.3), take $k = 1$ to obtain a pseudocontractive map. In (1.4), take $k = 0$ to obtain an accretive map.

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Let B be a nonempty and convex subset of X , $T : B \rightarrow B$ and $x_0, u_0 \in B$. The Mann iteration (see [3]) is defined by

$$u_{n+1} = (1 - \alpha_n)u_n + \alpha_n T u_n. \quad (1.5)$$

The Ishikawa iteration is defined (see [2]) by

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n T y_n, \\ y_n &= (1 - \beta_n)x_n + \beta_n T x_n, \end{aligned} \quad (1.6)$$

where $\{\alpha_n\} \subset (0, 1)$ and $\{\beta_n\} \subset [0, 1)$.

Let $s \geq 2$ be fixed. Let $T_i : B \rightarrow B$, $1 \leq i \leq s$, be a family of functions. We consider the following multistep procedure:

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n T_1 y_n^1, \\ y_n^i &= (1 - \beta_n^i)x_n + \beta_n^i T_{i+1} y_n^{i+1}, \quad i = 1, \dots, s-2, \\ y_n^{s-1} &= (1 - \beta_n^{s-1})x_n + \beta_n^{s-1} T_s x_n. \end{aligned} \quad (1.7)$$

Let $A, b \in (0, 1)$ be fixed. The sequence $\{\alpha_n\} \subset (0, 1)$ satisfies

$$0 < A \leq \alpha_n \leq b < 2(1 - k), \quad \forall n \in \mathbb{N}, \quad (1.8)$$

$$\{\beta_n^i\} \subset [0, 1), \quad i = 1, \dots, s-1. \quad (1.9)$$

Let $F(T_1, \dots, T_s)$ denote the common fixed points set with respect to B for the family T_1, \dots, T_s . In this paper, we will prove convergence results for iteration (1.7), for strongly pseudocontractive (accretive) maps when $\{\alpha_n\}$ satisfies (1.8). These results improve the recently obtained results from [6], in which $\{\alpha_n\}$ and $\{\beta_n\}$ converge to zero. We give two numerical examples in which iteration (1.7), when $\{\alpha_n\}$ satisfies (1.8), converges faster as in [6]. Note that, in both cases, iteration (1.7) converges faster than Ishikawa iteration.

LEMMA 1.3 [4]. *Let X be a real Banach space, and let $J : X \rightarrow 2^{X^*}$ be the duality mapping. Then for any given $x, y \in X$,*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y) \rangle, \quad \forall x, y \in X, \quad \forall j(x + y) \in J(x + y). \quad (1.10)$$

LEMMA 1.4 [7]. *Let $\{a_n\}$ be a nonnegative sequence which satisfies the inequality*

$$a_{n+1} \leq (1 - t)a_n + \sigma_n, \quad (1.11)$$

where $t \in (0, 1)$ is fixed, $\lim_{n \rightarrow \infty} \sigma_n = 0$. Then $\lim_{n \rightarrow \infty} a_n = 0$.

2. Main result

THEOREM 2.1. *Let $s \geq 2$ be fixed, X a real Banach space, and B a nonempty, closed, convex subset of X . Let $T_1 : B \rightarrow B$ be a strongly pseudocontractive operator and $T_2, \dots, T_s : B \rightarrow B$,*

with $T_i(B)$ bounded for all $1 \leq i \leq s$, such that $F(T_1, \dots, T_s) \neq \emptyset$. If $A, b \in (0, 1)$, $\{\alpha_n\} \subset (0, 1)$ satisfies (1.8), $x_0 \in B$, and the following condition is satisfied:

$$\lim_{n \rightarrow \infty} \|T_1 x_{n+1} - T_1 y_n^1\| = 0, \tag{2.1}$$

then iteration (1.7) converges to the unique common fixed point of T_1, \dots, T_s , which is the unique fixed point of T_1 .

Proof. Any common fixed point of T_1, \dots, T_s , in particular, is a fixed point of T_1 . However, T_1 can have at most one fixed point since it is strongly pseudocontractive. Let $x^* = F(T_1, \dots, T_s)$. Denote

$$M = \sup_{n \in \mathbb{N}} \{\|T_1 y_n^1\|, \|x_0\|, \|x^*\|\}. \tag{2.2}$$

Then if we assume $\|x_n\| \leq M$, by

$$\|x_{n+1}\| \leq (1 - \alpha_n)\|x_n\| + \alpha_n\|T_1 y_n^1\| \leq M, \tag{2.3}$$

we get $\|x_{n+1}\| \leq M$.

From (1.2) and (1.10), with

$$\begin{aligned} x &:= (1 - \alpha_n)(x_n - x^*), \\ y &:= \alpha_n(T_1 y_n^1 - T_1 x^*), \\ x + y &= x_{n+1} - x^*, \end{aligned} \tag{2.4}$$

we get

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &= \|(1 - \alpha_n)(x_n - x^*) + \alpha_n(T_1 y_n^1 - T_1 x^*)\|^2 \\ &\leq (1 - \alpha_n)^2 \|x_n - x^*\|^2 + 2\alpha_n \langle T_1 y_n^1 - T_1 x^*, j(x_{n+1} - x^*) \rangle \\ &= (1 - \alpha_n)^2 \|x_n - x^*\|^2 + 2\alpha_n \langle T_1 x_{n+1} - T_1 x^*, j(x_{n+1} - x^*) \rangle \\ &\quad + 2\alpha_n \langle T_1 y_n^1 - T_1 x_{n+1}, j(x_{n+1} - x^*) \rangle \\ &\leq (1 - \alpha_n)^2 \|x_n - x^*\|^2 + 2\alpha_n k \|x_{n+1} - x^*\|^2 \\ &\quad + 2\alpha_n \langle T_1 y_n^1 - T_1 x_{n+1}, j(x_{n+1} - x^*) \rangle \\ &\leq (1 - \alpha_n)^2 \|x_n - x^*\|^2 + 2\alpha_n k \|x_{n+1} - x^*\|^2 \\ &\quad + 2\alpha_n \|T_1 y_n^1 - T_1 x_{n+1}\| \|x_{n+1} - x^*\| \\ &\leq (1 - \alpha_n)^2 \|x_n - x^*\|^2 + 2\alpha_n k \|x_{n+1} - x^*\|^2 \\ &\quad + 4\alpha_n \|T_1 y_n^1 - T_1 x_{n+1}\| M. \end{aligned} \tag{2.5}$$

Using (1.8), we obtain

$$(1 - \alpha_n)^2 \leq 1 - 2\alpha_n + \alpha_n b < 1 - 2\alpha_n + \alpha_n 2(1 - k) = 1 - 2\alpha_n k, \tag{2.6}$$

thus,

$$\|x_{n+1} - x^*\|^2 \leq \frac{(1 - \alpha_n)^2}{1 - 2\alpha_n k} \|x_n - x^*\|^2 + \frac{4\alpha_n M}{1 - 2\alpha_n k} \|T_1 y_n^1 - T_1 x_{n+1}\|. \tag{2.7}$$

The following inequality is satisfied:

$$\begin{aligned} \frac{(1 - \alpha_n)^2}{1 - 2\alpha_n k} &= \frac{(1 - \alpha_n)^2 (1 - 2\alpha_n k + 2\alpha_n k)}{1 - 2\alpha_n k} = (1 - \alpha_n)^2 \left(1 + \frac{2\alpha_n k}{1 - 2\alpha_n k}\right) \\ &= (1 - \alpha_n)^2 + \frac{2\alpha_n k (1 - \alpha_n)^2}{1 - 2\alpha_n k} \leq (1 - \alpha_n)^2 + 2\alpha_n k \leq 1 - 2\alpha_n + \alpha_n b + 2\alpha_n k \\ &= 1 - (2(1 - k) - b)\alpha_n \leq 1 - (2(1 - k) - b)A. \end{aligned} \tag{2.8}$$

Substituting (2.6) and (2.8) into (2.7), we obtain

$$\|x_{n+1} - x^*\|^2 \leq (1 - (2(1 - k) - b)A) \|x_n - x^*\|^2 + \frac{4bM}{1 - 2bk} \|T_1 y_n^1 - T_1 x_{n+1}\|. \tag{2.9}$$

Set

$$\begin{aligned} a_n &:= \|x_n - x^*\|^2, \\ t &:= (2(1 - k) - b)A \in (0, 1), \\ \sigma_n &:= \frac{4bM}{1 - 2bk} \|T_1 y_n^1 - T_1 x_{n+1}\|. \end{aligned} \tag{2.10}$$

From (2.1), we know that $\lim_{n \rightarrow \infty} \sigma_n = 0$; all the assumptions of Lemma 1.4 are fulfilled and consequently we have $\lim_{n \rightarrow \infty} \|x_n - x^*\| = 0$. □

In Theorem 2.1, $\{\alpha_n\}$ does not converge to zero, while in [6], $\{\alpha_n\}$ converges to zero.

THEOREM 2.2 [6]. *Let $s \geq 2$ be fixed, X a real Banach space with a uniformly convex dual, and B a nonempty, closed, convex subset of X . Let $T_1 : B \rightarrow B$ be a strongly pseudocontractive operator and $T_2, \dots, T_s : B \rightarrow B$, with $T_i(B)$ bounded for all $1 \leq i \leq s$, such that $F(T_1, \dots, T_s) \neq \emptyset$. If $\{\alpha_n\} \subset (0, 1)$ satisfies $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = +\infty$, and $\{\beta_n^i\} \subset [0, 1)$, $i = 1, \dots, s - 1$, satisfy $\lim_{n \rightarrow \infty} \beta_n^1 = 0$, then iteration (1.7) converges to a fixed point of T_1, \dots, T_s .*

The Banach space in Theorem 2.1 contains no restrictions.

3. Further results

Denote by I the identity map.

Remark 3.1. Let $T, S : X \rightarrow X$ and let $f \in X$ be given. Then,

- (i) a fixed point for the map $Tx = f + (I - S)x$, for all $x \in X$, is a solution for $Sx = f$;
- (ii) a fixed point for $Tx = f - Sx$ is a solution for $x + Sx = f$.

Remark 3.2 [5]. The following are true.

- (i) The operator $T : X \rightarrow X$ is a (strongly) pseudocontractive map if and only if $(I - T) : X \rightarrow X$ is (strongly) accretive.
- (ii) If $S : X \rightarrow X$ is an accretive map, then $T = f - S : X \rightarrow X$ is a strongly pseudocontractive map.

We consider iteration (1.7), with $T_i x = f_i + (I - S_i)x$, $1 \leq i \leq s$ and $s \geq 2$, $\{\alpha_n\} \subset (0, 1)$, $\{\beta_n^i\} \subset [0, 1]$, $i = 1, \dots, s - 1$ satisfying (1.8):

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n(f_1 + (I - S_1)y_n^1), \\ y_n^i &= (1 - \beta_n^i)x_n + \beta_n^i(f_{i+1} + (I - S_{i+1})y_n^{i+1}), \quad i = 1, \dots, s - 2, \\ y_n^{s-1} &= (1 - \beta_n^{s-1})x_n + \beta_n^{s-1}(f_{s-1} + (I - S_s)x_n). \end{aligned} \tag{3.1}$$

Theorem 2.1, Remark 3.1(i), and Remark 3.2(i) lead to the following result.

COROLLARY 3.3. *Let $s \geq 2$ be fixed, X a real Banach space, and $S_1 : X \rightarrow X$ a strongly accretive operator, $S_2, \dots, S_s : X \rightarrow X$, such that the equations $S_i x = f_i$, $1 \leq i \leq s$, have a common solution and $T_i(X)$, $1 \leq i \leq s$, are bounded. If $A, b \in (0, 1)$, $\{\alpha_n\} \subset (0, 1)$ satisfies (1.8), and condition (2.1) is satisfied, then iteration (3.1) converges to a common solution of $S_i x = f_i$, $1 \leq i \leq s$.*

We consider iteration (1.7), with $T_i x = f_i - S_i x$, $1 \leq i \leq s$, and $s \geq 2$, $\{\alpha_n\} \subset (0, 1)$, $\{\beta_n^i\} \subset [0, 1]$, $i = 1, \dots, s - 1$, satisfying (1.8):

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n(f_1 - S_1 y_n^1), \\ y_n^i &= (1 - \beta_n^i)x_n + \beta_n^i(f_{i+1} - S_{i+1} y_n^{i+1}), \quad i = 1, \dots, s - 2, \\ y_n^{s-1} &= (1 - \beta_n^{s-1})x_n + \beta_n^{s-1}(f_{s-1} - S_s x_n). \end{aligned} \tag{3.2}$$

Theorem 2.1, Remark 3.1(ii), and Remark 3.2(ii) lead to the following result.

COROLLARY 3.4. *Let $s \geq 2$ be fixed, X a real Banach space, and $S_1 : X \rightarrow X$ an accretive operator, $S_2, \dots, S_s : X \rightarrow X$, such that the equations $x + S_i x = f_i$, $1 \leq i \leq s$, have a common solution and $S_i(X)$, $1 \leq i \leq s$, are bounded. If $A, b \in (0, 1)$, $\{\alpha_n\} \subset (0, 1)$ satisfies (1.8), and condition (2.1) is satisfied, then iteration (3.2) converges to a common solution of $x + S_i x = f_i$, $1 \leq i \leq s$.*

4. Numerical examples

Let $X = \mathbb{R}^2$ be the euclidean plane, consider $x = (a, b) \in \mathbb{R}^2$, with $x^\perp = (b, -a) \in \mathbb{R}^2$. We know that $\langle x, x^\perp \rangle = 0$, $\|x\| = \|x^\perp\|$, $\langle x^\perp, y^\perp \rangle = \langle x, y \rangle$, $\|x^\perp - y^\perp\| = \|x - y\|$, and $\langle x^\perp, y \rangle + \langle x, y^\perp \rangle = 0$, for all $x, y \in \mathbb{R}^2$. Denote by B the closed unit ball. In [1], we can get the following example in which Ishikawa iteration (1.6) converges and (1.5) is not convergent.

Table 4.1

\Iteration (1.7)	Case 1	Case 2
Step 10	(0.2217, 0.1480)	(0.0151, -0.0023)
Step 15	(0.1837, 0.1184)	(0.0017, -0.0006)
Step 20	(0.1603, 0.1015)	(0.0002, -0.0001)
Step 21	(0.1566, 0.0989)	$10^{-3} \cdot (0.1156, -0.0686)$
Step 22	(0.1531, 0.0965)	$10^{-4} \cdot (0.7406, -0.4641)$
Step 23	(0.1499, 0.0942)	$10^{-4} \cdot (0.4743, -0.3129)$
Step 24	(0.1468, 0.0921)	$10^{-4} \cdot (0.3037, -0.2103)$
Step 25	(0.1440, 0.0902)	$10^{-4} \cdot (0.1945, -0.1409)$

Example 4.1 [1]. Let $H = \mathbb{R}^2$ and let

$$B_1 = \left\{ x \in \mathbb{R}^2 : \|x\| \leq \frac{1}{2} \right\}, \quad B_2 = \left\{ x \in \mathbb{R}^2 : \frac{1}{2} \leq \|x\| \leq 1 \right\}. \quad (4.1)$$

The map $T : B \rightarrow B$ is given by

$$Tx = \begin{cases} x + x^\perp, & x \in B_1 \\ \frac{x}{\|x\|} - x + x^\perp, & x \in B_2. \end{cases} \quad (4.2)$$

Then the following are true:

- (i) T is Lipschitz and pseudocontractive;
- (ii) for all $(\alpha_n)_n \subset (0, 1)$, the Mann iteration does not converge to the fixed point of T (which is the point $(0, 0)$ and it is unique).

The main result from [2] assures the convergence of the Ishikawa iteration (1.6) applied to the map T given by (4.2). The convergence is very slow. In [6], for the same T , it was shown that iteration (1.7) converges faster. Here, we give an example for which (1.7) with $\{\alpha_n\}$ satisfying (1.8) converges even faster as in [6].

Case 1 [6]. Consider now $T_1(x, y) = 0.5 \cdot (x, y)$, for all $(x, y) \in B$, $T_2 = T$, and $s = 2$, where T is given by (4.2), the initial point $x_0 = (0.5, 0.7)$, and $\alpha_n = \beta_n = 1/(n+1)$ in (1.7). The main result from [6] assures the convergence of (1.7).

Case 2. Consider $T_1(x, y) = 0.5 \cdot (x, y)$, for all $(x, y) \in B$, $T_2 = T$, and $s = 2$, where T is given by (4.2), the initial point $x_0 = (0.5, 0.7)$, $\alpha_n = 0.7$, for all $n \in \mathbb{N}$, and $\beta_n = 1/(n+1)$ in (1.7). The fixed point for both functions is $(0, 0)$. Observe that $k = 0.5$, and $\{\alpha_n\}$ satisfies (1.8):

$$A = 0.7 = \alpha_n = b \leq 2(1 - k) = 1, \quad \forall n \in \mathbb{N}. \quad (4.3)$$

Note that Mann iteration does not converge for any $\{\alpha_n\} \subset (0, 1)$. Using a Matlab program, we obtain Table 4.1.

Case 3. Consider in (1.7) the same T_1 , T_2 , $s = 2$, and x_0 as in Case 1 and $\alpha_n = \beta_n = 1/\sqrt{n+1}$.

Table 4.2

\Iteration	Case 3 (1.7)	Case 4 (1.7)	Ishikawa iteration
Step 10	(0.0631, -0.0333)	(0.0044, -0.0164)	(0.4545, 0.2689)
Step 15	(0.0256, -0.0221)	(-0.0010, -0.0018)	(0.1289, -0.4827)
Step 20	(0.0117, -0.0139)	$10^{-5} \cdot (-22.6516, -11.0267)$	(-0.4456, -0.1532)
Step 11	(0.0101, -0.0126)	$10^{-5} \cdot (-15.5657, -5.4373)$	(-0.4651, -0.0274)
Step 22	(0.0087, -0.0115)	$10^{-5} \cdot (-10.5234, -2.3727)$	(-0.4511, 0.0941)
Step 23	(0.0075, -0.0105)	$10^{-5} \cdot (-7.0134, -0.7743)$	(-0.4077, 0.2037)
Step 24	(0.0066, -0.0096)	$10^{-5} \cdot (-4.6140, -0.0022)$	(-0.3407, 0.2954)
Step 25	(0.0057, -0.0088)	$10^{-5} \cdot (-2.9993, 0.3215)$	(-0.2562, 0.3654)
Step 1500	—	—	(0.0790, -0.0311)

Case 4. Consider in (1.7) $T_1, T_2, s = 2$, and x_0 as above and $\alpha_n = 0.7$, for all $n \in \mathbb{N}$, $\beta_n = 1/\sqrt{n+1}$.

Also, consider the Ishikawa iteration with the same T as in (4.2), $x_0 = (0.5, 0.7)$, $\alpha_n = \beta_n = 1/\sqrt{n+1}$, for all $n \in \mathbb{N}$. The main result from [2] assures the convergence of Ishikawa iteration. Note that in this case the convergence is very slow. Eventually, Example 4.1 assures that for the same map, Mann iteration does not converge. A Matlab program leads to the evaluations illustrated in Table 4.2.

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