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On the Approximation of the Global Extremum of a Semi-Lipschitz Function

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Abstract. In this paper one obtains a sequential procedure for determining the global extremum of a semi-Lipschitz real-valued function defined on a quasi-metric (asymmetric metric) space.

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1. Introduction

For a function from a specified class, a method for seeking its extremum deals with the problem of estimating the global maximum or/and minimum values of the function and locating the points where the extremum is attained.

An important class of such methods is the class of sequential methods i.e. in which the choice of each evaluation point, except for the first one, depends on the location and the values of the function at the previous points and, possibly, on the number n of the evaluations to be performed. In the latter case the method is called an n-step method. In the following, a sequential method is obtained for evaluating the global maximum and the global minimum of a semi-Lipschitz real-valued function defined on a subset of a quasi-metric space, sometimes called asymmetric metric space (see [7], [27]).

In order to determine the absolute maximum M_f of a real semi-Lipschitz function f, the algorithm we propose determines a decreasing sequence of numbers $(M_n)_{n\geq 1}$, having the limit M_f. Each number M_n $(n = 1, 2, ...)$ is the absolute maximum of a special semi-Lipschitz function $U_n(f)$. This function has a very simple analytical expression compared to the given function f (which is assumed only to be semi-Lipschitz). For determining $U_n(f)(x)$ one requires on one hand

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the computation of the value of f at a certain point, and the values of f at the n point from the previous step, and on the other hand the quasi-distances from the current point x to the $n+1$ points. One can see therefore that the determining of the maximum M_{n+1} of $U_{n+1}(f)$ requires a small amount of computation.The absolute minimum of f is given by the absolute maximum of $-f$.

We present in the following the framework of the described method.

Let X be a non-empty set. A function $d : X \times X \to [0, \infty)$ is called a *quasi-metric* on X [21] (see also [7], [27]) if the following conditions hold

$$AM1) \quad d(x, y) = d(y, x) = 0 \text{ iff } x = y,$$
$$AM2) \quad d(x, z) \le d(x, y) + d(y, z),$$

for all $x, y, z \in X$.

The function $\overline{d} : X \times X \to [0, \infty)$ defined by $\overline{d}(x, y) = d(y, x)$ for all $x, y \in X$ is also a quasi-metric on X, called the *conjugate* quasi-metric of d. A pair (X, d), where X is a non-empty set and d a quasi-metric on X, is called a quasi-metric space. Obviously, the function $d^s(x, y) = \max\{d(x, y), \overline{d}(x, y)\}$ is a metric on X. Each quasi-metric d on X induces a topology $\tau(d)$ on X which has as a base the family of balls *(forward open balls* [7]).

$$B^+(x, \varepsilon) := \{y \in X : d(x, y) < \varepsilon\},\ x \in X,\ \varepsilon > 0.$$

This topology is called the *forward topology* of X ([7], [15]) and is denoted by τ_+. Analogously, the quasi-metric $\overline{d}$ induces the topology $\tau(\overline{d})$ on X which has as a base the family of *backward open balls* ([7])

$$B^-(x, \varepsilon) := \{y \in X : d(y, x) < \varepsilon\},\ x \in X,\ \varepsilon > 0.$$

This topology is called the *backward topology* of X ([7], [15]) and is denoted by τ_-.

Note that the topology τ_+ is a T_0-topology. If the condition AM1) is replaced by the condition: AM0) $d(x, y) = 0$ iff $x = y$, then τ_+ is a T_1-topology. The pair (X, d) is called a T_0 quasi-metric space, respectively a T_1 quasi-metric space (see [21] and [22]).

Let (X, d) be a quasi-metric space. A sequence $(x_k)_{k\ge1}$ *d-converges* to $x_0 \in X$ (respectively *$\overline{d}$-converges* to $x_0 \in X$) iff

$$\lim_{k\to\infty} d(x_0, x_k) = 0, \text{ respectively } \lim_{k\to\infty} d(x_k, x_0) = \lim_{k\to\infty} \overline{d}(x_0, x_k) = 0.$$

A set $K \subset X$ is called *d-compact* if every open cover of K with respect to the forward topology has a finite subcover. We say that K is *d-sequentially compact* if every sequence in K has a d-convergent subsequence with limit in K (Definition 4.1 in [7]). Finally, the set Y in (X, d) is called *$(d, \overline{d})$-sequentially compact* if every sequence $(y_n)_{n\ge1}$ in Y has a subsequence (y_{n_k}) d-convergent to $u \in Y$ and $\overline{d}$-convergent to $v \in Y$.

Observe that, if (X, d) is a quasi-metric space $(d, \overline{d})$-sequentially compact and T_0-separated, then it is possible to find sequences with all subsequences both d-convergent and $\overline{d}$-convergent, but to different limits. For example, let $X = [0, 1]$

and $d(x,y) = (y-x) \vee 0, x, y \in [0,1]$. Then $\overline{d}(x,y) = (x-y) \vee 0$ and the sequence $(\frac{1}{n})_{n\geq 1}$ satisfies the property that every subsequences d-converges to 0 and $\overline{d}$-converges to 1. But if (X,d) is $(d,\overline{d})$-sequentially compact and T_1-separated, then by Lemma 3.1 of [7] it follows that if $(x_n)_{n\geq 1} \subset X$ is d-convergent to $x_0 \in X$ and $\overline{d}$-convergent to $y_0 \in X$, then $x_0 = y_0$. This fact is essential in the proof of Theorem 3.1 from bellow.

Definition 1.1 ([21]**).** Let Y be a non-empty subset of a quasi-metric space (X,d). A function $f : Y \to \mathbb{R}$ is called d-semi-Lipschitz if there exists $L \geq 0$ (named a d-semi-Lipschitz constant for f) such that

$$f(x) - f(y) \leq Ld(x,y), \text{ for all } x, y \in Y. \tag{1.1}$$

A function $f : Y \to \mathbb{R}$ is called $\leq_d$-increasing if $f(x) \leq f(y)$ whenever $d(x,y) = 0$.

Denote by $\mathbb{R}^Y_{\leq_d}$ the set of all $\leq_d$-increasing functions on Y. This set is a cone in the linear space $\mathbb{R}^Y$ of all real-valued functions defined on Y, i.e., for each $f, g \in \mathbb{R}^Y_{\leq_d}$ and $\lambda \geq 0$ it follows that $f + g \in \mathbb{R}^Y_{\leq_d}$ and $\lambda f \in \mathbb{R}^Y_{\leq_d}$.

For a d-semi-Lipschitz function f on Y, put [21]

$$\|f|_d = \sup_{\substack{d(x,y)>0 \\ x,y\in Y}} \frac{(f(x) - f(y)) \vee 0}{d(x,y)}. \tag{1.2}$$

Then $\|f|_d$ is the *smallest d-semi-Lipschitz constant* for f ([18]).

For a fixed element $\theta \in Y$ denote

$$d\text{-}SLip_0Y := \{f \in \mathbb{R}^Y_{\leq_d} : \|f|_d < \infty \text{ and } f(\theta) = 0\}. \tag{1.3}$$

If (X,d) is a T_1 quasi-metric space, then every $f \in \mathbb{R}^X$ is $\leq_d$-increasing ([21]).

The set defined by (1.3) is a subcone of the cone $\mathbb{R}^Y_{\leq_d}$, and the functional $\|\ |_d$: d-$SLip_0Y \to [0,\infty)$ defined by (1.2) is an asymmetric norm, i.e., it is subadditive, positive homogeneous and $\|f|_d = 0$ iff $f \equiv 0$. The pair $(d\text{-}SLip_0Y, \|\ |_d)$ is called the normed cone of real semi-Lipschitz functions on Y, vanishing at the fixed point $\theta \in Y$([22]).

In [22] some properties of the normed cone $(d\text{-}SLip_0\ Y, \|\ |_d)$ are presented. Similar properties in the case of semi-Lipschitz functions on a quasi-metric space with values in a quasi-normed linear space (space with asymmetric norm) are discussed in [24]. For more information concerning other properties of quasi-metric spaces and their applications, see [7], [8], [13], [20], [26].

2. Results

Let $f \in d$-$SLip_0Y$. A function F in d-$SLip_0X$ satisfying the inequality

$$F(u) - F(v) \leq \|f|_d\, d(u,v),$$

for all $u, v \in X$ and such that $F(y) = f(y)$ for all $y \in Y$ is called an *extension* of f (preserving the asymmetric norm $\|f|_d$).

It follows that each extension $F \in d\text{-}SLip_0 X$ of $f \in d\text{-}SLip_0 Y$ satisfies

$$F|_Y = f \text{ and } \|F|_d = \|f|_d. \tag{2.1}$$

The existence of such an extension for each $f \in d\text{-}SLip_0\ Y$ follows from the following theorem proved in [18]. For the sake of completeness we include the proof.

Theorem 2.1. *Let (X, d) be a quasi-metric space, $\theta \in X$ a fixed element, and Y a subset of X with $\theta \in Y$. Then for every $f \in d\text{-}SLip_0 Y$ there exists at least a function $F \in d\text{-}SLip_0 X$ such that $F|_Y = f$ and $\|F|_d = \|f|_d$.*

Proof. For $f \in d\text{-}SLip_0 Y$ let

$$F_d(f)(x) = \inf_{y \in Y} [f(y) + \|f|_d\, d(x, y)\}, x \in X. \tag{2.2}$$

First we show that $F_d(f)$ is well defined.

Let $x \in X$. For any $y \in Y$ we have

$$\begin{aligned} f(y) + \|f|_d\, d(x, y) &= \|f|_d\, d(x, y) - (f(\theta) - f(y)) \\ &\geq \|f|_d\, d(x, y) - \|f|_d\, d(\theta, y) \\ &= \|f|_d\, (d(x, y) - d(\theta, y)) \geq -\|f|_d\, d(\theta, x), \end{aligned}$$

showing that for every $x \in X$ the set

$$\{f(y) + \|f|_d\, d(x, y) : y \in Y\}$$

is bounded from below and, consequently, the infimum in (2.2) is finite.

Now we show that $F_d(f)|_Y = f$, $F_d(f) \in d\text{-}SLip_0\ X$ and $\|F_d(f)|_d = \|f|_d$.

For every $y \in Y$ we have

$$F_d(f)(x) \leq f(y) + \|f|_d\, d(x, y),\ x \in X,$$

which for $x = y$ yields

$$F_d(f)(y) \leq f(y).$$

On the other hand, for $y \in Y$ and all $y' \in Y$,

$$f(y) - f(y') \leq \|f|_d\, d(y, y')$$

implies

$$f(y) \leq f(y') + \|f|_d\, d(y, y').$$

Taking the infimum with respect to $y' \in Y$ one obtains $f(y) \leq F_d(f)(y)$, so that

$$F_d(f)(y) = f(y),\ y \in Y.$$

Let $x_1, x_2 \in X$ and $\varepsilon > 0$. Choosing $y \in Y$ such that

$$F_d(f)(x_1) \geq f(y) + \|f|_d\, d(x_1, y) - \varepsilon$$

we get

$$\begin{aligned} F_d(f)(x_2) - F_d(f)(x_1) &\leq f(y) + \|f|_d \cdot d(x_2, y) - (f(y) + \|f|_d\, d(x_1, y) - \varepsilon) \\ &= \|f|_d\, (d(x_2, y) - d(x_1, y)) + \varepsilon. \end{aligned}$$

Because $d(x_2, y) - d(x_1, y) \le d(x_2, x_1)$ it follows that

$$F_d(f)(x_2) - F_d(f)(x_1) \le \|f|_d\, d(x_2, x_1).$$

This means that $F_d(f) \in d\text{-}SLip_0\ X$, and by the last inequality

$$\|F_d(f)|_d \le \|f|_d\,.$$

By the definitions of an asymmetric norm

$$\|F_d(f)|_d \ge \|F_d(f)|_Y\,|_d = \|f|_d\,,$$

so that the equality $\|F_d(f)|_d = \|f|_d$ holds. □

The following Remarks 2.2 and 2.3 are taken from [18] and [19].

Remark 2.2. By Theorem 2.1 it follows that for every $f \in d\text{-}SLip_0\ Y$, the set of all extensions preserving the asymmetric norm $\|f|_d$, i.e.

$$\mathcal{E}_d(f) = \{H \in d\text{-}SLip_0 X : H|_Y = f \text{ and } \|H|_d = \|f|_d\} \tag{2.3}$$

is nonempty, because $F_d(f) \in \mathcal{E}_d(f)$ where $F_d(f)$ is given by (2.2).

Analogously, one proves that the function

$$G_d(f) = \sup_{y\in Y}\{f(y) - \|f|_d\, \overline{d}(x, y)\}, x \in X, \tag{2.4}$$

is in $\mathcal{E}_d(f)$.

Remark 2.3. Obviously, the set $\mathcal{E}_d(f)$ is convex, i.e. for every $H_1, H_2 \in \mathcal{E}_d(f)$ and $\lambda \in [0, 1]$ it follows $\lambda H_1 + (1 - \lambda)H_2 \in \mathcal{E}_d(f)$. Moreover for every $H \in \mathcal{E}_d(f)$ we have:

$$G_d(f)(x) \le H(x) \le F_d(f)(x),\ \ x \in X. \tag{2.5}$$

The function $F_d(f)$ defined by (2.2) is called the maximal extension of f, and $G_d(f)$ defined by (2.4) is called the minimal extension of f.

Remark 2.4. If $\theta \in Y_1 \subset Y_2 \subset Y$ and $f \in d\text{-}SLip_0 Y$, then for each $u \in Y$ we can easily obtain:

$$\inf_{y\in Y_1}\{f(y) + \|f|_d\, d(u, y)\} \ge \inf_{y\in Y_2}\{f(y) + \|f|_d\, d(u, y)\}$$

and

$$\sup_{y\in Y_1}\{f(y) - \|f|_d\, \overline{d}(u, y)\} \le \sup_{y\in Y_2}\{f(y) - \|f|_d\, \overline{d}(u, y)\}.$$

Remark 2.5. Observe that Theorem 2.1 is the "nonsymmetric" analog of McShane's theorem [14] for metric spaces.

Theorem 2.6. *Let (X, d) be a quasi-metric space and $Y \subseteq X$. Then*

(a) *Every $f \in d\text{-}SLipY$ is upper semicontinuous on $(Y, \overline{d})$;*
(b) *If Y is $\overline{d}$-sequentially compact, then every $f \in d\text{-}SLipY$ attains its maximum value on Y.*

Proof. Let $f \in d\text{-}SLipY$. If $\|f|_d = 0$ then $f(y) = constant$ for all $y \in Y$ and this function is upper semicontinuous. Let $y_0 \in Y$ and $\|f|_d > 0$. Then the inequality

$$f(y) - f(y_0) \le \|f|_d\, d(y, y_0)$$

implies

$$f(y) \le f(y_0) + \|f|_d\, d(y, y_0).$$

For $\varepsilon > 0$ and $y \in Y$ such that $d(y, y_0) < \frac{\varepsilon}{\|f|_d}$ it follows

$$f(y) \le f(y_0) + \varepsilon,$$

showing that f is upper semicontinuous on $(Y, \overline{d})$.

Let Y be $\overline{d}$-sequentially compact in (X, d) and $M = \sup f(Y)$, where $M \in \mathbb{R} \cup \{+\infty\}$. Then there exists a sequence $(y_n)_{n\ge1}$ in Y such that $\lim\limits_{n\to\infty} f(y_n) = M$. Because Y is $\overline{d}$-sequentially compact there exists $y_0 \in Y$ and a subsequence $(y_{n_k})_{k\ge1}$ of $(y_n)_{n\ge1}$ such that $\lim\limits_{k\to\infty} d(y_{n_k}, y_0) = 0$. By the upper semicontinuity of f in y_0 it follows:

$$M = \lim_{k\to\infty} f(y_{n_k}) = \limsup f(y_{n_k}) \le f(y_0) \le M$$

implying $M < \infty$ and $f(y_0) = M$. □

By Theorem 2.6 it follows that for Y $\overline{d}$-sequentially compact, the functional $\|\ |_\infty\colon d\text{-}SLip_0Y \to [0, \infty)$ defined by

$$\|f|_\infty = \max\{f(y) : y \in Y\}$$

is an asymmetric norm on $d\text{-}SLip_0Y$.

Indeed, for every f in $d\text{-}SLip_0Y$ we have $\|f|_\infty \ge f(\theta) = 0$. If $\|f|_\infty > 0$ then there exists $y_0 \in Y$ such that $f(y_0) > 0 = f(\theta)$. Consequently, because $f \in \mathbb{R}^Y_{\le_d}$ it follows $d(y_0, \theta) > 0$, and

$$\|f|_d \ge \frac{f(y_0) - f(\theta)}{d(y_0, \theta)} > 0.$$

It follows $f \neq 0$, because $\|\ |_d$ is asymmetric norm on $d\text{-}SLip_0Y$. Obviously, $\|f + g|_\infty \le \|f|_\infty + \|g|_\infty$ and $\|\lambda f|_\infty = \lambda \|f|_\infty$ for all $f, g \in d\text{-}SLip_0Y$ and $\lambda \ge 0$.

3. The sequential method

Let (X, d) be a quasi-metric space, $\theta \in X$ a fixed element, and $Y \subset X$ with $\theta \in Y$. Suppose that Y is $\overline{d}$-sequentially compact, and $f \in d\text{-}SLip_0Y$. Let

$$M_f = \sup\{f(y) : y \in Y\}$$

and

$$E_f = \{y \in Y : f(y) = M_f\}.$$

We want to find the maximum value M_f of f and a point $y_0 \in E_f$.

For this goal we consider the following sequential method, supposing that $q > 0$ is an upper bound for $\|f|_d$ on Y, i.e. $\|f|_d \le q$.

Firstly, let Z be a nonempty subset of Y with $\theta \in Z$. From the proof of Theorem 2.1, the functions

$$U(f)(y) = \inf\{f(z) + qd(y, z) : z \in Z\},\ y \in Y$$

and

$$u(f)(y) = \sup\{f(z) - qd(z, y) : z \in Z\},\ y \in Y$$

satisfy the conditions:

$$U(f)|_Z = u(f)|_Z = f|_Z$$

and

$$\|U(f)|_d = \|u(f)|_d = q \geq \|f|_d \text{ on } Y.$$

Moreover

$$u(f)(y) \leq f(y) \leq U(f)(y),\ y \in Y.$$

Indeed, for $y \in Y$ and each $z \in Z \subset Y$ we have

$$f(y) - f(z) \leq \|f|_d\, d(y, z) \leq qd(y, z)$$

and

$$f(y) \leq f(z) + qd(y, z).$$

Taking the infimum with respect to $z \in Z$ it follows

$$f(y) \leq U(f)(y),\ y \in Y.$$

Analogously,

$$f(z) - f(y) \leq qd(z, y),$$

implies

$$f(y) \geq f(z) - qd(z, y).$$

Taking the supremum with respect to $z \in Z$ one obtains

$$u(f)(y) \leq f(y),\ y \in Y.$$

If

$$M_U := \max\{U(f)(y) : y \in Y\},$$

then

$$M_f \leq M_U.$$

We define now two sequences $(y_n)_{n\geq 0}$ in Y and $(M_n)_{n\geq 0}$ in $\mathbb{R}$ in the following way.

Let

$$U_0(f)(y) = f(\theta) + qd(y, \theta) = qd(y, \theta),\ y \in Y,$$

i.e. $U_0(f)$ is an extension (the maximal extension) of $f|_{\{\theta\}}$ with the semi-Lipschitz constant q. Then, by the above considerations, it follows

$$f(y) \leq U_0(f)(y),\ y \in Y,$$
$$U_0(f) \in d\text{-}SLip_0 Y.$$

If $y_0 \in Y$ is such that

$$U_0(f)(y_0) = M_0 := \sup U_0(f)(Y),$$

then

$$M_f \le M_0.$$

Let $Z_1 = \{\theta, y_0\}$ and let

$$U_1(f)(y) = \inf_{z \in Z_1} \{f(z) + qd(y,z)\},\ y \in Y,$$

be the maximal extension of $f|_{Z_1}$ with semi-Lipschitz constant q. Then $U_1 \in d$-$SLip_0Y$ and by Remark 2.3, it follows:

$$f(y) \le U_1(f)(y) \le U_0(f)(y),\ y \in Y,$$
$$f|_{Z_1} = U_1(f)|_{Z_1} = U_0(f)|_{Z_1}.$$

If $y_1 \in Y$ is such that

$$U_1(f)(y_1) = M_1 := \sup U_1(f)(Y),$$

then

$$M_f \le M_1 \le M_0.$$

Let now

$$Z_2 = \{\theta, y_0, y_1\}.$$

Supposing that, following the described procedure, we have constructed the sets

$$Z_n = \{\theta, y_0, y_1, \dots, y_{n-1}\} \text{ and } \{M_0, M_1, M_2, \dots, M_{n-1}\}.$$

Put

$$U_n(f)(y) = \inf_{z \in Z_n} \{f(z) + qd(y,z)\},\ y \in Y.$$

It follows

$$f(y) \le U_n(f)(y) \le \dots \le U_1(f)(y) \le U_0(f)(y)$$

for all $y \in Y$.

Choose $y_n \in Y$ such that

$$U_n(f)(y_n) = M_n := \sup U_n(Y).$$

Continuing in this manner we obtain the sequences

$$\begin{aligned} &\{\theta, y_0, y_1, \dots, y_n, \dots\} \subset Y, \text{ and} \\ &\{M_0, M_1, \dots, M_n, \dots\} \subset \mathbb{R}. \end{aligned} \tag{3.1}$$

The following theorem contains the properties of these two sequences, if Y is $(d, \bar{d})$-sequentially compact.

Theorem 3.1. *Let (X,d) be a T_1 quasi-metric space, $\theta \in X$ fixed, and Y a $(d,\bar{d})$-sequentially compact subset of X with $\theta \in Y$. Let $f \in d - SLip_0Y$, $q \ge \|f|_d$ and let (y_n) and (M_n) be the sequences in* (3.1). *Then*

(a) *(M_n) converges to M_f;*

(b) $\lim_{n\to\infty} \inf\{d(y_n, y) : y \in E_f\} = 0.$

Proof. (a). Since for every $n \geq 1$

$$U_n(f)(y) \leq U_{n-1}(f)(y), \ y \in Y,$$

it follows

$$M_n = \sup U_n(f)(Y) \leq \sup U_{n-1}(f)(Y) = M_{n-1}.$$

Therefore, the sequence (M_n) is decreasing. Since $U_n(f)(\theta) = 0$ we have $M_n \geq 0$ for all $n \geq 0$. It follows that there exists $M \geq 0$ such that

$$M = \lim_{n\to\infty} M_n.$$

Since Y is $(d, \overline{d})$ -sequentially compact, the sequence (y_n) contains a subsequence $(y_{n_k})_{k\geq 1}$ which is d- and $\overline{d}$-convergent to an element $\overline{y} \in Y$, i.e.,

$$\lim_{k\to\infty} d(y_{n_k}, \overline{y}) = 0 \text{ and } \lim_{k\to\infty} d(\overline{y}, y_{n_k}) = 0.$$

Furthermore

$$\lim_{k\to\infty} U_{n_k}(f)(y_{n_k}) = \lim_{k\to\infty} M_{n_k} = M.$$

On the other hand, by the upper semicontinuity of the function f, it follows

$$\limsup_{k\to\infty} f(y_{n_k}) \leq f(\overline{y}) \leq M_f.$$

By the definitions of the extensions $U_n(f)$ $(n \geq 1)$ we have

$$\begin{aligned} U_{n_k}(f)(y_{n_k}) - U_{n_k}(f)(y_{n_k-1}) &\leq q d(y_{n_k}, y_{n_k-1}) \\ &\leq q(d(y_{n_k}, \overline{y}) + d(\overline{y}, y_{n_k-1})) \to 0, \text{ as } k \to \infty. \end{aligned}$$

It follows that for every $\varepsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$,

$$U_{n_k}(f)(y_{n_k}) - f(y_{n_k-1}) < \varepsilon,$$

or equivalently,

$$U_{n_k}(f)(y_{n_k}) < f(y_{n_k-1}) + \varepsilon.$$

Taking $\limsup$ as $k \to \infty$, we get

$$M \leq \lim_{k\to\infty} \sup f(y_{n_k-1}) + \varepsilon \leq M_f + \varepsilon.$$

As $\varepsilon > 0$ was arbitrarily chosen, we obtain $M \leq M_f$. Because the inequality $M_f \leq M$ is also true, it follows that *(a)* holds.

(b). For the proof of (b), supposing that the sequence

$$(\inf\{d(y_n, y) : y \in E_f\})_{n\geq 1}$$

does not converge to 0, then there exist $\varepsilon > 0$ and an infinite sequence $n_1 < n_2 < \ldots n_k < \ldots$ such that

$$\inf\{d(y_{n_k}, y) : y \in E_f\} \geq \varepsilon, \ \forall k \in \mathbb{N}.$$

By the $(d, \overline{d})$-sequentially compactness of Y, the sequence $(y_{n_k})_{k\geq 1}$ contains a subsequence $(y_{n_{k_i}})_{i\geq 1}$ that converges to an element $\overline{y} \in Y$ such that $f(\overline{y}) = M_f$, i.e. $\overline{y} \in E_f$, in contradiction to the inequality

$$\inf\{d(y_{n_k}, y) : y \in E_f\} \geq \varepsilon.$$

The theorem is proved. □

Remark 3.2. Let $\overline{M}_n = \max\{f(\theta),\ f(y_0),\ f(y_1),\dots,f(y_n)\}$. Then $\overline{M}_n \le M_f \le M_n$ for every $n = 1,2,3,\dots.$ It follows that

$$M_f - \overline{M}_n \le M_n - \overline{M}_n,\ n = 1,2,\dots.$$

The last inequality is a convenient upper bound for the error $M_f - \overline{M}_n$.

Because

$$U_n(f)(y) = \inf_{z\in\{\theta,y_0,y_1,\dots,y_{n-1}\}=Z_n.} \{f(z) + qd(y,z)\},\ y \in Y$$

has a simple expression depending essentially on $d(y,z)$, $z \in Z_n$ and $y \in Y$, it is easy - at least in principle - to compute the number

$$M_n = \max U_n(f)(Y).$$

Also

$$0 \le U_{n+1}(f)(y_{n+1}) - U_{n+1}(f)(y_n) \le qd(y_{n+1}, y_{n)}$$

i.e.

$$0 \le M_{n+1} - f\,(y_n) \le qd(y_{n+1,}y_n)\,,$$

and because $\overline{M}_{n+1} \ge f(y_n)$ it follows that

$$0 \le M_{n+1} - \overline{M}_{n+1} \le M_{n+1} - f(y_n) \le qd(y_{n+1,}y_n).$$

This means that

$$M_{n+1} - \overline{M}_{n+1} = O(d(y_{n+1,}y_n))$$

and, consequently,

$$M_f - M_n = O(d(y_n, y_{n-1})).$$

Remark 3.3. A function f belongs to d-$SLip_0Y$ if and only if $-f$ belongs to $\overline{d}$-$SLip_0Y$ and for every $f \in d$-$SLip_0Y$, $\|f|_d = \|-f|_{\overline{d}}$ ([22], Corollary 1, page 59).

It follows that $-f$ is upper semicontinuous on (Y,d) and attains its maximum on Y, if Y is d-sequentially compact (see Theorem 2.6).

By Theorem 2.1 and Remark 2.2 it follows that the maximal extension of $-f$ in $\overline{d}$-$SLip_0Y$ is

$$F_{\overline{d}}(-f)(x) = \inf\{(-f)(y) + \|f|_d\,\overline{d}(x,y)\},\ x \in X, \tag{3.2}$$

i.e.

$$(-f)|_Y = F_{\overline{d}}\,(-f)|_Y \ \text{ and }\ \|f|_d = \|-f|_{\overline{d}} = \|F_{\overline{d}}(-f)|_{\overline{d}}\,.$$

The algorithm described above may be applied for searching the global maximum of $-f$, i.e. the global minimum of f, if the set Y is $(d,\overline{d})$-sequentially compact, and X is T_1-separated.

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