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Results in Mathematics

On the Extensions Preserving the Shape of a Semi-Hölder Function

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Abstract. We present some results concerning the extension of a semi-Hölder real-valued function defined on a subset of a quasi-metric space, preserving some shape properties: the smallest semi-Hölder constant, the radiantness and the global minimum (maximum) of the extended function.

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1. Introduction

Let X be a nonvoid set. A mapping $d : X \times X \to [0, \infty)$ satisfying the following conditions:

(QM_1) $d(x, y) = d(y, x) = 0$ iff $x = y$,
(QM_2) $d(x, y) \leq d(x, z) + d(z, y)$,

for all $x, y, z \in X$ is called a quasi-metric (asymmetric metric) on X, and the pair (X, d) is called a quasi-metric space [17,18].

Because, in general, $d(x, y) \neq d(y, x), x, y \in X$, one defines the conjugate $\bar{d}$ of quasi-metric d as the quasi-metric $\bar{d}(x, y) = d(y, x), x, y \in X$.

For example, an asymmetric norm $\| \, |$ on a linear space X (see [6], Ch. IX, § 5) or [2], where a functional analysis in asymmetric normed space is presented) defines a quasi-metric $d_{\||}$ through the formula:

$$d_{\| \, |}(x, y) = \|y - x| \, , \; x, y \in X.$$

Let (X, d) be a quasi-metric space. A sequence $(x_k)_{k\geq 1}$ is d-convergent to $x_0 \in X$ (or forward convergent to $x_0 \in X$) if

$$\lim_{k\to\infty} d(x_0, x_k) = 0,$$

and $\bar{d}$-convergent to $x_0 \in X$ (or backward convergent to $x_0 \in X$) if

$$\lim_{k\to\infty} d(x_k, x_0) = \lim_{k\to\infty} \bar{d}(x_0, x_k) = 0.$$

We say that the set $Y \subset X$ is d-closed ($\bar{d}$-closed) if every d-convergent ($\bar{d}$-convergent) sequence $(y_n)_{n\geq 1} \subset Y$ has limit in Y.

We say that a set $Y \subset X$ is d-sequentially compact (forward sequentially compact) if every sequence in Y has a d-convergent (forward convergent) subsequence with limit in Y (Definition 4.1 in [3]). Finally, the set Y in (X, d) is called $(d, \bar{d})$-sequentially compact if every sequence $(y_n)_{n\geq 1}$ in Y has a subsequence $(y_{n_k})_{k\geq 1}$ d-convergent to $u \in Y$ and $\bar{d}$-convergent to $v \in Y$. For other properties and results in asymmetric metric spaces, see also [3,5,7,11–18].

2. Extension of Semi-Hölder Functions

Let (X, d) be a quasi-metric space, $Y \subset X$ be a nonvoid subset of X and $\alpha \in (0, 1]$ a given number.

Definition 1. *A function $f : Y \to \mathbb{R}$ is called d -semi-Hölder (of exponent α) if there exists a constant $K_Y(f) \geq 0$ such that*

$$f(x) - f(y) \leq K_Y(f) d^\alpha(x, y), \tag{1}$$

for all $x, y \in Y$.

For f a function d-semi-Hölder on Y denote by

$$\|f|^\alpha_{Y,d} := \sup\Big\{\frac{(f(x) - f(y)) \vee 0}{d^\alpha(x, y)} : d(x, y) > 0,\ x, y \in Y\Big\}, \tag{2}$$

the smallest constant $K_Y(f)$, satisfying the inequality (1).

A function $f : Y \to \mathbb{R}$ is called $\leq_d$-increasing if $f(x) \leq f(y)$ whenever $d(x, y) = 0, x, y \in Y$.

The set $\mathbb{R}^Y_{\leq_d}$ of all $\leq_d$-increasing functions on Y is a cone in the linear space $\mathbb{R}^Y$ of all real-valued functions defined on Y.

One denotes by

$$\Lambda_\alpha(Y, d) := \{f \in \mathbb{R}^Y_{\leq_d} \,|f \text{ is } d\text{-semi-Hölder}\}, \tag{3}$$

the set of all d-semi-Hölder functions on Y. This set is a subcone of the cone $\mathbb{R}^Y_{\leq_d}$.

For $y_0 \in Y$ fixed, let

$$\Lambda_{\alpha,0}(Y, d) := \{f \in \Lambda_\alpha(Y, d), f(y_0) = 0\}. \tag{4}$$

The functional $\| \, |_{Y,d}^{\alpha} : \Lambda_{\alpha,0}(Y,d) \to [0,\infty)$ defined by (2) satisfies the axioms of an asymmetric norm, and $(\Lambda_{\alpha,0}(Y,d), \| \, |_{Y,d}^{\alpha})$ is an asymmetric normed cone (compare with [18]).

Observe that $f \in \Lambda_{\alpha}(Y,d)$ if and only if $-f \in \Lambda_{\alpha}(Y,\bar{d})$; moreover $\|f|_{Y,d}^{\alpha} = \|-f|_{Y,\bar{d}}^{\alpha}$.

Example 1. Let Y be a set in a quasi-metric space (X,d) and let $y_0 \in Y$ be fixed. For a number $\alpha \in (0,1]$ one considers the function $f : Y \to \mathbb{R}, f(y) = d^{\alpha}(y,y_0)$. Then $f \in \Lambda_{\alpha,0}(Y,d)$. Indeed, for all $y_1, y_2 \in Y$,

$$f(y_1) - f(y_2) = d^{\alpha}(y_1,y_0) - d^{\alpha}(y_2,y_0) \le d^{\alpha}(y_1,y_2).$$

The last inequality follows by the following simple lemma:

Lemma 1. *Let a, b, c be real nonnegative numbers such that $a \le b + c$. Then for $\alpha \in (0,1]$ it follows $a^{\alpha} \le b^{\alpha} + c^{\alpha}$.*

Since $d(y_1,y_0) \le d(y_1,y_2) + d(y_2,y_0)$ and $\alpha \in (0,1]$, Lemma 1 yields $d^{\alpha}(y_1,y_0) \le d^{\alpha}(y_1,y_2) + d^{\alpha}(y_2,y_0)$, i.e., $d^{\alpha}(y_1,y_0) - d^{\alpha}(y_{2,}y_0) \le d^{\alpha}(y_1,y_2)$.

Example 2. Let $(X, \| \, |)$ be an asymmetric normed space. For a fixed $y_0 \in X$ and $\alpha \in (0,1]$ the function $h(x) = \|x - y_0|^{\alpha}$ is $d_{\| \, |}$-semi-Hölder, where $d_{\| \, |}(y_0,x) = \|x - y_0|$, $x \in X$. Using Lemma 1 it follows $h(x_1) - h(x_2) \le \|x_1 - x_2|^{\alpha}$, $x_1, x_2 \in X$, and $h(y_0) = 0$. This means that $h \in \Lambda_{\alpha,0}(X, d_{\| \, |})$.

Remark 1. By Lemma 1 it follows that if d is a quasi-metric on X, then $d^{\alpha}(\alpha \in (0,1])$ is also a quasi-metric on X. In fact a d-semi-Hölder function f (of exponent $\alpha \in (0,1]$) on Y is a d^{α}-semi-Lipschitz function on (Y,d^{α}) (see [17], for the definition of semi-Lipschitz functions).

The following theorem holds.

Theorem 1. *Let (X,d) be a quasi-metric space, let Y be a nonvoid subset of X, let $\alpha \in (0,1]$, and let $f \in \Lambda_{\alpha}(Y,d)$. Further, let $\mathcal{E}_d(f)$ be defined by*

$$\mathcal{E}_d(f) := \{H \in \Lambda_{\alpha}(X,d) : H|_Y = f, \ \|H|_{X,d}^{\alpha} = \|f|_{Y,d}^{\alpha}\}. \tag{5}$$

Then the following statements hold:

1^0 *The function $F_d(f) : X \to \mathbb{R}$, defined by*

$$F_d(f)(x) := \inf_{y \in Y} \left\{ f(y) + \|f|_{Y,d}^{\alpha} \, d^{\alpha}(x,y) \right\}, \tag{6}$$

belongs to $\mathcal{E}_d(f)$.

2^0 *The function $G_d(f) : X \to \mathbb{R}$, defined by*

$$G_d(f)(x) := \sup_{y \in Y} \left\{ f(y) - \|f|_{Y,d}^{\alpha} \cdot d^{\alpha}(y,x) \right\}, \tag{7}$$

belongs to $\mathcal{E}_d(f)$.

3⁰ Each $H \in \mathcal{E}_d(f)$ satisfies

$$G_d(f)(x) \leq H(x) \leq F_d(f)(x), \tag{8}$$

whenever $x \in X$.

Proof. If $f \in \Lambda_\alpha(Y, d)$ then f is d^α-semi-Lipschitz on (Y, d^α). By ([12], Theorem 2) it follows that the functions $F_d(f)$ defined by (6), and $G_d(f)$ defined by (7) satisfy

$$F_d(f)|_Y = G_d(f)|_Y = f, \quad \|F_d(f)\|_{X,d}^\alpha = \|G_d(f)\|_{X,d}^\alpha = \|f|_{Y,d}^\alpha. \tag{9}$$

Consequently, the statements 1^0 and 2^0 are proved.

The inequalities (8) are proved in [12] for semi-Lipschitz functions (see also ([15], Remark 3), so that the statement 3^0 holds too. □

The set $\mathcal{E}_d(f)$ defined in (5) is called the set of extensions of $f \in \Lambda_\alpha(Y, d)$ (preserving the smallest constant $\|f|_{Y,d}^\alpha$). The functions $F_d(f)$, respectively $G_d(f)$ are called the maximal extension, respectively the minimal extension of f [see (8)].

Remark 2. In [15] one gives a direct proof of Theorem 1, by considering the function $G_d(f)$ defined by (7) and proving that $G_d(f)$ is well defined, $G_d(f)|_Y = f$ and $\|G_d(f)|_{X,d}^\alpha = \|f|_{Y,d}^\alpha$ (see also [10,12]).

A natural problem is the following: If $f \in \Lambda_\alpha(Y, d)$ has some supplementary properties, does there exist $H \in \mathcal{E}_d(f)$ preserving these properties? Such a problem is considered in [9] for Lipschitz functions.

We shall consider two problems of such kind.

For the first one, in the sequel (X, d) is a quasi-metric linear space and $Y \subset X$ is a subset of X.

The set Y is said to be radiant if it has the following properties:

(i) Y is nonvoid;

(ii) $\lambda y \in Y$ for all $y \in Y$ and all $\lambda \in [0, 1]$.

Let Y be a radiant set in X, and let $f : Y \to \mathbb{R}$, and let $\alpha \in (0, 1]$. The function f is said to be α-radiant if

$$f(\lambda y) \leq \lambda^\alpha f(y), \tag{10}$$

for all $y \in Y$ and all $\lambda \in (0, 1]$.

The 1-radiant functions are called, simply, radiant.

Observe that all radiant sets in a linear space X contain the null element θ of X, and every α-radiant function satisfies $f(\theta) \leq 0$. We consider only functions satisfying $f(\theta) = 0$.

The function $f : Y \to \mathbb{R}$ is said to be α-co-radiant ($\alpha \in (0, 1]$) if

$$f(\lambda y) \geq \lambda^\alpha f(y), \tag{11}$$

for all $y \in Y$ and $\lambda \in [0, 1]$. The 1-co-radiant functions are called co-radiant [4,8].

The function $f : Y \to \mathbb{R}$ is called α-inverse co-radiant ($\alpha \in (0,1]$ is fixed) if

$$f(\lambda y) \leq \frac{1}{\lambda^\alpha} f(y), \tag{12}$$

for all $y \in Y$ and $\lambda \in (0,1]$. The 1-inverse co-radiant functions are called inverse co-radiant.

Obviously, every nonnegative α-co-radiant function is co-radiant, and every inverse co-radiant function is α-inverse co-radiant.

If Y is a convex set in X, a function $f : Y \to \mathbb{R}$ is called α-convex ($\alpha \in [0,1]$) if

$$f(\lambda x + (1-\lambda)y) \leq \lambda^\alpha f(x) + (1-\lambda)^\alpha f(y), \tag{13}$$

for all $x, y \in Y$ and $\lambda \in [0,1]$ (see [1]).

If $\theta \in Y$ and Y is convex, then every convex function ($\alpha = 1$) on Y with $f(\theta) = 0$ is radiant, and every α-convex function on Y such that $f(\theta) = 0$ is α-radiant, because

$$f(\lambda x) = f(\lambda x + (1-\lambda)\theta) \leq \lambda^\alpha f(x) + (1-\lambda)^\alpha f(\theta) = \lambda^\alpha f(x),$$

for all $x \in Y$ and $\lambda \in [0,1]$.

A quasi-metric d on a quasi-metric linear space X is called positively homogeneous if

$$d(\lambda x, \lambda y) = \lambda d(x,y), \tag{14}$$

for all $x, y \in X$ and $\lambda \geq 0$. Such a quasi-metric is for example $d_{\|\ |}$, generated by an asymmetric norm $\|\ |$.

The following result holds.

Theorem 2. *Let (X,d) be a quasi-metric linear space with d positively homogeneous, let Y be a radiant subset of X, let $\alpha \in (0,1]$ and let $f \in \Lambda_\alpha(Y,d)$. Then the following statements hold:*

1^0 If f is α-radiant, then $F_d(f)$ is α-radiant.
2^0 If f is α-co-radiant, then $G_d(f)$ is α-co-radiant.
3^0 If f is inverse co-radiant, then $F_d(f)$ is inverse co-radiant.

Proof. Let $f : Y \to \mathbb{R}$ be radiant, and $f \in \Lambda_\alpha(Y,d)$. Let us consider the maximal extension $F_d(f)$. Then for all $\lambda \in [0,1]$ and $y \in Y$,

$$\begin{aligned} F_d(f)(\lambda x) &\leq f(\lambda y) + \|f|^\alpha_{Y,d}\, d^\alpha(\lambda x, \lambda y) \\ &\leq \lambda^\alpha f(y) + \lambda^\alpha \|f|^\alpha_{Y,d}\, d^\alpha(x,y) \\ &= \lambda^\alpha [f(y) + \|f|^\alpha_{Y,d}\, d^\alpha(x,y)]. \end{aligned}$$

Taking the infimum with respect to $y \in Y$ one gets

$$F_d(f)(\lambda x) \leq \lambda^\alpha F_d(f)(x),$$

for every $x \in X$, showing that $F_d(f)$ is α-radiant.

Now, let $x \in X, \lambda \in [0,1]$, and $f \in \Lambda_\alpha(Y,d)$ be α-co-radiant. Then, for every $y \in Y$, by considering the minimal extension $G_d(f) \in \mathcal{E}_d(f)$ one gets:

$$\begin{aligned} G_d(f)(\lambda x) &\geq f(\lambda y) - \|f|_{Y,d}^\alpha \, d^\alpha(\lambda y, \lambda x) \\ &\geq \lambda^\alpha f(y) - \lambda^\alpha \, \|f|_{Y,d}^\alpha \, d^\alpha(y,x) \\ &= \lambda^\alpha [f(y) - \|f|_{Y,d}^\alpha \, d^\alpha(y,x)]. \end{aligned}$$

Taking the supremum with respect to $y \in Y$, one obtains

$$G_d(f)(\lambda x) \geq \lambda^\alpha G_d(f)(x), \;\; x \in X,$$

and the statement 2^0 is proved.

Finally, if $f \in \Lambda_\alpha(Y,d)$ is inverse co-radiant on the radiant set Y, then for every $y \in Y$ and $\lambda \in (0,1]$, and for the maximal extension $F_d(f)$ one obtains:

$$\begin{aligned} F_d(f)(x) &\leq f(\lambda y) + \|f|_{Y,d}^\alpha \, d^\alpha(\lambda x, \lambda y) \\ &\leq \frac{1}{\lambda} f(y) + \lambda^\alpha \, \|f|_{Y,d}^\alpha \cdot d^\alpha(x,y) \\ &= \frac{1}{\lambda}[f(y) + \lambda^{\alpha+1} \, \|f|_{Y,d}^\alpha \, d^\alpha(x,y)] \\ &\leq \frac{1}{\lambda}[f(y) + \|f|_{Y,d}^\alpha \, d^\alpha(x,y)]. \end{aligned}$$

Taking the infimum with respect to $y \in Y$ it follows

$$F_d(f)(\lambda x) \leq \frac{1}{\lambda} F_d(f)(x), \;\; x \in X,$$

and the statement 3^0 holds. □

Another property preserved by extensions is the global minimum (maximum) of a function $f \in \Lambda_\alpha(Y,d)$.

Let (X,d) be a quasi-metric space, and let $Y \subset X$ be a nonempty subset of X. An element $y_0 \in Y$ is called a global minimum (maximum) point of $f \in \Lambda_\alpha(Y,d)$ if

$$f(y_0) \leq f(y) \;\; (f(y_0) \geq f(y)),$$

for all $y \in Y$.

Theorem 3. *Let (X,d) be a quasi-metric space, let Y be a nonvoid subset of X, let $y_0 \in Y$, let $\alpha \in (0,1]$, and let $f \in \Lambda_\alpha(Y,d)$. Then the following statements hold:*

1^0 If Y is d-closed, then $y_0 \in Y$ is a global minimum point for f in Y if and only if y_0 is a global minimum point of $F_d(f)$ in X.

2^0 If Y is $\bar{d}$-closed, then $y_0 \in Y$ is a global maximum point for f in Y if and only if y_0 is a global maximum point of $G_d(f)$ in X.

Proof. 1^0 Let $y_0 \in Y$ be a global minimum point of $f \in \Lambda_\alpha(Y,d)$. For every $y \in Y$ we have

$$F_d(f)(y) = f(y) \geq f(y_0) = F_d(f)(y_0).$$

If $x \notin Y, Y$ being d-closed, there exists $\delta > 0$ such that $d(x, y) > \delta$ for all $y \in Y$. Consequently,

$$\begin{aligned} F_d(f)(x) &= \inf_{y\in Y}\{f(y) + \|f|^{\alpha}_{Y,d}\, d^{\alpha}(x,y)\} \\ &\geq \inf_{y\in Y}\{f(y) + \|f|^{\alpha}_{Y,d}\, \delta^{\alpha}\} \\ &= f(y_0) + \|f|^{\alpha}_{Y,d}\, \delta^{\alpha} \geq f(y_0), \end{aligned}$$

so that for every $x \in X, F_d(f)(x) \geq f(y_0) = F_d(f)(y_0)$.

Conversely, suppose that $y_0 \in X$ is a global minimum point for $F_d(f)$ in X. If we would show that $y_0 \in Y$, then, as $F_d(f)|_Y = f$, it would follow that y_0 is a global minimum point for f in Y.

Case I: $\|f|^{\alpha}_{Y,d} = 0$.

In this case there exists $c \in \mathbb{R}$ such that $f(y) = c$ for all $y \in Y$. It follows $\|F_d(f)|^{\alpha}_{X,d} = 0$, so that $F_d(f) = const$ on X. Since $F|_Y = f$ we must have $F_d(f)(x) = const$, for all $x \in X$.

Case II: $\|f|^{\alpha}_{Y,d} > 0$.

Let $y_0 \in X$ such that $F_d(f)(y_0) \leq F_d(f)(x)$, for all $x \in X$. Since $F_d(f)(y_0) = \inf[f(y) + \|f|^{\alpha}_{Y,d}\, d^{\alpha}(y_0, y)]$, for every $n \in \mathbb{N}$ there exist $y_n \in Y$ such that

$$f(y_n) + d^{\alpha}(y_0, y_n)\, \|f|^{\alpha}_{Y,d} < F_d(f)(y_0) + \frac{\|f|^{\alpha}_{Y,d}}{n^{\alpha}}.$$

The inequalities $F_d(f)\,(y_0) \leq F_d(f)(y_n) = f(y_n)$, imply $f(y_n) + \|f|^{\alpha}_{Y,d}\, d^{\alpha}(y_0, y_n) < f(y_n) + \|f|^{\alpha}_{Y,d}\, n^{-\alpha}$, so that $d(y_0, y_n) < \frac{1}{n}, n \in \mathbb{N}$, i.e., $d(y_0, y_n) \to 0$. The sequence $(y_n)_{n\geq 1}$ in Y is d-convergent to y_0, and since Y is d-closed, $y_0 \in Y$.

2^0 Let $y_0 \in Y$ be a global maximum point of $f \in \Lambda_{\alpha}(Y)$.

Then for every $y \in Y$,

$$G_d(f)(y) = f(y) \leq f(y_0) = G_d(f)(y_0).$$

If $x \notin Y, Y$ being $\bar{d}$-closed, there exists $\eta > 0$ such that $\bar{d}(x, y) > \eta$ (i.e. $d(y, x) > \eta$) for all $y \in Y$. Therefore

$$\begin{aligned} G_d(f)(x) &= \sup_{y\in Y}\{f(y) - \|f|^{\alpha}_{Y,d}\, \bar{d}^{\alpha}(x,y)\} \\ &= \sup_{y\in Y}\left\{f(y) - \|f|^{\alpha}_{Y,d}\, d^{\alpha}(y,x)\right\} \\ &\leq \sup_{y\in Y}\left\{f(y) - \|f|^{\alpha}_{Y,d}\, \eta^{\alpha}\right\} \\ &\leq f(y_0) - \|f|^{\alpha}_{Y,d}\, \eta^{\alpha} \leq f(y_0). \end{aligned}$$

It follows that $G_d(f)(x) \leq f(y_0)$, for all $x \in X$.

Conversely, suppose that $y_0 \in X$ is a global maximum point for $G_d(f)$ in X.

Case I $\|f|_{Y,d}^{\alpha} = 0$.

In this case, because $f = G_d(f)|_Y$ and $\|G_d(f)|_{X,d}^{\alpha} = \|f|_{Y,d}^{\alpha} = 0$ it follows that f and $G_d(f)$ are equal with the same constant.

Case II $\|f|_{Y,d}^{\alpha} > 0$.

Let $y_0 \in X$ such that $G_d(f)(y_0) > G_d(f)(x), x \in X$.
Since

$$G_d(f)(y_0) = \sup_{y\in Y}[f(y) - \|f|_{Y,d}^{\alpha}\, d^{\alpha}(y, y_0)],$$

for every $n \in \mathbb{N}$, there exists $y_n \in Y$ such that

$$f(y_n) - \|f|_{Y,d}^{\alpha}\, d^{\alpha}(y_n, y_0) > G_d(f)(y_0) - \frac{\|f|_{Y,d}^{\alpha}}{n^{\alpha}}.$$

The inequalities $G_d(f)(y_0) \geq G_d(f)(y_n) = f(y_n)$, imply $f(y_n) - \|f|_{Y,d}^{\alpha}\, d^{\alpha}(y_n, y_0) > f(y_n) - \frac{1}{n^{\alpha}}\|f|_{Y,d}^{\alpha}$, so that $d(y_n, y_0) < \frac{1}{n}, n \in \mathbb{N}$, i.e., $d(y_n, y_0) \to 0$. This means that the sequence $(y_n)_{n\geq 1}$ is $\bar{d}$-convergent to y_0. Since Y is $\bar{d}$-closed, $y_0 \in Y$. □

Remark 3. By Theorem 2.6 in [14], it follows that every $f \in \Lambda_{\alpha,0}(Y, d)$ is lower semicontinuous on (Y, d^{α}) and attains its minimum on Y, provided that Y is d^{α}-sequentially compact. Also, every $f \in \Lambda_{\alpha,0}(Y, d)$ is upper semicontinuous on $(Y, \bar{d}^{\alpha})$ and attains its maximum value on Y whenever Y is $\bar{d}^{\alpha}$-sequentially compact.

If Y is $(d^{\alpha}, \bar{d}^{\alpha})$-sequentially compact and (X, d^{α}) is a T_1-topological space, every $f \in \Lambda_{\alpha,0}(Y, d)$ attains both the global minimum and the global maximum on Y. Moreover, the sequential method for the calculation of the global extremum (maximum and/or minimum) of f, ([14], Th. 3.1) is applicable.

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