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C O N T E N T S

One can show that in this case the sequence $(x_n)_{n\geq 0}$ converges to the maximum point $x^* = 0.12944$ (the solution of the equation $tg(1/x) = 1/x$, $x \in (1/3\pi , 1/2\pi)$).

The maximum of f is $M_f = f(x^*) = 0.128374$. After 300 iteration we have obtained the following results :

$$x_{300} = 0.129982$$

$$f(x_{300}) = 0.128309$$

$$M_{299} = F^{299}(x_{300}) = 0.144181$$

The errors are in this case

$$M_f - f(x_{300}) = 0.000065$$

$$x_{300} - x^* = 0.000537$$

R E F E R E N C E S

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NORMAL CONE VALUED METRICS AND NONCONVEX VECTOR MINIMIZATION PRINCIPLE

by A.B. Németh

0. Introduction. The extension in (N1-2) of the nonconvex minimization principle in (E) does not be formally a generalisation. In order to formulate a theorem which contains as particular cases both the classical formulation due to EKELAND (E) as well as its extension for ordered spaces in (N2), we have to introduce cone valued metrics. It turns out that these metrics offer useful technical facilities in various problems of the nonlinear functional analysis. The generalisation of the classical metric spaces in this direction was initiated by KUREPA (KU) and then used by KANTOROVIC, VULICH and PINSKER (KVP), SCHRÖDER (SchR), COLLATZ (CO), ANTONOVSKII, BOLTJANSKII and SARYMSAKOV (ABS) etc. Are noteworthy the applications in fixed point theory. We have got an excellent guide for surprising this orientation in the monography (R) of RUS and the literature cited there. The first relevant results in this direction seem to be these due to PEROV (PE) and PEROV and KIBENKO (PK) (see also (R)). The regular cone valued metrics present for us a special interest. Their importance for the fixed point theory was discovered by EISENFELD and LAKSHMIKANTHAM who have developed in (EL1-3) various aspects of fixed point theory for spaces with metric having values in a regular cone of a Banach space.

In our next considerations occur naturally some theoretical problems. First of all we have to verify if our principal result is or not a generalization of that in (N2). Further, we have to characterize the topological spaces which are metrizable in this generalised way. Concerning this and some related topics a deep investigation is that due to ANTONOVSKII, BOLTJANSKII and SARYMSAKOV (ABS), who develop the theory of topological semifields. This theory can be used to give answers to the above two questions. However, we must consider ordered vector spaces which does not be topological semifields. Hence the necessity of developing of an independent theory although it runs in some places parallel with the considerations in the above cited monography.

The organization of the material in which follows answers to the mentioned requirements. It begins with some investigations about normal cone valued metrics showing that every normal cone valued metric induces in a natural way a uniformizable topology. The results in (ABS) are then outlined to show that every completely regular Hausdorff topological space is metrizable by a metric with values in the positive cone of a topological semifield. Hence the topological spaces which are metrizable in this sense are quite those which are uniformizable (see e.g. (IS) I.15). It follows in particular that every locally convex Hausdorff vector topology can be induced by such a metric.

It is important that the positive cone of a topological semifield is regular. Hence all the metrizability results can be formulated for regular cone valued metrics. In particular the sublinear operator which occurs in Proposition 23 of (N2) can be interpreted as to be a regular cone valued quasi metric and hence this proposition follows from the principal result which will be proved in 12.

The proof of the principal result uses the method of BISHOP and PHELPS (BP), wich is also the origin of the techniques used

by EKELAND (E), BRÉZIS and BROWDER (BB), ISAC (I), II, §3, (N1-2) an also by EISENFELD and LAKSHMIKANTHAM (EL3). Our proof uses HAUSDORFF's theorem (an equivalent of the axiom of choice, (DSch) I, 2.6). It will be showed elsewhere that this can be avoid when the ordered vector space considered is metrizable. Of a major techniqual importance in the proof is also the use of summation criteria for regular cones even in the case when the space does not be metrizable (N2).

1. *Normal cones*. All the vector spaces considered below will be over the reals. The subset K of the vector space E is said to be a *cone* if $K + K \subset K$; $tK \subset K$ for each non-negative real number t, and if $K \cap (-K) = \{0\}$, where 0 stands for the zero element of E. The cone K introduces a reflexive, transitive and antisymmetric order relation $\leqslant$ in E by putting $x \leqslant y$ if $y-x$ is in K. This relation relates to the linear structure of E by the properties: $x \leqslant y$ implies $x+z \leqslant y+z$ for every z, and $x \leqslant y$ implies $tx \leqslant ty$ for every non-negative t. The vector space E endowed with an order relation of this kind will be called *ordered vector space*. The cone K inducing the order relation in E will be called the *positive cone of* E. One will be supposed throughout that $K \neq \{0\}$. We shall use the terms "K order bounded", "K increasing" etc. for order bounded, increasing etc. with respect to the order relation induced by K. We shall use also simply the terms order bounded, increasing etc. when no ambiguity can occur.

The set A in the ordered vector space E with the positive cone K will be called *full* if $A = (A+K) \cap (A-K)$. A full set A can be characterised also with the property that for every a and b in

A the set $\{z : a \leqslant z \leqslant b\}$ is contained in A.

If the ordered vector space E with the positive cone K is endowed with a vector space topology, then K will be said to be a <u>normal cone</u> if the zero element of E has a neighbourhood basis $B(0)$ consisting of full sets. If E is an ordered vector space which is locally convex and Hausdorff then it has a neighbourhood basis $B(0)$ consisting of full sets if and only if it has a generating family $\{p_i\}$ of seminorms which are monotone in the sense that from $0 \leqslant x \leqslant y$ it follows $p_i(x) \leqslant p_i(y)$ for every i (see (P), II, Proposition 1.5).

2. <u>Normal cone valued metrics.</u> Let E be an ordered vector space with the positive cone K.

A K <u>quasi semimetric</u> on a set V is a mapping r of $V \times V$ in K satisfying the following conditions for all u, v, z in V :

(m1) $\quad r(u,u) = 0$;

(m2) $\quad r(u,z) \leqslant r(u,v) + r(v,z)$,

where $\leqslant$ stands for the order relation induced by K.

The K quasi semimetric r is said a K <u>semimetric</u> if it satisfies the condition

(m3) $\quad r(u,v) = r(v,u)$

for all u and v.

The K semimetric r on V is called K <u>metric</u> if it satisfies the condition

(m4) $\quad r(u,v) = 0$ implies $u = v$.

The K quasi semimetric r is called K <u>quasi metric</u> if it satisfies condition (m4).

If E is endowed with a topology like structure (a topology, a convergence structure, etc.) then r will induce in a natural way something similar in V. We shall consider here only the simplest case : when E is a topological vector space and K is a normal cone (K metric spaces for a such K were considered in (AB6), (H1-3) etc.).

Let us consider first the axioms for a neighbourhood basis in order to it define a topology (see (B), I, 1, 2) :

Let X be a set. For every x in X we consider a family $B(x)$ of subsets in X, which satisfies the axioms :

(t1) If $M \in B(x)$ and $M \subset N$, then $N \in B(x)$;

(t2) If M, $N \in B(x)$, then $M \cap N \in B(x)$;

(t3) If $M \in B(x)$, then $x \in M$;

(t4) For every $M \in B(x)$ there exists $N \in B(x)$ with $N \subset M$, such that for every y in N there exists an N' in $B(y)$ such that $N' \subset M$.

The family $B(x)$ will be called the <u>neighbourhood filter</u> of x. If we define the open sets in X as to be the sets $G \subset X$ such that for every x in G there exists M in $B(x)$ such that $M \subset G$, then we have defined a topology on X (see (B) I, 1, 2, Proposition 2.).

It is easy to show that if we consider the families $B(x)$ satisfying only

(t'2) If M, N are in $B(x)$ then there exists R in $B(x)$

(2.1) $\quad$ with the property that $R \subset M \cap N$,

and the conditions (t3) and (t4),

then if we complete the families $B(x)$ with the sets S having the property that there exists M in $B(x)$ such that $M \subset S$, the obtained new families will satisfy the axioms (t1) - (t4). The family $B(x)$ satisfying (2.1) will be called a *neighbourhood basis* at x. Accordingly, a neibhouhood basis satisfying (2.1) induces a topology.

Let E be a topological vector space and let K be a normal cone in E. Denote by $B(0)$ a neighbourhood basis of 0 in E consisting of full sets. Then we have the assertion

2.1. *If r is a K quasi semimetric on V then the sets*

$$(2.2) \qquad V(U,a) = \{u \in V : r(a,u) \in U\}, \quad U \in B(0)$$

for a in V, form a neighbourhood basis which induces a topology on the set V.

We have to verify (t'2), (t3) and (t4) for the family (2.2). Our family inherits obviously (t'2) and (t3) from $B(0)$.

To prove (t4) we have to check that for every $V(U,a)$ we can get a set $V(U',a)$ such that for every b in $V(U',a)$ there exists $V(U'',b)$ such that $V(U'',b) \subset V(U,a)$ (where U, U' and U'' are in $B(0)$).

Let U be given in $B(0)$ and consider U_0 in $B(0)$ with the property that $U_0 + U_0 \subset U$. For b an arbitrary element in $V(U_0,a)$ and for c in $V(U_0,b)$ we have by the axiom (m2) :

$$(2.3) \qquad r(a,c) \leq r(a,b) + r(b,c).$$

Since $r(a,b)$ and $r(b,c)$ are in U_0, it follows that $r(a,b) + r(b,c) \in U_0 + U_0 \subset U$. The neighbourhood U is full and hence this last relation and (2.3) imply that $r(a,c)$ is in U, that is, c is in $V(U,a)$. Accordingly, by putting $U' = U'' = U_0$ we get the relation which we need.

Q.E.D.

We shall refer to the topology defined by the neighbourhood bases (2.3) as to the *topology induced by the K quasi semimetric* r. The following assertion is immediate:

2.2. *The topology induced by a K quasi metric or by a K metric is Hausdorff.*

3. *Every normal cone valued semimetric is uniformizable.* A *quasiuniformity* on a set V is a collection $\mathcal{U}$ of subsets of $V \times V$ which satisfies the following axioms :

(u1) For all M in $\mathcal{U}$, $\Delta \subset M$, where $\Delta = \{(v,v) : v \in V\}$;

(u2) For M in $\mathcal{U}$ and N in $\mathcal{U}$, $M \cap N$ is in $\mathcal{U}$;

(u3) If M is in $\mathcal{U}$ and $M \subset N \subset V \times V$, then N is in $\mathcal{U}$;

(u4) For all M in $\mathcal{U}$ there is N in $\mathcal{U}$ such that $N \circ N$ is in $\mathcal{U}$, where $\circ$ is defined by $S \circ T = \{(u,s) :$ there is a v in V such that (u,v) is in T and (v,s) is in $S\}$.

(see (IS)). The quasiuniformity is a *uniformity* if the following additional condition is satisfied

(u5) If M is in $\mathcal{U}$, then M^{-1} is in $\mathcal{U}$, where $M^{-1} = \{(v,u) : (u,v) \in M\}$.

A *basis* of a quasiuniformity (uniformity) is a family $\mathcal{U}$ of subsets of $V \times V$ which satisfy (u1), (u4) and

(u'2) For M in $\mathcal{U}$ and N in $\mathcal{U}$ there exists R in $\mathcal{U}$ such that $R \subset M \cap N$.

(and respectively (u1), (u4), (u5) and (u'2) for the uniformity). If we complete a basis of a quasi uniformity (uniformity) $\mathcal{U}$ with the sets N in $V \times V$ having the property that there is M in $\mathcal{U}$ such that $N \subset M$, then the completed family satisfies (u1)-(u4) (or respectively (u1)-(u5)).

Every quasiuniformity basis $\mathcal{U}$ on a set V generates a basis of a topology on V by taking as a neighbourhood basis for V the sets $M(u)$, where $u \in V$ and $M(u) = \{v : (u,v) \in M\}$. The notion of quasiuniformity generalizes the notion of the (real) quasi metric space.

The following assertion relates the K quasi semimetric to quasi uniform structure. Let us suppose that K is a normal cone in the topological vector space E. Then we have :

3.1 <u>The topology induced in the set</u> V <u>by a</u> K <u>quasi semimetric</u> r <u>is quasi uniformizable in the sense that it can be induced also by a quasiuniformity</u>.

Let us define a family of sets in $V \times V$ by putting

$$\mathcal{U} = \{ r^{-1}(U) : U \in B(0) \},$$

where B(0) is the neighbourhood basis consisting of the full sets used in the definition of the sets in (2.2). We shall show that $\mathcal{U}$ satisfies the axioms of a base for a quasiuniformity on V, that is, the axioms (u1), (u4) and (u'2).

The property (u'2) for $\mathcal{U}$ is inherited obviously from a similar property for B(0). Since $0 \in U$ for each U in B(0), we have directly also (u1).

We have to show that $\mathcal{U}$ verifies (u4). For let us consider $r^{-1}(U)$ arbitrarily in $\mathcal{U}$, and suppose $U' \in B(0)$ has the property $U' + U' \subset U$. Denote $M = r^{-1}(U)$ and $N = r^{-1}(U')$. Now, $(u,z) \in N \circ N$ if there is v in V such that (u,v) is in N and (v,z) is in N. That is, for every (u,z) in $N \circ N$ there exists a v in V such that $r(u,v) \in U'$ and $r(v,z) \in U'$. From the axiom (m2) we have

$$0 \leqslant r(u,z) \leqslant r(u,v) + r(v,z) \in U' + U' \subset U.$$

Since U is full, it follows that $r(u,z)$ is in U and hence $(u,z) \in r^{-1}(U)$. That is, it was shown that for arbitrary M in $\mathcal{U}$ there exists N in $\mathcal{U}$ so as to have $N \circ N \subset M$, which completes the proof that $\mathcal{U}$ is a basis for a quasiuniformity on V.

A neighbourhood of a in V defined by the above defined quasiuniformity basis $\mathcal{U}$ is by definition the set

$$\{u \in V : (a,u) \in M\}, \ M \text{ in } \mathcal{U},$$

which is in fact $V(U,a)$ defined by (2.2), if $M = r^{-1}(U)$. That is, $\mathcal{U}$ induces quite the topology induced by r.

Q.E.D.

Using the axiom (m3) and 3.1 one can be verified directly the following assertion :

3.2. <u>The topology induced by a</u> K <u>semimetric is uniformizable</u>.

Similarly, we have using the axiom (m4) the assertions :

3.3. <u>The topology induced by a</u> K <u>quasi metric is a quasiuniformizable Hausdorff topology</u>.

3.4. <u>The topology induced by a</u> K <u>metric is a uniformizable Hausdorff topology</u>.

3.5. <u>Remark</u>. From the above considered constructions one can be expected that the topology induced by a K metric is situated between the (real) metrizable and the uniformizable topologies. It is a result of ANTONOVSKII, BOLTJANSKII and SARYMSAKOV (ABS) that in fact K metrizable for K a given special normal cone, and uniformizable Hausdorff topologies are the same (for an explicite statement see paragraph 5). This together with our above assertion will imply that K metrizable and uniformizable Hausdorff topologies are the same for general normal cones K.

4. Normal cone valued norms. Denote by E_1 a vector space and by E an ordered vector space with the positive cone K. The mapping $\|.\|$ from E_1 to K will be called a K quasi seminorm if the following axioms hold :

(n0) $\|tx\| = t\|x\|$ for every x in E_1 and every $t \geq 0$;

(n1) $\|0\| = 0$;

(n2) $\|x + y\| \leq \|x\| + \|y\|$, for every x and y in E_1.

The K quasi seminorm $\|.\|$ is called a K seminorm if it satisfies the condition :

(n3) $\|x\| = \|-x\|$, for every x in E_1 .

The K seminorm $\|.\|$ is called K norm if it satisfies the condition :

(n4) $\|x\| = 0$ implies $x = 0$.

The K quasi seminorm $\|.\|$ is called K quasi norm if it satisfies the condition (n4).

Putting $r(x,y) = \|x-y\|$ the K quasi seminorm (K seminorm, K norm, K quasi norm) $\|.\|$ defines a K quasi semimetric (K semimetric, K metric, K quasi metric).

The operator P from E_1 to E will be called sublinear if it is positively homogeneous (i.e., if $P(tx) = tP(x)$ for every x in E_1 and every $t \geq 0$) and if it satisfies the condition

$$P(x + y) \leq P(x) + P(y)$$

for every x and y in E_1. If the sublinear operator P has the

property that $P(E_1) \subset K$, then it is a K quasi seminorm. If in plus, $P(x) = P(-x)$ for every x, then P is a K seminorm. If $P(x) = P(-x)$ for every x and $P(x) = 0$ implies $x = 0$, then P is a K norm. If the sublinear operator P satisfies the condition $P(E_1) \subset K$ and the condition: $P(x) = 0$ implies $x = 0$, then P is a quasi norm.

Let P be a sublinear operator from the topological vector space E_1 into the ordered topological vector space E with the positive cone K. Then P will be said to be coercive if there exists a neighbourhood basis B(0) of 0 in E, such that the filter basis $\{P^{-1}(U) : U \in B(0)\}$ refines the filter of neighbourhoods of 0 in E_1, in the sense that for every element U_1 of this last filter there exists a U in B(0) with $P^{-1}(U) \subset U_1$. In fact, if this condition holds for a given filter basis of the neighbourhoods of the origin in E, then it holds also for any other filter basis of neighbourhoods of 0.

4.1. The topology induced in E_1 by the K quasi norm (norm) $\|.\|$ defined by $\|x\| = P(x)$, where P is sublinear operator from E_1 to E with $P(E_1) \subset K$, coincides with the original one if and only if P is continuous and coercive.

Indeed, from the condition that P is continuous at a in E_1 it follows that the filter basis consisting of the sets

(4.1) $$\{y \in E_1 : P(a-y) \in U\},\ U \text{ in } B(0)$$

is coarser that the neighbourhood filter in the original topology of E_1. If P is coercive, then this filter basis is also finer than the neighbourhood filter of the original topology at a. Hence the two topologies are equivalent.

Conversely, in order to P be continuous at a the filter basis

(4.1) must be coarser, in order to P be coercive, the filter basis (4.1) must be finer than the neighbourhood filter at a in the original topology of E_1.

Q.E.D.

Assume that E is a normed space and let K be a normal cone in E. Let k be an element in K different from 0. A simple form of a sublinear operator P which is continuous, coercive and has the property that $P(E) \subset K$ is given by

$$P(x) = \|x\| k.$$

This operator considered as a K norm induces obviously the original topology in E. Other sublinear operators satisfying the conditions in 4.1 are the so called correct operators considered in the paper (N3).

The ordered vector space E with the positive cone K will be called a vector lattice if for every pair of elements x and y in E there exists an infimum $\inf\{x,y\}$, i.e., a greatest lower bound for x and y. In this case it follows also the existence of $\sup\{x,y\}$, a smallest upper bound for x and y. The positive cone K of a vector lattice is called minihedral (see (K) and (KL3)). For x in E let us use the notations $x^+ = \sup\{x,0\}$, and $x^- = \sup\{-x,0\}$. Then $x = x^+ - x^-$ (see e.g. (P), I.), hence K generates E in the sense that $E = K - K$.

Let E be a topological vector space ordered by a normal cone K which is also minihedral. The set M in E is called solid if $x \in M$ and $|y| \leq |x|$, where $|z| = z^+ + z^-$, imply that y is in M. Suppose that the space E has a neighbourhood basis of 0 consisting of solid sets (in this case E is called locally solid). Then

the sublinear operator

(4.2) $$P : x \mapsto |x|$$

is continuous ((Sch) V, 7.1). We shall show that P is also coercive. Indeed, if U is a solid neighbourhood of 0 in E, then x is in U if and only if |x| is in U. Hence

(4.3) $$P^{-1}(U) = \{x \in E : |x| \in U\} = U.$$

We observe that this relation proves both the continuity and the coercivity of P.

From the above considerations we have using 4.1 the assertion:

4.2. Let E be a topological vector space ordered by a normal minihedral cone K. Assume that E is locally solid. Then the K norm $\|\cdot\|$ defined by $\|x\| = x^+ + x^-$ induces the original topology on E.

5. K metrisability and K normability. Let S be a set. Denote by R^S the vector space of all functions defined on S with values in R. Equipped with the direct product topology the space R^S (where R stands for the real line) becomes a locally convex Hausdorff space. Let R^S_+ denote the set of all non-negative functions in R^S. Then R^S_+ is a normal cone since the generating family of seminorms defined on R^S by $p_s(x) = |x(s)|$ is obviously monotone on R^S_+.

We remark that R^S_+ is completely minihedral (see (K) and (KL3)) in the sense that every lower bounded set in the ordered vector space R^S with the positive cone R^S_+ has an infimum, i.e., a greatest lower bound. An ordered vector space having this property is called laticially complete or order complete vector lattice (P).

In the space R^S can be introduced a product: the pointwise multiplication of the functions. Then R^S endowed with the closed positive cone R^S_+ becomes a topological semifield in the sense of ANTONOVSKII, BOLTJANSKII and SARYMSAKOV (ABS) (see 10.1 in the cited monography). This topological semifield has also an important universality property: every topological semifield can be imbedded in it (see 10.1 in (ABS)). Since R^S_+ is normal and the subcone of every normal cone is normal, it follows that the positive cone of every topological semifield is normal.

In 11.4 and 11.5 of (ABS) it was shown that a topological space V is K metrizable, where K is the closed cone of a topological semifield, if and only if it is completely regular. Wherefrom, according (IS), I, 15, it follows that V is K metrizable if and only if it is uniformizable. From this assertion and the results in paragraph 3 it follows :

5.1. The topological space V is K metrizable, where K is a normal cone of a topological vector space, if and only if it is uniformizable.

Let E be a locally convex Hausdorff space and let $\{p_s : s \in S\}$ be a generating family of seminorms on E. Then the sublinear operator P from E to R^S defined by

$$P(x)(s) = p_s(x)$$

is an R^S_+-norm which induces the original topology on E. It holds in fact the following sharper result (see 12.4 of (ABS)) :

5.2. The locally convex Hausdorff space E is R^S_+ normable if and only if there exists a neighbourhood basis of 0 in E obtained from the positive multiples of a family of neighbourhoods whose

cardinal is not exceeding S.

5.3. Remark. We observe that there is a technical problem only to transpose the above results to quasi metrizable and quasi normable cases respectively.

6. Completness of K quasi metric spaces. Let K be a normal cone of the topological vector space E. The absence of the symmetry of a K quasi metric needs special considerations in introducing K Cauchy nets and K completness with respect to these quasi metrics.

Let V be a K quasi metric space with the K quasi metric r. A net $(v_\nu)_{\nu \in I}$ in V indexed by the totally ordered set I will be called a Cauchy net if for every U in B(0) there exists a μ in I such that whenever $\mu \le \nu \le \eta$, it holds $r(v_\nu, v_\eta) \in U$. The K quasi metric space V will be called complete if every Cauchy net in it is convergent.

We observe that if V is a K metric space which is complete in usual sense, then it is obviously complete in the sense considered above. We have also the following assertion:

6.1. If P is a continuous and coercive sublinear operator from the locally convex Hausdorff space E_1 to E, with the property that $P(E_1) \subset K$, and if V is a complete subset in E_1, then V is complete with respect to the K quasi norm defined by $\|x\| = P(x)$.

Suppose that $(x_\nu)_{\nu \in I}$ is a Cauchy net in V with respect to the K quasi norm. Let U_1 be a given neighbourhood of 0 in E_1. Since P is coercive there exists a neighbourhood U of 0 in E such that $P^{-1}(U) \subset U_1$. Since (x_ν) is Cauchy, there exists a μ in I such that for $\mu \le \nu \le \eta$ it follows that

$$P(x_\nu - x_\eta) \in U.$$

Let ξ and ζ be two arbitrary elements in I such that $\xi \ge \mu$ and $\zeta \ge \mu$. Then since I is totally ordered, we can suppose $\xi \le \zeta$ and hence

$$v_\xi - v_\zeta \in P^{-1}(U) \subset U_1$$

and also $v_\zeta - v_\xi \in U_1$. That is (v_ν) is a Cauchy net in the sense of the topology of E_1 and hence converges to an element of V. By 4.1 the topology in E_1 coincides with the topology induced by the K quasi norm P and hence (x_ν) converges to the same element in this last topology too.

Q.E.D.

7. Regular cones. Let W be a cone of the topological vector space E and let K be a subcone in W. The cone K is said W bound regular if every K totally ordered K increasing net in K which is W bounded, is convergent to an element of K. If K is itself K bound regular, it is called regular. Every W bound regular subcone of W is obviously regular. In Lemma 3 of (N2) it was shown that a complete cone K is W bound regular if and only if it is sequentially W bound regular, i.e., if every K increasing W bounded sequence in K converges to an element of K. This equivalence permits the following characterisation of the W bound regular complete subcones K of W (see (N2), Criterion 6) :

7.1. CRITERION. The complete subcone K of the cone W is W bound regular if and only if for every sequence (y_i) in K the condition $y_i \notin U$ for every i and for some neighbourhood U of 0, implies that

the set $\{s_n : n \in N\}$, where N stands for the natural numbers and we have put $s_n = \sum_{i=1}^{n} y_i$, cannot be W order bounded.

The cone K in the topological vector space E is called fully regular (M) if every totally ordered K monotonically increasing topologically bounded net of its is convergent to an element in K. The Criterion 7 in (N2) is a characterisation similar to 7.1 of the fully regular cones.

Spaces with two cones were considered by KRASNOSEL'SKII in (K) § 8. Our notion of W bound regularity is different from the relativised regularity considered there. We have used W bound regular cones in (N3) and (N2).

8. Examples of regular cones.

8.1. Let c denote the Banach space of convergent sequences of the real numbers equipped with the sup norm. The cone $W = c^+$ of sequences with non-negative terms in c is normal. Let c_0 the subspace in c of sequences converging to 0. Put $K = W \cap c_0$. We shall show in example 8.4 for a more general case that K is regular. But K is neither W bound regular nor fully regular.

8.2. Let W be a normal cone of the locally convex Hausdorff space E. Suppose that K is a complete subcone of W having the property that the linear span of its is complete, metrizable and does not contain any subspace isomorphic with the Banach space c_0 of all the real sequences converging to 0 equipped with the sup norm. According to Theorem 1 in the paper (K) of McARTHUR, every closed normal cone in a Fréchet space, which contains no subspace isomorphic with c_0 is fully regular. Hence K is fully regular and since every K increasing W bounded sequence in W is topologically bounded according to the normality of W (see e.g. (P) Proposition II. 1.4), it is convergent. This means that K is sequentially W bound regular and hence also W bound regular ((H2), Lemma 3).

8.3. Suppose that B is a closed, convex and bounded subset of the locally convex Hausdorff space E, which does not contain 0. Let K be the cone generated by B. Then K is a W bound regular subcone of every normal cone W containing it. Indeed, if we consider the subspace spK of E, then this space can be normed considering as unit ball of its the set ΓB, the closed and circled hull of B. The induced norm topology on spK is finer than that induced by the original topology of E. In this normed space K admits a plastering and hence it is, by Theorem 1.12 of (K), fully regular. Now, since W is normal, we can repeat the reasoning at the end of 8.2 to conclude that K is W bound regular.

8.4. Let S be a set and let $c_0(S)$ denote the subspace of the vector space R^S of all the real functions on S, having the property that for every positive real ε the set $\{s\in S : |x(s)| \geq \varepsilon\}$ is finite. We can define in $c_0(S)$ a norm by putting $\|x\| = \max_s |x(s)|$. The space $c_0(S)$ equipped with this norm becomes a Banach space ((D) II. 2). If we denote by $K = c_0^+(S)$ the cone of the non-negative functions in $c_0(S)$, then we observe that K is normal since the norm on it is monotone. We shall show that it is regular too. Let (x_n) be an increasing sequence in K, bounded from above by x. Let ε be an arbitrary positive real. The set $S_0 = \{s\in S : |x(s)| \geq \varepsilon\}$ is finite. Since (x_n) is an increasing sequence bounded from above by x, the sequence of real numbers $(x_n(s))$ is increasing

and bounded from above by x(s) for every s in S. Accordingly these real sequences are convergent and hence there is a number n, so as to have for every s in S_0 :

(8.1) $$|x_n(s) - x_p(s)| < \varepsilon$$

as soon as $n,p \geq n$. We have for every s in S

$$|x_n(s) - x_p(s)| \leq x(s),$$

and since $x(s) < \varepsilon$ for each $s \notin S_0$, it follows that 8.1 holds for every s in S. Therefore

$$\|x_n - x_p\| < \varepsilon$$

for any $n, p \geq n$. That is, the sequence (x_n) is Cauchy and hence convergent to an element of K.

8.5. Let the vector space R^S be endowed with the direct product topology. Then R^S is a complete locally convex Hausdorff space. We have seen in paragraph 5 that the cone $K = R^S_+$, where R_+ is the closed positive half line, is normal and completely minihedral. It is easy to see that R^S_+ is also regular. Since the convergence in the space R^S is in fact the pointwise convergence of functions, the regularity of R^S_+ is a direct consequence of the regularity of R_+ in its natural topology.

From the imbedding theorem 10.2 in (ABS) it follows that the positive cone of every topological semifield is regular, since the regularity inherits by closed subcones.

9. Regular cones of linear operators. Denote by L(E) the vector space of the linear and continuous operators acting in the locally convex space E. We shall suppose that L(E) is endowed

with the vector space topology of the simple convergence ((Sch), III.3).

9.1. <u>If</u> K <u>is a generating closed normal cone in the barelled space</u> E, <u>then the set</u> $L^+(E)$ <u>of all the positive operators in</u> $L(E)$, <u>i.e., the set of operators</u> A <u>with the property that</u> $A(K) \subset K$, <u>forms a regular cone.</u>

The set $L^+(E)$ is obviously a cone. Suppose that $(A_i : i \in I)$ is a totally ordered $L^+(E)$ increasing net in $L^+(E)$ bounded from above by A. Then $(A_i x : i \in I)$ is for every x in K a totally ordered K increasing net of K with the upper bound Ax. Since K is normal, it follows that the set $\{A_i x : i \in I\}$ is topologically bounded, and hence $\{A_i : i \in I\}$ is bounded with respect to the topology of the simple convergence ((Sch), III. 3.3). Since the space E is barelled, the set $\{A_i : i \in I\}$ is by Theorem 4.2 in (Sch) III equally continuous. The closure in E^E of this set is contained in $L(E)$ by III. 4.3 in (Sch). Consider the filter of the upper sections of the net $(A_i : i \in I)$. Let A_0 be an element of the closure of this filter. This means that there exists a subnet (A_{i_j}) confinal with (A_i) such that for every x in K the net $(A_{i_j}(x))$ is totally ordered, increasing and converges to $A_0(x)$. Now, since $(A_i(x))$ is increasing and K is a normal cone, we can conclude that $(A_i(x))$ converges to $A_0(x)$. Because K is a generating cone, it follows that this convergence holds for every x in E. This means that the filter of upper sections of the net (A_i) converges to A_0 in the topology considered, and the regularity of $L^+(E)$ follows.

Q.E.D.

Let H denote a Hilbert space. The operator A in $L(H)$ is called <u>positive</u> if $(Ax,x) \geq 0$ for all x in H, where $(\cdot,\cdot)$ denotes

the scalar product in H.

9.2. <u>The set</u> L^+ <u>of all the positive operators in the vector space of all linear and continuous operators acting in</u> H <u>equipped with the topology of simple convergence is a regular cone.</u>

The set L^+ is obviously a cone. According to Theorem V. 6.3 in (KA) every L^+ increasing norm bounded sequence (A_n) in L^+ is convergent with respect to the simple convergence to an element A_0 of L^+. The considerations in the proof of the above statement in (KA) yield the same conclusion for a totally ordered L^+ increasing net in L^+. Let us assume that (A_i) is a totally ordered L^+ increasing net of L^+ bounded from above by the linear and continuous operator A. We have for an arbitrary T in L^+ the relation

$$|(Tx,y)|^2 \leqslant (Tx,x)(Ty,y)$$

(relation (6) in V. 6.2 of (KA)). Using this and the fact that $(Ax,x) \leqslant \|A\|\|x\|^2$ for each x, we conclude that

$$|(A_i x,y)|^2 \leqslant (A_i x,x)(A_i y,y) \leqslant (Ax,x)(Ay,y) \leqslant \|A\|^2\|x\|^2\|y\|^2,$$

wherefrom for $y = A_i x$ it follows that

$$\|A_i x\|^2 \leqslant \|A\|^2\|x\|^2,$$

that is, $\|A_i\| \leqslant \|A\|$ for every i. Since the net (A_i) is norm bounded we can apply the theorem in (KA) cited above to conclude that it converges to A_0 in L^+ with respect to the considered topology. Accordingly, the cone L^+ is regular.

Q.E.D.

10. <u>Regular cone valued metrics.</u> Let W be a normal cone of the locally convex space E and let K be a W bounded regular subcone

of W. We have the following assertion :

10.1. Let V be a set and let r be a quasi metric on V with values in the W bound regular subcone K of the normal cone W. Suppose that v_n, $n \in N$ are elements of V such that the set

$$\left\{ \sum_{n=1}^{m} r(v_{n+1}, v_n) \; : \; m \in N \right\}$$

is W bounded. Then (v_n) is a Cauchy sequence with respect to the topology induced by r in the set V.

Let us assume the contrary : the sequence (v_n) does not be Cauchy. Then by the definition in 6 there exists a neighbourhood U of 0 in E so as to have for every m some natural n and p, $n \leq p \leq m$, such that $r(v_m, v_p) \notin U$. Let us assume that n_1 and n_2 are two natural numbers such that $1 < n_1 < n_2$ and $r(v_{n_2}, v_{n_1}) \notin U$. Then consider two natural numbers $n_3 < n_4$ such that $n_3 > n_2$ and $r(v_{n_4}, v_{n_3}) \notin U$, and so on. Thus we have got a subsequence (v_{n_k}) of (v_n) such that $r(v_{n_{2k}}, v_{n_{2k-1}}) \notin U$ for every k. Since

$$\sum_{k=1}^{m} r(v_{n_{2k}}, v_{n_{2k-1}}) \leq \sum_{n=1}^{n_{2m}-1} r(v_{n+1}, v_n),$$

it follows the W order boundedness of the sums in the left hand side. But this contradicts Criterion 7.1. The obtained contradiction completes the proof.

Q.E.D.

11. Monotone semicontinuity of the operators with values in ordered vector spaces. The theory of the vector minimization needs the introduction of some concepts of weakened semicontinuity. We have for instance the following one (see (N1) and (N2)) :

11.1. Let V be a topological space and let F be an operator with values in the topological vector space E ordered by a regular cone K. Then F is called lower monotone semicontinuous (N2) if from the conditions

(i) $\lim_{\nu} v_\nu = v$, where $(v_\nu)_{\nu \in I}$ is a net indexed by the totally ordered set I ;

(ii) the set $\{ F(v_\nu) : \nu \in I \}$ is lower bounded ;

(iii) $F(v_\nu) \leq F(v_\mu)$ whenever $\nu \geq \mu$,

it follows that

$$\lim_{\nu} F(v_\nu) \geq F(v).$$

This notion of semicontinuity is basic in the following assertion proved in (N2) (Proposition 23) :

11.2. Let E be a locally convex Hausdorff space ordered by the normal and regular cone K.

Let F be a lower monotone semicontinuous operator defined on the complete subset V of the locally convex Hausdorff space E_o with values in E, and let P be a continuous and coercive sublinear operator from E_o to E with the property that $P(E_o) \subset K$.

If F has order bounded lower sections, i.e., if there exists some z in V such that the set

(11.1) $\quad (F(z) - K) \cap F(V)$ has a lower bound,

then for every positive real number ε there existian element v in V such that

(11.2) $\quad F(v) \leq F(z) - \varepsilon P(z-v)$ and

(11.3) $\quad F(v) - F(w) - \varepsilon P(v-w) \notin K$ whenever $w \in V \setminus \{v\}$.

If H is a subset in K with the property that $E \setminus H$ is a neighbourhood of 0, then in the condition (11.1) there is an element u in V such that

(11.4) $$F(u) \leqslant F(z) \quad \text{and}$$

(11.5) $$(F(u) - \complement E - K) \cap F(V) = \emptyset .$$

For every u with this property there is a v in V so as to have (11.3), the condition (11.2) with u in place of z, and

(11.6) $$u - v \in F^{-1}(K \setminus E) .$$

The inequality $\leqslant$ stands for the order relation induced by K. Sublinearity and coercivity are the notions defined in paragraph 4. (In (N1) and (N2) we have used the term "definite operator" for coercive operator.)

The conditions on K and F can be weakened in a generalised form of the assertion 11.2. Let us consider here the definition of a weakened form of lower monotone semicontinuity of F.

11.3. Let V be a topological space and let F be an operator from V to the ordered vector space E. We shall say that F is submonotone if from the conditions

(i) $\lim_{\nu} v_{\nu} = v$, where $(v_{\nu})_{\nu \in I}$ is a net in V indexed by the totally ordered set I ;

(ii) $F(v_{\nu}) \leqslant F(v_{\mu})$ whenever $\nu \geqslant \mu$;

it follows that

(11.7) $$F(v) \leqslant F(v_{\nu}) \text{ for every } \nu \text{ in } I.$$

11.4 Remark. We observe that if E is an ordered topological vector space with regular positive cone, then from (iii) of 11.3 it follows that $\lim F(v_{\nu})$ exists and by Proposition II.3.1 of (P) $\lim F(v_{\nu}) \leqslant F(v_{\mu})$ for every μ in I, and this limit is the infimum of the set $\{F(v_{\nu}) : \nu \in I\}$ and hence, by (11.7) it holds

$F(v) \leqslant \lim F(v_{\nu})$. This shows that "submonotone" generalises "lower monotone semicontinuous" for the case when the positive cone in E does not be regular.

11.5. Remark. Another semicontinuity notion was used for other purposes in (NL2) and (NL3). There an operator acting in an ordered regular Banach space X is upper semicontinuous from the right if from (x_n) and $(F(x_n))$ both convergent and nonincreasing sequences it follows that

$$\lim F(x_n) \leqslant F(\lim x_n).$$

Extending this terminology we can say that F is lower semicontinuous from the right if from (x_n) and $(F(x_n))$ both convergent and nonincreasing sequences in X it follows that

$$\lim F(x_n) \geqslant F(\lim x_n).$$

In the special case of $V = E = X$ this notion is a weaker one as lower monotone semicontinuous considered at 11.1.

12. The main result.

12.1. THEOREM. Let E be a locally convex Hausdorff space and let W be a closed normal cone in E. Suppose that K is a W bound regular subcone of W.

Let V be a complete K quasi metric space with the K quasi metric denoted by r and let $F : V \to E$ be a submonotone operator with respect to the order induced by W.

Suppose that F has W order bounded W lower sections, i.e., there is a z in V such that the set

(12.1) $$(F(z) - W) \cap F(V) \text{ has a } W \text{ lower bound}.$$

Then for every positive real ε there is a v in V so as to have

$$F(s) - F(v) - \varepsilon r(s,v) \in W \quad \text{and} \tag{12.2}$$

$$F(v) - F(w) - \varepsilon r(v,w) \notin W \quad \text{whenever} \quad w \in V \setminus \{v\}. \tag{12.3}$$

Let U be a neighbourhood of 0 in E. If $H = K \setminus U$, then in the condition (12.1) there exists an element u in V such that

$$F(s) - F(u) \in W \quad \text{and} \tag{12.4}$$

$$(F(u) - \varepsilon H - W) \cap F(V) = \emptyset. \tag{12.5}$$

For every u with this property there is an element v in V satisfying (12.3) and the condition (12.2) with u in place of s, and such that

$$r(u,v) \in U\ . \tag{12.6}$$

The term "minimization principle" for this theorem is motivated by the relation (12.3) which is in fact a minimum condition: it shows that v is a W minimum poit of the operator $F(v) - F(.) - \varepsilon r(v,.)$

Proof. Define the relation $\prec$ on $F(V)$ by putting $F(p) \prec F(q)$ if

$$F(q) - F(p) - \varepsilon r(p,q) \in W.$$

This relation is reflexive, transitive and antisymmetric.

The reflexivity of $\prec$ is immediate.

In order to verify the transitivity let us suppose that $F(p) \prec F(q)$ and $F(q) \prec F(s)$, that is,

$$F(q) - F(p) - \varepsilon r(p,q) \in W \quad \text{and}$$
$$F(s) - F(q) - \varepsilon r(q,s) \in W,$$

wherefrom by summation we have

$$F(s) - F(p) - \varepsilon (r(p,q) + r(q,s)) \in W\ . \tag{12.7}$$

From the axiom (m2) we have

$$r(p,q) + r(q,s) - r(p,s) \in K \subset W\ ,$$

wherefrom multiplying with ε and summing with (12.7) we get

$$F(s) - F(p) - \varepsilon r(p,s) \in W\ ,$$

that is, $F(p) \prec F(s)$ and the transitivity of $\prec$ is proved.

Let us assume now that $F(p) \prec F(q)$ and $F(q) \prec F(p)$, that is,

$$F(q) - F(p) - \varepsilon r(p,q) \in W \quad \text{and}$$
$$F(p) - F(q) - \varepsilon r(q,p) \in W.$$

Summing these relations we get

$$-\varepsilon(r(p,q) + r(q,p)) \in W$$

and since $r(p,q),\ r(q,p) \in W$ it follows that

$$r(p,q) = r(q,p) = 0.$$

These relations imply that $p = q$, since r is a quasi metric (see axiom (m4)). Wherefrom the antisymmetry of $\prec$.

Apply HAUSDORFF's theorem ((DSch) I.2.6) to determine a subset Z in $F(V)$ which is totally ordered with respect to the relation $\prec$, has $F(s)$ as supremum, and is maximal with respect to the set theoretic inclusion. We shall show that Z contains its infimum with respect to the relation $\prec$.

Let $\{v_\nu\}$ the set of elements in V with the property that $F(v_\nu) \in Z$. In the set I of indices ν we introduce an ordering $\leq$ by putting $\mu \leq \nu$ if $F(v_\nu) \prec F(v_\mu)$. Since Z is totally ordered, I will be totally ordered too.

We shall show in what follows that $(v_\nu)_{\nu \in I}$ is Cauchy net in V in the sense of the paragraph 6. Let us assume the contrary. Then there is a neighbourhood U' of 0 in E such that for every μ in I there exists ν, η in I such that $\mu \leq \nu \leq \eta$ and $r(v_\nu, v_\eta) \notin U'$. Fix $\mu \in I$ and let ν and η as above. Put

$v_\nu = q_1$, $v_\eta = q_2$. Then

$$r(q_2,q_1) \notin U' \quad \text{and}$$
$$F(q_1) - F(q_2) - \varepsilon r(q_2,q_1) \in W.$$

Starting with η in place of μ we can by a similar way get q_3 and q_4 in V with $F(q_3)$, $F(q_4)$ in Z, and such that

$$F(q_2) - F(q_3) - \varepsilon r(q_3,q_2) \in W,$$
$$r(q_4,q_3) \notin U' \quad \text{and}$$
$$F(q_3) - F(q_4) - \varepsilon r(q_4,q_3) \in W.$$

Repeating this procedure we can determine the sequence (q_n) in V such that $F(q_n)$ is in Z for every n and

(12.8) $\quad F(q_n) - F(q_{n+1}) - \varepsilon r(q_{n+1},q_n) \in W$ for every n,

and such that

(12.9) $$r(q_{2k},q_{2k-1}) \notin U' \quad \text{for every } k.$$

Summing the relation (12.8) from n = 1 to n = m, we get

$$F(q_1) - F(q_{m+1}) - \varepsilon \sum_{n=1}^{m} r(q_{n+1},q_n) \in W.$$

Now, all the $F(q_n)$-s are in the set (12.1) by the definition of Z, and since this set has a W lower bound, say y_0, the above relation yields, by adding with $F(q_{m+1}) - y_0 \in W$,

$$F(q_1) - y_0 - \varepsilon \sum_{n=1}^{m} r(q_{n+1},q_n) \in W.$$

Hence the sums

$$\sum_{n=1}^{m} r(q_{n+1},q_n), \quad m \in N$$

are W order bounded, wherefrom we get, via the assertion 10.1, a contradiction with (12.9) since K is a W bound regular subcone

of W. The obtained contradiction proves that (v_ν) is a Cauchy net and hence it converges to an element v in V, according the completity of this set.

Since F is submonotone with respect to the order induced by W, we have

(12.10) $$F(v_\nu) - F(v) \in W$$

for every ν in I. Let $F(v_\mu)$ be arbitrary in Z. For every $\nu > \mu$ we have

(12.11) $$F(v_\mu) - F(v_\nu) - \varepsilon r(v_\nu,v_\mu) \in W,$$

wherefrom adding with (12.10) we get

$$F(v_\mu) - F(v) - \varepsilon r(v_\nu,v_\mu) \in W.$$

Since $\lim_\nu r(v_\nu,v_\mu) = r(v,v_\mu)$ and since W is closed, we conclude that

$$F(v_\mu) - F(v) - r(v,v_\mu) \in W,$$

that is, $F(v) \prec F(v_\mu)$ for each $F(v_\mu)$ in Z. Now, since Z is maximal, F(v) must be in Z and it is the infimum of Z with respect to the relation $\prec$.

The last assertion implies also that there does not exist any w in $V \setminus \{v\}$ so as to have $F(w) \prec F(v)$. Thus we have proved the relations (12.2) and (12.3).

If H is the set defined in the theorem, then according Proposition 21 in (N2) it follows the existence of a u in V with the properties (12.4) and (12.5).

If we proceed as above taking u in the place of s, we can get a v in V so as to have (12.3) and (12.2) with u in place of s, that is, to have the relation

$$(12.12) \qquad F(v) \in F(u) - \varepsilon r(u,v) - W .$$

We assume now that (12.6) does not hold. Then we have $r(u,v) \in K \setminus U = H$, that is,

$$F(u) - \varepsilon r(u,v) - W \subset F(u) - \varepsilon H - W .$$

From this relation and (12.12) we get a contradiction with (12.5).

Q.E.D.

12.2. Remark. We observe that Theorem 12.1 generalizes the assertion 11.2. Indeed, putting $W = K$ and assuming K is normal and regular, considering $r(a,b) = P(a-b)$, r will be a quasi K metric wich according 4.1 defines the original topology on V by the continuity and coercivity of P. Finally, in the conditions of 11.2 the submonotunity of F implies lower monotone semicontinuity of it, as we have observed this at 11.4.

13. An application to fixed point theory. After various applications to the fixed point theory of the cone valued metrics (see e.g. (PE), (PK), (CO) etc.) EISENFELD and LAKSHMIKANTHAM were the first authors who discovered the importance of regular cones for this theory. The above named authors have developed systematically in the series of papers (EL1-3) the application of the regular cone valued metrics to the fixed point theory. They have proved among others an extension of CARISTI's fixed point theorem (C) for the case of K metrics, where K is a regular minihedral cone with interior points in a Banach space. According to the relation of the nonconvex minimization principle due to EKELAND (E) and the CARISTI's fixed point theorem, one can be

expected that from Theorem 12.1 it can be deduced a result similar with the above mentioned theorem of EISENFELD and LAKSHMIKANTHAM (EL3). We begin, after some introduction, with reproducing this last result here.

From (K) it follows that a minihedral, regular closed cone K in the Banach space E is completely minihedral. If E is separable, then the infimum of a set M of E can be obtained as a limit of a sequence of decreasing elements $y_n = \inf \{x_1, \dots, x_n\}$ with x_i in M. All these considerations are used in (EL3) in the proof and to define semicontinuous operators from a topological space X to K. Let F be an operator from X to K. If (x_n) is an arbitrary sequence in X, then $\liminf F(x_n)$ is defined as to be $\inf \{F(x_n) : n \in N\}$. As we have remarked above this limit can be obtained as a sequence of infima of increasing finite sets in $\{F(x_n) : n \in N\}$ fact used in the proofs in (EL3). The operator $F : X \to E$ is said to be lower semicontinuous (EL3), if $x_n \to x$ implies

$$F(x) \leqslant \liminf_n F(x_n) .$$

A lower semicontinuous operator in the above sense is obviously lower monotone semicontinuous too and hence also submonotone in the sense of 11.1 and respecively, 11.3.

The fixed point theorem in (EL3) can be formulated as follows (see Lemma 3.3 of the cited paper) :

13.1 Let V be a complete K metric space where K is a closed regular, minihedral cone with a non empty interior in a separable Banach space. Suppose $f : V \to V$ admits a majorant, i.e., an operator F from V to K such that

$$r(f(x),x) \leq F(x) - F(f(x)),$$

and assume that F is lower semicontinuous. Then for arbitrary z in V there is a fixed point v of f such that

$$F(v) \leq F(z).$$

We shall deduce from Theorem 12.1 the following generalization of this theorem :

13.2. THEOREM. Let E be a locally convex Hausdorff space, W a closed normal cone in E and let K be a W bound regular subcone of W. Let V be a complete K metric space with the K metric r and let F be an operator from V to E which is W submonotone and has W order bounded lower sections, i.e., there is a z in V such that the set

$$(F(z) - W) \cap F(V)$$

has a W lower bound. If $f : V \to V$ has the property that

$$r(f(u),u) \leq F(u) - F(f(u)) \tag{13.1}$$

for every u in V, then f has a fixed point v such that

$$F(v) \leq F(z) - r(v,z),$$

where $\leq$ stands for the W order.

Proof. Put $\varepsilon = 1$ and apply Theorem 12.1 to V and F in the above theorem. Then it follows the existence of a v in V such that $F(v) \leq F(z) - r(v,z)$ and

$$F(v) - F(w) - r(w,v) \notin W \text{ whenever } w \in V \setminus \{v\} . \tag{13.2}$$

From the other hand (13.1) implies

$$F(v) - F(f(v)) - r(f(v),v) \in W .$$

If would have $f(v) \neq v$, this relation would contradict (13.2).

Q.E.D.

The condition (13.1) on the mapping f has a few interesting consequences even in the absence of the submonotonity of F. To formulate these consequences we consider first some notions.

13.3. The operator G from the K metric space V to the regular cone K will be said upper (lower) semicontinuous from the right (compare with the notion used in (HL3) and 11.5), if from $v_n \to v$ and $G(v_{n+1}) \leq G(v_n)$ for every n, it follows that $\lim G(v_n) \leq G(v)$ (respectively, $\lim G(v_n) \geq G(v)$). Here $\leq$ stands for the K ordering.

If f is a mapping acting in V, then V will be called f orbitally complete if the Cauchy subsequences of the sets $\{f^n(u) : n \in N\}$ with u in V, are convergent. We can now use reasonings similar with those in Lemmas 3.1 and 3.2 of (HL3) to conclude the following assertion :

13.4. Let K be a W bound regular subcone of the normal cone W. If V is a K metric space, if $f : V \to V$ has a majorant $F : V \to E$, i.e., an operator with the property (13.1), if $F(V)$ has a W lower bound y_0, and if V is f orbitally complete, then

(a) the operator $G : V \to K$ given by

$$G(u) = \sum_{i=0}^{\infty} r(f^i(u), f^{i+1}(u)) \tag{13.3}$$

is defined throughout in V ;

(b) $(f^i(u))$ is a Cauchy sequence ;

(c) $G(f^i(u))$ is K decreasing and converges to 0 for every u in V ;

(d) G *is* K *upper semicontinuous from the right on the closure of every orbit* $M(u) = \{f^i(u) : i \in N\}$ *of* f.

Indeed, by (13.1) it follows

$$r(f^i(u), f^{i+1}(u)) \leq F(f^i(u)) - F(f^{i+1}(u)),$$

which, by summing from $i=0$ to $i=m$, yields

$$S_m = \sum_{i=0}^{m} r(f^i(u), f^{i+1}(u)) \leq F(u) - F(f^{i+1}(u)) \leq F(u) - F_0,$$

Since (S_m) is K increasing and K bounded sequence in K, it converges to a limit $G(u)$ in K and we have (a).

The assertion 10.1 implies (b).

We have further

$$G(f^i(u)) - G(f^{i+1}(u)) = r(f^i(u), f^{i+1}(u)) \in K,$$

which shows that the sequence $(G(f^i(u)))$ is K decreasing. Its limit is 0, since $G(f^{i+1}(u))$ is the i^{th} remainder term of the convergent series (13.3) and we have (c).

From (b) and from f orbitally completity of V it follows that $\lim\limits_i f^i(u) = v$ and $G(\lim\limits_i f^i(u))$ exist and accordingly using (c) we have

$$G(\lim_i f^i(u)) = G(\lim_i f^i(u)) - \lim_i G(f^i(u)) \in K,$$

which is in fact (d).

Q.E.D.

The above proved assertions permits us to estabilish some orbital variants of the fixed point theorem 13.2. The advantage of these results is that they yield constructive procedures for finding the fixed points.

13.5. *In the conditions of* 13.4 *the following assertions are equivalent* :

(a) G *is* K *lower semicontinuous from the right on* $\overline{M(u)}$.

(b) G *is continuous on* $\overline{M(u)}$.

(c) f *is continuous on* $\overline{M(u)}$.

(d) $\lim\limits_i f^i(u) = v$ *is a fixed point of* f.

Since G is K upper semicontinuous from the right on $\overline{M(u)}$ by 13.4 (d), it follows according (a) that

$$\lim_i G(f^i(u)) = G(\lim_i f^i(u)), \tag{13.4}$$

that is, G is continuous on $\overline{M(u)}$.

If (b) holds, that is, if we have (13.4) then by (c) of 13.4, $G(\lim f^i(u)) = 0$, and by the definition (13.3) of G, $r(\lim\limits_i f^i(u), f(\lim\limits_i f^i(u)) = 0$, that is,

$$f(\lim_i f^i(u)) = \lim_i f^i(u), \tag{13.5}$$

which is (d). The relation (13.5) proves also the continuity of the function f on $\overline{M(u)}$, that is, the assertion (c).

Suppose now that f is continuous on $\overline{M(u)}$. Then we have (13.5) and hence, via (13.3), $G(v) = 0$. This together with 13.4 (c) implies that G is continuous on $\overline{M(u)}$.

Q.E.D.

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VECTOR MINIMIZATION PRINCIPLES WITH AND WITHOUT THE AXIOM OF CHOICE

by A. B. Németh

0. At the beginnings of investigations about some conceptual problems of the set theory it have become clear by the results due to ZORN, ZERMELO, HAUSDORFF, KURATOWSKI and others that the existence of the minimal elements in some semiordered sets, the existence of the fixed points of some mappings as well as the famuous axiom of choice are related questions. It is the merit of BOURBAKI the pointing out accurately this interconnection in (B) (see also (DSch) I. 2). For intuitionistic's sake it is of a major importance when the considered ordering principles and the related fixed point results work without any reference to the axiom of choice. This question have been investigated independently from the classical literature in some application oriented papers of CARISTI and KIRK (KC) and BRÉZIS and BROWDER (BB) (see also (E)). The relation with the results in (B) was pointed out then by BRØNDSTED (BR). The mentioned results have a constructive character and in them a real valued "comparation" function plays a fundamental role. When this is changed into an operator with values into an ordered vector space, then as have observed EISENFELD and LAKSHMIKANTHAM (EL) there appear some problems of a conceptually new feature. We have proved in (N2) an ordered vector space variant of EKELAND's minimization principle (E) the later being closely related to the result in (BB). Our note aims to complete the paper (N2) showing the relation of the main result of its with the classical ordering principles ((B) and