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VECTOR MINIMIZATION PRINCIPLES WITH AND WITHOUT THE AXIOM OF CHOICE

by A. B. Németh

0. At the beginnings of investigations about some conceptual problems of the set theory it have become clear by the results due to ZORN, ZERMELO, HAUSDORFF, KURATOWSKI and others that the existence of the minimal elements in some semiordered sets, the existence of the fixed points of some mappings as well as the famuous axiom of choice are related questions. It is the merit of BOURBAKI the pointing out accurately this interconnection in (B) (see also (DSch) I. 2). For intuitionistic's sake it is of a major importance when the considered ordering principles and the related fixed point results work without any reference to the axiom of choice. This question have been investigated independently from the classical literature in some application oriented papers of CARISTI and KIRK (KC) and BRÉZIS and BROWDER (BB) (see also (E)). The relation with the results in (B) was pointed aut then by BRØNDSTED (BR). The mentioned results have a constructive character and in them a real valued "comparation" function plays a fundamental role. When this is changed into an operator with values into an ordered vector space, then as have observed EISENFELD and LAKSHMIKANTHAM (EL) there appear some problems of a conceptually new feature. We have proved in (N2) an ordered vector space variant of EKELAND's minimization principle (E) the later being closely related to the result in (BB). Our note aims to complete the paper (N2) showing the relation of the main result of its with the classical ordering principles ((B) and

(DSch) I.2) as well as with the recently developed constructive ordering principles and fixed point theorems ((KO), (BR), (BB) and (T)).

After the pointing aut of the role of HAUSDORFF's theorem (one of the equivalents of the axiom of choice quoted somewhere as ZORN's theorem; for our terminology see (DSch) I. 2.6) in the proof of Theorem 12.1 in (N2) we show that the nonconvex vector minimization principle comprised by the theorem is equivalent with the regularity of the cone inducing the considered order relation. Then is showed that for E a Fréchet space (an essential part of) the quoted result can be deduced from an ordered vector space variant of the BRÉZIS-BROWDER ordering principle. The note ends with showing that our denumerable ordering principle does not work when E isn't metrizable.

We observe finally that TURINICI have recently presented in (T) an abstract variant of BRÉZIS-BROWDER's principle which, for the case of E being metrizable, can also be adapted for a vector space variant like that presented here.

We shall use for all the notions and the terminology as a reference material the paper (N2).

1. Even if one perhaps did not be explicitely stated, it is known that the following ordering principle is an equivalent of the axiom of choice.

1.1. Let $(X, \leq)$ and $(Y, \leq)$ be two ordered sets and let F be a monotone map from X to Y in the sense that u and v in X, $u \leq v$ implies $F(u) \leq F(v)$. If every totally ordered set in X has a lower bound, then there exists a v in X such that for every $z \leq v$ it holds $F(z) = F(v)$.

We observe that order relation or ordering means throughout a reflexive, transitive and antisymmetric relation.

The above principle plays a fundamental role in our next considerations. Hence we shall supply here a proof of its equivalence with the following theorem of HAUSDORFF, which is one of the equivalents of the axiom of choice (see (DSch), I. 2.6).

1.2. Every ordered set contains a maximal totally ordered set.

We prove first that 1.2 implies 1.1. For let we consider the relation $\prec$ on F(X) defined as follows: $F(u) \prec F(v)$ if $F(u) \leq F(v)$ and $u \leq v$ simultaneously. Then $\prec$ is an order relation on F(X). Let Z be a maximal $\prec$-totally ordered subset of F(X) which exists by 1.2. If we consider

$$X_0 = \{x \in X : F(x) \in Z\},$$

then X_0 will be a totally ordered set in X. By the hypothesis in 1.1, X_0 has a lower bound v. Assume that $z \leq v$. By the monotonicity of F we have $F(z) \leq F(v) \leq F(x)$ and hence $F(z) \prec F(v) \prec F(x)$ for every x in X_0. Since Z is a maximal $\prec$-ordered set in F(X), we have $F(v) \in Z$ and it is the infimum of Z with respect to $\prec$. From the maximality of Z, $F(z) \prec F(v)$ implies $F(z) = F(v)$, that is, the assertion in 1.1 follows.

Consider now the implication $1.1 \Rightarrow 1.2$. Let X be an arbitrary ordered set. Consider the family $\mathcal{P}(X)$ of parts of X with the order relation $\leq$ defined as follows: $A \leq B$ if $B \subset A$. Denote by $\mathcal{T}(X)$ the subfamily of $\mathcal{P}(X)$ of totally ordered subsets in X. Let us consider $\mathcal{T}(X)$ to be ordered by the relation $\leq$ induced from $\mathcal{P}(X)$. Define the mapping

$$F : \mathcal{T}(X) \to \mathcal{P}(X)$$

to be the inclusion. Then F is obviously monotone. Consider the totally ordered subfamily $\{A_i : i \in I\}$ in $\mathcal{T}(X)$ and put $A = \bigcup_{i \in I} A_i$. Then A is a member of $\mathcal{T}(X)$ since for any u and v in A there is some A_i containing u and v and hence these elements are comparable with respect to the ordering in X. Now, $A \supset A_i$ and hence $A \leqslant A_i$ for every i. That is, every totally ordered subset in $\mathcal{T}(X)$ has a lower bound. Thus we have checked the condition of 1.1 and hence this can be applied to conclude that there exists a V in $\mathcal{T}(X)$ such that for every $Z \leqslant V$, Z in $\mathcal{T}(X)$ (i.e., for every totally ordered set Z in X such that $Z \supset V$) it follows that $F(Z) = F(V)$. But $Z = F(Z)$ and $V = F(V)$ by definition. In conclusion we have for every totally ordered set Z in X which contains V that $Z = V$. This shows that V is a maximal totally ordered set in X.

Q.E.D.

1.3. Observe that if we consider only Y to be an ordered set and put $u \leqslant v$ (u, v in X) if only $F(u) \leqslant F(v)$, then for the case F an injection, 1.1 becomes a tautology. (When F isn't an injection the relation $\leqslant$ induced in X by this way is not generally an ordering.) However, a similar procedure applies in the cases considered next, where the domain of F isn't apriori ordered, but there the order relation on the range of F is constructed by mean of the topological structure of its domain, and this structure is subsequently used for verifying the hypothesis in 1.1.

2. As we have remarked above, the ordering principle 1.1 applies to situations when the ordering in X does not be apriori given and it can be defined with the aid of the ordering in Y and the topology in X. In this case special conditions are needed in order to ensure the existence of a lower bound for every totally ordered subset in X. In this paragraph we shall consider

as range space Y an ordered locally convex Hausdorff space E. Let the ordering in E be induced by a closed normal cone W. The W bound regular subcone K of W has the role to realise the connection between the order relation in E and the topology of the domain of definition of an operator F from the nonvoid set V to E, assuming that V is a complete K metric space. (For the considered notions see the paragraphs 1, 2 and 6 in (N2)). Let F be a W submonotone operator from V to E (see for definition 11.3 in (N2)). Denote by r the K metric on V and let ε be an arbitrary positive real. Then we have the following assertion:

2.1. *Using the above introduced notations, if we put* $F(p) \prec F(q)$ *in the case when*

$$F(q) - F(p) - \varepsilon\, r(p,q) \in W$$

and define $\leqslant$ *in* V *considering* $p \leqslant q$ *if* $F(p) \prec F(q)$, *then in the condition* F *has a lower bound, it follows that every totally ordered set in* V *has a lower bound.*

Indeed, if V_0 is a totally ordered set in V, it follows that $F(V_0)$ is a $\prec$ totally ordered in $F(V)$. Since $F(V_0)$ has a W lower bound in E, we can use the method in the proof of Theorem 12.1 in (N2) to conclude that the filter of lower sections in V_0 is convergent to a limit v, and from the submonotonity of F, $v \leqslant u$ for every u in V_0, that is V_0 has a lower bound.

Q.E.D.

2.2. *If we accept the axiom of choice, then the general ordering principle* 1.1 *will imply, in conditions of* 2.1, *the existence of an element* v *in* V *such that*

(2.1) $\quad F(v) - F(w) - \varepsilon\, r(v,w) \notin W$ *whenever* $w \in V \setminus \{v\}$.

Let we consider the order relation $\prec$ defined on $F(V)$ and the order relation $\leqslant$ defined on V at 2.1. Then all the conditions of 1.1 are satisfied according to the assertion in 2.1. Hence there exists a v in V such that for every s in V with $s \; v$ it holds $F(s) = F(v)$. Assume that for some w in V it holds

$$(2.2) \qquad F(v) - F(w) - \varepsilon\, r(v,w) \in W.$$

Then from the definition of the order relation on V it would be $w \leqslant v$ which by the condition on v implies $F(w) = F(v)$. Hence by (2.2), $-\varepsilon\, r(v,w) \in W$ and then $r(v,w) = 0$.

Q.E.D.

3. In this paragraph we shall show that the nonconvex vector minimization principle comprised in 2.2 is equivalent in some sense with the condition to K be a W bound regular subcone of W. More precisely, we have the assertion :

3.1. Let W be a closed normal cone of the locally convex Hausdorff space E, and let K be a subcone of W. Then the minimization principle 2.2 holds for every complete K metric space V and every W submonotone mapping $F : V \to E$ which is W lower bounded if and only if K is W bound regular.

The if part of this assertion is in fact the content of 2.2. For the converse, let us suppose that K does not be W bound regular. Then by Criterion 6 in (N1) (see also Criterion 7.1 in (N2)), there exists a neighbourhood U of 0 in E and a K increasing sequence (x_n) in K which is W order bounded and for which $x_{n+1} - x_n \notin U$ for each n. Put $V = \{x_n : n \in N\}$. Define a K metric r on V by putting $r(x_k,x_h) = x_m - x_n$, where $m = \max\{h,k\}$, $n = \min\{h,k\}$. Then V is trivially r complete since it is a discrete space. Let us define

$F : V \to E$ by putting $F(x) = -x$. Then F is W lower bounded (since $V = \{x_n\}$ is W order bounded from above). Because V is discrete, F is trivially W submonotone.

Put $\varepsilon = 1/2$ and consider x_n to be arbitrary in V. Let $m > n$. Then

$$F(x_n) - F(x_m) - \tfrac{1}{2}\, r(x_n,x_m) = -x_n + x_m - \tfrac{1}{2}(x_m - x_n) = \tfrac{1}{2}(x_m - x_n) \in W.$$

We have in conclusion for every v $(=x_n)$ in V that there exists some w $(=x_m$ with $m > n)$ in $V \setminus \{v\}$ such that the relation (2.1) fails.

Q.E.D.

3.2. In (N1), Proposition 14, we have given a characterization of a W bound regular subcone K of the cone W by means of the so called near to minimality property of the W lower bounded sets. We observe the parallelism in form and in content between the mentioned result and the assertion 3.1.

4. In proof of Theorem 12.1 in (N2) as well as in the outline of the proof of 2.2 above we have used the theorem 1.2 of HAUSDORFF. The ordered set E considered at 2 has a special structure, hence is natural the question : when can be avoid the reference to 1.2 or other equivalent conditions in the proof of 2.2. The model for the ordering principle 1.1 can be considered to be the BRÉZIS-BROWDER ordering principle (BB) which for the special case of E being the real numbers can be used for the proof of 2.2 and it furnishes this proof without any reference to the axiom of choice or any other equivalents of its. The proof in (BB) is purely constructive, and this is another advantage. Hence this principle is sometimes considered as to be a constructive version

of ZORN's lemma (E). An inductive or denumerable version of the latter result can be get already in (B).

In (N1) we have proved some denumerable criterions for regular cones even in non metrizable locally convex spaces. The problem is : wheter or not a similar procedure can be used to produce a proof for 2.2, or similarly, in order to deduce the principal statement of Theorem 12.1 of (N2) from an ordered vector space generalization of the BRÉZIS-BROWDER ordering principle . Our aim is to show that this can be done when the ordered locally convex space E is metrizable, while our construction does not work for a more general case.

4.1. Let E be a metrizable and complete locally convex space (a Fréchet space) ordered by a closed regular cone K. Let V be an ordered set with the property that every decreasing sequence in it has a lower bound. Let $F : V \to E$ be monotone in the sense that $F(u) \leq F(v)$ whenever $u \leq v$, and assume it has a K lower bound. Then for every x_0 in V there exists a $v \leq x_0$ such that $w \leq v$ implies $F(w) = F(v)$.

By Theorem 3 of McARTHUR in (M) in the above conditions K is also a normal cone. Denote by q a (real) metric which generates the topology of E. Consider x_0 in V be given and put

$$d_1 = \sup \{ q(F(u),F(x_0)) \; : \; u \leq x_0 \} .$$

d_1 is finite since F has a K lower bound and K is a normal cone (see Proposition II. 1.4 in (P)). If $d_1 = 0$, then $v = x_0$ satisfies the condition in 3.1. If not, consider $x_1 \leq x_0$ such that

$$\frac{d_1}{2} < q(F(x_1),F(x_0)) \leq d_1 .$$

Assume $x_1,\dots,x_{n-1}$ were determined and put

$$(4.1) \qquad d_n = \sup \{ q(F(u),F(x_{n-1})) \; : \; u \leq x_{n-1} \} .$$

If $d_n = 0$ we are done with $v = x_{n-1}$. If not, then choose x_n with $x_n \leq x_{n-1}$ and

$$(4.2) \qquad \frac{d_n}{2} < q(F(x_n),F(x_{n-1})) \leq d_n .$$

If this procedure ends for some x_m, then $x_m = v$ will satisfy the condition. If not, we have determined a decreasing sequence (x_n) in V which satisfies (4.2) for each n. Since F is monotone, $(F(x_n))$ is a decreasing, K lower bounded sequence which converges since K is regular. Now, (4.2) shows that $d_n \to 0$. The sequence (x_n) has a lower bound, say v in V. Suppose $w \leq v$. Since $w \leq v \leq x_n$, it follows that $q(F(v),F(x_{n-1})) \leq d_n$ and $q(F(w),F(x_{n-1})) \leq d_n$ for every n. Hence

$$q(F(v),F(w)) \leq q(F(v),F(x_{n-1})) + q(F(w),F(x_{n-1})) \leq 2d_n ,$$

wherefrom $q(F(v),F(w)) = 0$ and the proof is complete.

Q. E. D.

4.2. The method of proof in 4.1 is similar with the argument in (BB). The same argument can be adapted in order to produce an abstract variant of this principle similar to that due to TURINICI (T), Theorem 3, when E replaces R.

By resonings similar to those in the proofs of 2.1 and 2.2 we can deduce (of course, without using 1.2) the assertion :

4.3. Let K be a closed regular cone of the Fréchet space E and let (V,r) be a sequentially complete K metric space. Suppose $F : V \to E$ is a K submonotone and K lower bounded operator. Then for every positive real number ε there is a v in V such that

$$F(v) - F(w) - \varepsilon r(v,w) \notin W \quad \text{whenever} \quad w \in V \setminus \{v\} .$$

4.4. We observe that 4.3 does not be the full statement of 2.2 for the special case of E metrizable and complete. This because the method of the proof does not work for this general case.

We shall show in the following example that 4.1 does not hold for a particular F which isn't metrizable.

4.5 Example. Suppose S is a set with non denumerable cardinal. Consider the locally convex Hausdorff space R^S of all real valued functions defined on S, endowed with the direct product topology. Let R^S_+ be the cone of the non negative functions in R^S. We have seen at 8.5 of (N2) that R^S_+ is regular. Let us consider the following set in R^S_+ :

$$A = \{x \in R^S_+ : x(s) \leqslant 1 \text{ for each } s, \text{ and the set of elements } s \text{ with } x(s) > 0 \text{ is at most denumerable}\}.$$

If $x_1 \leqslant x_2 \leqslant \ldots \leqslant x_n \leqslant \ldots$, x_n in A for every n, then there exists an y in A such that $x_n \leqslant y$ for each n : it can be taken to be the function with the value 1 on all the elements s for which there exists at least an n such that $x_n(s) > 0$, and with the value 0 elsewhere. The set A has the property that for every $y \in A$ there exists $y' \in A$ such that $y \leqslant y'$ and $y' \neq y$.

Consider now $V = -A$, and $F : V \to E$ to be the identity mapping. Then all the conditions of 4.1, except E being metrizable, are satisfied. From the above observation about A it follows that the conclusion of 4.1 does not hold.

The method of the proof of Theorem 13.2 in (N2) can be used in order to deduce from the assertion 4.3 the following generalization of a fixed point theorem due to EISENFELD and LAKSHMIKANTHAM (EL) :

4.6. Let K be a regular cone in the Fréchet space E. Suppose that f is a selfmapping of the complete K metric space (V,r) which satisfies the relation

$$r(f(u),u) \leqslant F(u) - F(f(u))$$

for every u in V, where F is a K submonotone operator from V to K. Then f has a fixed point.

This assertion is a particular case of Theorem 13.2 in (N2) for W = K and E a Fréchet space. We have pointed it aut since its proof can be done without referring to the axiom of choice.

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SUR LES MÉTHODES ITÉRATIVES DE TYPE INTERPOLATOIRE À VITESSE DE CONVERGENCE OPTIMALE

par

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Dans cet article nous étudions une classe de méthodes itératives pour la résolution des équations de la forme:

(1) $$f(x) = 0 ,$$

où $f : I \rightarrow R$ est une fonction réelle d'une variable réelle et I est un intervalle de l'axe réel.

Désignons par $x_1, x_2, \dots, x_{n+1}$, $n+1$ points distincts de l'intervalle I et par $\alpha_1, \alpha_2, \dots, \alpha_{n+1}$, $n+1$ nombres naturels tels que:

(2) $$\alpha_1 + \alpha_2 + \dots + \alpha_{n+1} = m + 1 ,$$

où $m \in N$.

Il est bien connu que quels que soient les nombres y_i^j, $j = 0, 1, \dots, \alpha_i - 1$; $i = 1, 2, \dots, n+1$, il existe un seul polynôme H_1 de degré au plus m qui vérifie les conditions:

(3) $$H_1^{(j)}(x_i) = y_i^j , \quad j = 0, 1, \dots, \alpha_i - 1 ; \quad i = 1, 2, \dots, n+1.$$

Le polynôme H_1 déterminé par les conditions (3) a la forme:

(4) $$H_1(x) = \sum_{i=1}^{n+1} \sum_{j=0}^{\alpha_i - 1} \sum_{k=0}^{\alpha_i - j - 1} y_i^j \frac{1}{k!\,j!} \left[\frac{(x-x_i)^{\alpha_i}}{\omega(x)} \right]^{(k)}_{x=x_i} \frac{\omega(x)}{(x-x_i)^{\alpha_i - j - k}} ,$$

où

(5) $$\omega(x) = \prod_{i=1}^{n+1} (x - x_i)^{\alpha_i} .$$

Si on suppose que la fonction f admet une dérivée d'ordre $m+1$ sur l'intervalle I et si $y_i^j = f^{(j)}(x_i)$, $j = 0, 1, \dots, \alpha_i - 1$; $i = 1, 2, \dots, n+1$, alors H_1, le polynôme d'Hermite de la fonction f, relativement aux noeuds x_i, aux ordres de multiplicité