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SEQUENTIAL REGULARITY AND THE DIRECTIONAL DIFFERENTIABILITY OF CONVEX OPERATORS ARE EQUIVALENT

by

A.B. Németh

0. Introduction. A regularity type property of an ordered topological vector space is a property which from the monotonity and order boundendess or topological boundendess of an arbitrary net or sequence concludes its convergence or fundamentality. In some additional hypotheses a property of this kind implies the directional differentiability of some nonlinear operators (see e.g. (B) or (N1)) and hence, as it was observed in (N1), also the subdifferentiability of the convex operators as well as the weak Hahn-Banach extension property of sublinear operators. In the present note we shall show that directional differentiability of the convex operators actually characterizes the sequentially regular spaces.

1. Definitions and results. Let E be a real vector space ordered by the (acute) cone K. The operator F from the vector space H to E is called convex if for every u and v in H and every real number t in the interval $[0,1]$ it holds the relation

$$F(tu + (1-t)v) \leqslant tF(u) + (1-t)F(v),$$

where $\leqslant$ stands for the order relation induced by the cone K in E.

Let G be an operator from H to E. The epigraph of G is by definition the subset of the Chartesian product $H \times E$ given by

$$\text{epi } G = \{(u,x) : G(u) \leqslant x\}.$$

It is easy to check that the operator $F : H \to E$ is convex if and only if epi F is a convex set in the vector space $H \times E$.

Suppose now that E is a locally convex space. Then E and equivalently its positive cone K is called <u>normal</u> if it has a neighbourhood basis consisting of sets U with the property $U = (U+K) \cap (U-K)$.

The ordered locally convex space E and equivalently its positive cone K is called <u>(sequentially) regular</u> (see e.g. (M1)) if every decreasing net (sequence) in K is convergent.

We observe that the term locally convex ordered regular space is used by various authors and papers in slightly different senses. We adopt here the above definition for the sake of simplicity. It coincides with that used by MCARTHUR (M1) when the cone K is closed. BORWEIN (B) uses for (sequentially) regular closed cones the term (countably) Daniell.

Consider the operator G from the vector space H to the ordered locally convex space E. If for some u and h in H there exists the limit

$$G'(u;h) = \lim_{t \searrow 0} t^{-1}(G(u+th) - G(u)) \quad (t \text{ in } (0,\infty)),$$

then it is called the <u>directional derivative of</u> G <u>at</u> u <u>in the direction</u> h.

If F is a convex operator from the vector space H to the locally convex ordered regular space E, then it has directional derivative at every point u and in any direction h (see e.g. (B) Proposition 3.7 (c)). This follows from the observation that for fixed u and h the function φ defined by

$$(1) \qquad \varphi(t) = t^{-1}(F(u+th) - F(u))$$

is increasing with t in $R \setminus \{0\}$ and bounded from below on $(0,\infty)$ according the convexity of F. If only sequential regularity of E is postulated, then the normality of E is also used in proving the existence of the directional derivatives of a convex operator (see e.g. (B)). The normality can be assumed to be a mild condition on a cone since the most of important ordered topological vector spaces have normal positive cones. In some cases regularity and sequential regularity are equivalent (N2). On the other hand it is known that in a Fréchet space every closed regular cone is normal (see (M2), Theorem 3). However, we can avoid the normality assumption in the proof of the result in the title. To show that this circumstance is consistent we show that there are ordered regular normed spaces which are not normal. We shall adapt for this an example due to BRECKNER and ORBÁN used for other purposes in (BO1), Remark 2.4.3.

Let E denote the vector space of all polynomials with real coefficients defined on the interval $[-1,0]$, equiped with the uniform norm. Let us consider the cone K defined by

$$K = \{x \in E : x^{(i)}(0) \geqslant 0, \text{ i non-negatve integer}\}.$$

The cone K is regular since if (x_n) is a decreasing sequence in K, then it is necessarily a sequence of polynomials of a given finite order, say m, and according the ordering introduced by K, their coefficients form decreasing non-negative sequences of real numbers. Since all the coefficients of the terms of order $> m$ are 0, the sequence (x_n) tends uniformly to a polynomial with non-negative coefficients and the regularity follows.

To prove that K is not a normal cone, consider the sequences (x_n) and (y_n) defined by

$$x_n(t) = \frac{1}{2n} \sum_{i=0}^{n} \binom{2n}{2i} t^{2i} \quad \text{and} \quad y_n(t) = \frac{1}{2n}(t+1)^{2n}, \quad t \in [-1,0].$$

One has $0 \leq x_n \leq y_n$ for each $n \in N$. The sequence (y_n) tends uniformly to 0, but (x_n) does not. This contradicts a well known criterion of normality (see (P), Proposition II. 1.3 or (E01), Theorem 2.4.1).

The aim of the present note is to show that the directional derivability of the convex operators with values in an ordered locally convex space characterizes the sequentially regular spaces. More precisely we shall prove the following

THEOREM. Let E be an ordered locally convex space. Then E is sequentially regular if and only if every convex operator defined on a vector space and taking values in E has directional derivative at every point and in every direction.

The necessity of the theorem is in fact known and follows from the monotonity of the operator φ defined in (1). The only problem in this direction is that we must use only sequential regularity to conclude the existence of directional derivatives.

It is immediate that we can state the theorem considering directional derivability of the convex operators defined on the vector space R only.

2. The proof of the theorem. We prove first the necessity, that is that if E is sequentially regular then every convex operator F from the vector space H to E has directional derivative at every point u and in every direction h. Consider the function φ defined on $R \setminus \{0\}$ by (1). Since φ is increasing with t and lower bounded on $(0, \infty)$, the sequence $(\varphi(1/n))$ decreases for $n \to \infty$ and is lower bounded. Since E is sequentially regular,

$$v = \lim_{n \to \infty} \varphi(1/n) \tag{2}$$

exists. We shall show that $\lim_{t \searrow 0} \varphi(t)$ also exists and equals v. If we assume by contradiction that the filter of lower sections $M_{t_o} = \{\varphi(t) : t \leq t_o\}$, $t_o > 0$ does not converge to v, we can get a neighbourhood U of 0 in E such that for every n there exists a t_n, $0 < t_n < 1/n$ such that

$$\varphi(t_n) - v \notin U. \tag{3}$$

Now, we can put together the sequences $(1/n)$ and (t_n) in order to get a decreasing sequence (s_m) in $(0, \infty)$ which converges to 0. Then $(\varphi(s_m))$ is decreasing for $m \to \infty$ and lower bounded. It must be convergent since E is sequentially regular. But this contradicts (2) and (3). Hence

$$F'(u;h) = \lim_{t \searrow 0} t^{-1}(F(u+th) - F(u))$$

exists.

To prove the sufficence, i.e., that directional derivability of the convex operators implies sequential regularity, we proceed by contradiction. We shall give indeed a constructive proof of the following proposition :

If the positive cone K in E contains a decreasing sequence of elements which is not convergent, then there exists a convex operator from R to E without directional derivative at $0 \in R$ in the direction $1 \in R$.

An essential role in our construction has the assertion :

(a) If K contains a decreasing sequence which is not convergent, then it contains a decreasing sequence (x_n) for which it exists a sequence (t_n) of numbers with the property $0 < t_n < 1$ for each n, such that

$$x_{n+1} \leqslant t_n x_n, \quad n \in N,$$

and (x_n) is not convergent.

Indeed, if (y_n) is a decreasing sequence in K which is not convergent, then if we put

$$x_n = \frac{n+1}{n} y_n, \quad n \in N,$$

then (x_n) will satisfy the condition of the assertion (a).

(b) Let (x_n) be a sequence with the property asserted in (a). Then there exists a strictly decreasing sequence (a_n) of positive real numbers a_n such that $\lim a_n = 0$ and so as to have

$$(4) \quad (a_{n-1}-a_n)(a_n x_n - a_{n+1}x_{n+1}) \leqslant (a_n - a_{n+1})(a_{n-1}x_{n-1} - a_n x_n)$$

for every $n \geqslant 2$.

We put $a_1 = 2$, $a_2 = 1$ and shall show that if $a_1, \ldots, a_n$, $n \geqslant 2$ were choosen, then a_{n+1} can be determined in order to have $0 < a_{n+1} < \min\{2^{-n+3}, a_n\}$ and

$$(5) \qquad a_n x_n \leqslant \frac{a_n - a_{n+1}}{a_{n-1}-a_{n+1}} a_{n-1}x_{n-1}.$$

Indeed, according the special feature of (x_n) there exists $0 < t_{n-1} < 1$ such that $x_n \leqslant t_{n-1}x_{n-1}$ and hence, in order to realise (5) it suffices to show that there exists arbitrarily small positive solution in a_{n+1} of the inequality

$$\frac{a_n - a_{n+1}}{a_{n-1}-a_{n+1}} \, \frac{a_{n-1}}{a_n} \geqslant t_{n-1}.$$

But this inequality is equivalent with

$$a_{n+1} \leqslant \frac{a_n a_{n-1}(1-t_{n-1})}{a_{n-1}-t_{n-1}a_n},$$

where the right hand side term is a positive real number. Hence a_{n+1} can be choosen to satisfy our requirements. Now (5) implies

$$a_n x_n \leqslant \frac{a_n - a_{n+1}}{a_{n-1}-a_{n+1}} a_{n-1}x_{n-1} + \frac{a_{n-1} - a_n}{a_{n-1}-a_{n+1}} a_{n+1}x_{n+1},$$

wherefrom it follows the relation (4).

(c) Let us define the operator $F : R \to E$ by

$$(6) \quad F(t) = \begin{cases} 0 & \text{if } \quad t \leqslant 0, \\ \dfrac{t - a_{n+1}}{a_n - a_{n+1}}(a_n x_n - a_{n+1}x_{n+1}) + a_{n+1}x_{n+1}, & t \in [a_{n+1}, a_n), n \geqslant 2, \\ (t-1)(2x_1 - x_2) + x_2 & \text{if } \quad t \in [1, \infty) \end{cases}$$

where (x_n) and (a_n) are the sequences satisfying the conditions in (b). Then the operator F is convex.

Let us define the affine operators $F_n : R \to E$, $n = 0, 1, 2, \ldots$, with $F_0(t) = 0$, $t \in R$,

$$F_1(t) = (t-1)(2x_1 - x_2) + x_2, \quad t \in R,$$

and

$$F_n(t) = \frac{t - a_{n+1}}{a_n - a_{n+1}}(a_n x_n - a_{n+1}x_{n+1}) + a_{n+1}x_{n+1}, \quad t \in R,$$

for $n \geqslant 2$. The operators F_n are obviously convex, hence the epigraphs

$$\text{epi } F_n = \{(t,x) \in R \times E : F_n(t) \leqslant x\}, \quad n \in N \cup \{0\}$$

are convex.

We shall show that

$$(7) \qquad (t, F_{n+1}(t)) \in \text{epi } F_n \quad \text{if} \quad t \in (-\infty, a_{n+1})$$

and

(8) $$(t,F_n(t)) \in \operatorname{epi} F_{n+1} \quad \text{if} \quad t \in [a_{n+1},\infty)$$

for n = 1,2,... . Indeed, we have

$$F_n(t) - F_{n+1}(t) = \left(\frac{a_nx_n-a_{n+1}x_{n+1}}{a_n-a_{n+1}} - \frac{a_{n+1}x_{n+1}-a_{n+2}x_{n+2}}{a_{n+1}-a_{n+2}}\right)(t-a_{n+1}).$$

According the relation (4),

$$\frac{a_nx_n - a_{n+1}x_{n+1}}{a_n- a_{n+1}} - \frac{a_{n+1}x_{n+1} - a_{n+2}x_{n+2}}{a_{n+1}- a_{n+2}} \geqslant 0$$

and hence

$$F_n(t) - F_{n+1}(t) \geqslant 0 \quad \text{for} \quad t \geqslant a_{n+1}$$

and

$$F_n(t) - F_{n+1}(t) \leqslant 0 \quad \text{for} \quad t < a_{n+1}.$$

From the second relation we have (7) while from the first it follows (8).

We shall show that

(9) $$\operatorname{epi} F = \bigcap_{m=0}^{\infty} \operatorname{epi} F_m .$$

To this end, we verify first that

$$\operatorname{epi} F \subset \operatorname{epi} F_m, \quad m \geqslant 1.$$

Consider the order relation $\prec$ in the vector space $R \times E$ induced by the cone $\{0\}\times K$, with K the positive cone in E. Then the relation (7) can be written as

$$(t,F_{n+1}(t)) \succ (t,F_n(t)) \quad \text{for} \quad t \in (-\infty,a_{n+1}).$$

Suppose that $t \in [a_{k+1},a_k)$ if $k > 1$ and that $t \in [1, \infty)$ if k = 1. Let $k > m$. Then using successively the relation above for n = m, m+1, ... , k-1, we get, since $t < a_{n+1}$, the relations

$$(t,F_k(t)) \succ (t,F_{k-1}(t)) \succ \dots \succ (t,F_m(t)).$$

Since $t \in [a_{k+1},a_k)$ for $k > 1$ and $t \in [1, \infty)$ for k = 1, we have $F(t) = F_k(t)$ and the above relation yields

$$(t,F(t)) \succ (t,F_m(t)),$$

that is,

$$(t,F(t)) \in \operatorname{epi} F_m.$$

Suppose now that $t \in [a_{k-1},a_k)$ if $k > 1$ and $t \in [1, \infty)$ if k = 1, and if $k \leqslant m$. The relation (8) can be written as

$$(t,F_{n+1}(t)) \prec (t,F_n(t)) \quad \text{for} \quad t \in [a_{n+1}, \infty).$$

Using this relation successively for n = k, ... , m, and taking into account that $t \geqslant a_{n+1}$ with n = k,...,m, we get

$$(t,F_m(t)) \prec (t,F_{m-1}(t)) \prec \dots \prec (t,F_k(t))$$

and since $F(t) = F_k(t)$, we have

$$(t,F_m(t)) \prec (t,F(t)),$$

that is, $(t,F(t)) \in \operatorname{epi} F_m$.

We have also $(t,F_0(t)) = (t,0) \prec (t,F(t))$ since by the definition of F, $F(t) \geqslant 0$ for every t. Henceforth it follows that

$$\operatorname{epi} F \subset \bigcap_{m=0}^{\infty} \operatorname{epi} F_m.$$

If for some t in R and some x in E we have $(t,x) \notin \operatorname{epi} F$, then by the definition of F, $(t,x) \notin \operatorname{epi} F_m$ with m = 0 if $t \leqslant 0$, m = k if $t \in [a_{k+1},a_k)$ and $k > 1$ and m = 1 if $t \in [1, \infty)$. That is,

$$(t,x) \notin \bigcap_{m=0}^{\infty} \operatorname{epi} F_m$$

and this completes the proof of the relation (9).

The relation (9) shows that epi F is a convex set and hence F is a convex operator.

(d) The operator F defined by (4) does not have directional derivative at the point $0 \in R$ in the direction $1 \in R$.

We have to show that

$$\lim_{t \searrow 0} t^{-1}(F(t) - F(0))$$

does not exist. Indeed, we have $a_n \searrow 0$ for $n \to \infty$, $F(0) = 0$ and $F(a_n) = a_n x_n$, and hence

$$a_n^{-1}(F(a_n) - F(0)) = x_n .$$

By hypothesis the sequence (x_n) is not convergent and our assumption follows.

The obtained result, which constitutes the proof of the proposition stated at the beginning of this paragraph, proves that if an ordered locally convex space E is not sequentially regular, then it exists a convex operator with values in this space which does not have directional derivative at some point and in some direction. Hence if every convex operator with values in E has directional derivatives at every point and in any direction, then the space E must be sequentially regular.

Q.E.D.

Remarks. We observe that in (N1) an other notion of directional derivability is used that in (B) is called minorability. The above construction can be adapted also for that case in order to conclude that sequentially chain completness of and ordered vector space and the minorability of the convex operators with

values in this space are equivalent.

If we postulate that K contains a topologically bounded decreasing sequence which is not convergent, then we can assert that the operator F constructed above is also continuous on R. The single problem in this case is to verify the continuity of F at 0. This follows from the local convexity of E and from the fact that $F(a_n) = a_n x_n$ converges to 0 when $n \to \infty$, for (x_n) a bounded sequence. The same conclusion follows also when we suppose to E have the boundendess property. We remind that an ordered topological vector space have the boundendess property if every order bounded set in it is topologically bounded. This condition is less restrictive than normality ((B01) Remark 2.4.3). In this case the continuity of F follows also from a more general result in (B02). We have also that the boundendess property is necessary to a space be sequentially regular ((N3) Lemma 4.4).

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SIMULTANEOUS TRANSFORMATION OF THE ORDER AND OF THE TOPOLOGY BY NONLINEAR OPERATORS

by

A.B. Németh

0. Introduction. In a recent paper (N1) we have considered a method of transformation of a wedge into another one so as to preserve some of its properties and to improve some others. Our construction is closely related to the norm of the space and permits to approximate from the inside every wedge, whose closure is not a subspace, by fully regular cones. The approximation is in operatorial sense : the wedge is approximated with its images by a net of nonlinear operators converging uniformly on bounded sets to the identity operator. The whole construction in (N1) depends essentially on the existence of the Fredholm resolvents of some sublinear operators. But the most of the results can be transposed for a more general case. We shall consider in this paper the similar problem for locally convex spaces and more general operators. We shall also admit the simultaneous transformation of the order and of the topology in the considered vector space. Although, our attention is focused also here on transforming wedges into regular, respectively fully regular cones. The approximability of a wedge by regular cones in this general context is limited by the fact that if a regular cone has nonempty interior, then the space must be normable. Hence the