„BABEŞ—BOLYAI" UNIVERSITY
FACULTY OF MATHEMATICS
RESEARCH SEMINARIES

SEMINAR OF

FUNCTIONAL ANALYSIS AND NUMERICAL METHODS

Preprint Nr. 1, 1984

CLUJ-NAPOCA
ROMANIA

C O N T E N T S

(BO2) BRECKNER, W.W.; ORBÁN, G., On the continuity of convex mappings,
Rev. d'Analyse Numer. Théor. l'Approx., 6 (1977), 117-123.

(M1) McARTHUR, C.W., In what spaces is every closed normal cone regular?,
Proc. Edinburgh Math. Soc. (Series III), 17 (1970), 121-125.

(M2) McARTHUR, C.W., Convergence of monotone nets in ordered topological vector spaces, Studia Math. 34 (1970), 1-16.

(N1) NÉMETH, A.B., Some differential properties of the convex maps, Mathematica (Cluj), 22 (45), (1980), 107-114.

(N2) NÉMETH, A.B., Summation criteria for regular cones with applications,
"Babeş-Bolyai" Univ. Faculty of Mathematics Research seminaries, Preprint Nr. 4, 1981, 99-124.

(N3) NÉMETH, A.B., Simultaneous transformation of the order and of the topology by nonlinear operators,
this volume.

(P) PERESSINI, A.L., Ordered Topological Vector Spaces,
Harper & Row, New York Evanston - London, 1967.

Seminar of Functional Analysis and
Numerical Methods, Preprint Nr. 1, 1984, pp. 135 - 158.

SIMULTANEOUS TRANSFORMATION OF THE ORDER AND OF THE TOPOLOGY BY NONLINEAR OPERATORS

by

A.B. Németh

0. Introduction. In a recent paper (N1) we have considered a method of transformation of a wedge into another one so as to preserve some of its properties and to improve some others. Our construction is closely related to the norm of the space and permits to approximate from the inside every wedge, whose closure is not a subspace, by fully regular cones. The approximation is in operatorial sense : the wedge is approximated with its images by a net of nonlinear operators converging uniformly on bounded sets to the identity operator. The whole construction in (N1) depends essentially on the existence of the Fredholm resolvents of some sublinear operators. But the most of the results can be transposed for a more general case. We shall consider in this paper the similar problem for locally convex spaces and more general operators. We shall also admit the simultaneous transformation of the order and of the topology in the considered vector space. Although, our attention is focused also here on transforming wedges into regular, respectively fully regular cones. The approximability of a wedge by regular cones in this general context is limited by the fact that if a regular cone has nonempty interior, then the space must be normable. Hence the

approximation problem in (N1) has its natural place in normable spaces. The investigations in the recent paper concern the theory of regular and fully regular cones which have become important in some application oriented considerations of EISENFELD and LAKSHMI-KANTHAM (EL), ISAC (I), the author (N3) etc.

We use the terminology in (N2) which is in fact that proposed by MCARTHUR (MC) and differs from that in our paper (N1) and in (I) by the use of the term "fully regular" in place of "completely regular" which is most frequently used for other notions in topology and in functional analysis.

All the considered vector spaces will be supposed to be real. The subset W in the vector space Y is said to be a wedge, if $W + W \subset W$ and if $tW \subset W$ for every non-negative real number t. A wedge W is called cone if $W \cap (-W) = \{0\}$. A wedge W introduces a translation invariant reflexive and transitive order relation in Y, if we put $u \leq v$ whenever $v-u$ is in W. This order relation is antisymmetric if and only if W is a cone.

The operator F from the vector space X to the vector space Y ordered by a wedge W is called W convex if for every x_1 and x_2 in X and every t in $[0,1]$ it holds the relation

$$F(tx_1+(1-t)x_2) \leq tF(x_1) + (1-t)F(x_2)$$

where $\leq$ stands for the order relation introduced by W.

An operator from X to Y is called W sublinear if it is W convex and positively homogeneous. An operator G from X to Y is called W superlinear, if $-G$ is W sublinear.

The operator Q from Y to Y is called W monotone, if $Q(u) \leq Q(v)$ whenever $u \leq v$.

The subset A in Y is called W full if $(A+W) \cap (A-W) = A$. This

relation is equivalent with the condition that for every a and b in A the set $\{z \in Y : a \leq z \leq b\}$ is contained in A.

Let Y be endowed with a vector space topology. Then Y is called W locally full if it contains a neighbourhood of 0 consisting of W full sets. If K is a cone and Y is K locally full, then it is called a normal space.

It is said that the topological vector space has the boundendess property with respect to the order induced W if every W order bounded set in it is topologically bounded too. Obviously, every W locally full topological vector space has the boundendess property with respect to the order induced by W.

Let W_o be a wedge contained in W. Then W_o is called quasi W bound regular (sequentially quasi W bound regular) if every W order bounded W_o increasing net (sequence) in W_o is Cauchy. The wedge W is called quasi regular (sequentially quasi regular)if it is W bound regular.

The wedge W_o in W is called fully quasi W regular (sequentially fully quasi W regular) if each W increasing and topologically bounded net (sequence) in W_o is Cauchy. The wedge W is called fully quasi regular (sequentially fully quasi regular) if it is fully quasi W regular (sequentially fully quasi W regular).

The terms locally full, normal, with boundendess property, quasi (sequentially)regular, fully quasi(sequentially) regular will be used equally for wedges and the topological vector spaces ordered by them.

1. Ordering transformation by superlinear operators. Let W be a nonempty wedge of the vector space Y. The following assertions summarize and generalize some results in (N1) concerning some algebraic aspects of order transformations by superlinear operators.

1. PROPOSITION. If G is a W superlinear operator acting in Y, then $G^{-1}(W)$ is a wedge. If $G^{-1}(W) \subset W$, then $G^{-1}(W) \cap (-G^{-1}(W))$ is the maximal subspace W_0 of W with the property that $(-I+G)W_0 \subset W$, where I denotes the identity operator of Y.

Proof. That $G^{-1}(W)$ is a wedge is straightforward.

Denote by W_0 the subset in W of the elements having the property that y, $-y$, $-y+G(y)$ and $y+G(-y)$ are in W. Then W_0 is a subspace of Y. Indeed, since G is positively homogeneous, from y in W_0 it follows that sy, $-sy$, $-sy+G(sy)$ and $sy+G(-sy)$ are all in W for every real s. Suppose that y_1 and y_2 are in W_0. Then y_1+y_2, $-(y_1+y_2)$, $-(y_1+y_2) + G(y_1) + G(y_2)$, $y_1+y_2 + G(-y_1) + G(-y_2)$ are in W. From the W superlinearity of G it follows that $G(y_1+y_2) - G(y_1) - G(y_2)$ and $G(-y_1-y_2) - G(-y_1) - G(-y_2)$ are in W. Summing these relations with the appropiate ones obtained above we conclude that $-(y_1+y_2) + G(y_1+y_2)$ and $y_1+y_2 + G(-y_1-y_2)$ are in W and therefore y_1+y_2 is in W_0. Hence W_0 is a subspace of Y and by its construction it follows that it is the maximal subspace of W with the above considered properties.

We observe that W_0 is contained in $G^{-1}(W) \cap (-G^{-1}(W))$. Indeed, $y \in W_0$ has the properties that y, $-y$, $-y + G(y)$ and $y+ G(-y)$ are in W. But then summing the appropriate terms we conclude that $G(y)$ and $G(-y)$ are in W, that is, $y \in G^{-1}(W) \cap (-G^{-1}(W))$.

Finally, if y and $-y$ are in $G^{-1}(W)$, then the condition $G^{-1}(W) \subset \subset W$ implies that y, $-y \in W$ and hence $-y + G(y)$ and $y + G(-y)$ are in W. This means that y is in W_0 and we conclude

$$W_0 = G^{-1}(W) \cap (-G^{-1}(W)). \qquad \text{Q.E.D.}$$

Remark. The inclusion $G^{-1}(W) \subset W$ was used only to estabilish that $G^{-1}(W) \cap (-G^{-1}(W))$ is contained in W_0. The converse of this relation follows without it. We are specially interested in considering superlinear operators G with the above propety. They were used in (N1) for "approximation" of the wedge W by cones with nice properties.

2. COROLLARY. In the conditions of Proposition 1, if W is a cone or if $-I + G$ carries no whole linear subspace of W into W, then $G^{-1}(W)$ is a cone.

3. PROPOSITION. Let G be of the form $G = I - S$, where I is the identity operator of Y and S is a W sublinear operator with the property that $S(Y) \subset W$. Then $G^{-1}(W) \subset W$.

Proof. We have $y \in (I-S)^{-1}(W)$ if and only if $y - S(y) \in W$. Q.E.D.

4. COROLLARY. Suppose that G is as in Proposition 3. If y, $-y$ and $-S(-y) \in W$ imply that $y = 0$, it follows that $G^{-1}(W)$ is a cone.

The wedge W in Y is said to be generating if $W - W = Y$.

5. PROPOSITION. If W is a generating wedge, if G is a W superlinear operator acting in Y and having the properties that $W \subset G(Y)$ and $(I-G)(-W) \subset W$, then the wedge $G^{-1}(W)$ is generating too.

Proof. The wedge $G^{-1}(W)$ is generating if for every w in W there exists a u in $G^{-1}(W)$ so as to have $u-w \in G^{-1}(W)$. Indeed, in this case we have for every z in Y the representation $y = w_1 - w_2$ with w_1 and w_2 in W. Let u_1 and u_2 be elements in $G^{-1}(W)$ such that $u_1-w_1 \in G^{-1}(W)$ and $u_2-w_2 \in G^{-1}(W)$. Put $u = u_1+u_2$. Then u, $z_1 = u-w_1 = u_2+u_1-w_1$ and $z_2 = u-w_2 = u_1+u_2-w_2$ are in $G^{-1}(W)$. But we have

$$z_2 - z_1 = (u-w_2) - (u-w_1) = w_1 - w_2 = y,$$

that is, $G^{-1}(W)$ is generating.

It remains to prove the existence for every w in W of a u in $G^{-1}(W)$ with the property that $u-w$ is in $G^{-1}(W)$. For w in W there exists a v_1 in $G^{-1}(W)$ such that $w = G(v_1)$ and there exists a v_2 in $G^{-1}(W)$ such that $(I-G)(-w) = G(v_2)$. Put $u = v_1+v_2$. Since G is W superlinear, we have

$$G(v_1+v_2) - G(v_1) - G(v_2) \in W,$$

that is,

$$G(u) + G(-w) \in W.$$

Using again the W superlinearity of G we get

$$G(u-w) \in W. \qquad \text{Q.E.D.}$$

6. COROLLARY. Let S be a W sublinear operator which carries Y in W. If $G = I - S$ contains W in its range, then G has the properties in Proposition 5, and $G^{-1}(W)$ is generating if W has this property.

The element e in W is said to be a W order unit in Y, if for every y in Y there exists a positive real s such that $se - y$ is in W.

7. PROPOSITION. If e is a W order unit in Y, then every e_1 in $G^{-1}(e)$ will be a $G^{-1}(W)$ order unit in Y.

Proof. Let y be arbitrary in Y. Since e is a W order unit in Y, there exists an $s>0$ with the property that

$$se + G(-y) \in W.$$

Because $e = G(e_1)$ and G is positively homogeneous, we have

$$G(se_1) + G(-y) \in W,$$

which together with the W superlinearity of G imply

$$se_1 - y \in G^{-1}(W). \qquad \text{Q.E.D.}$$

Remark. As we have observed in Introduction, the idea of order transformation by nonlinear operators originated in (N1) was developed for some operators of special feature . Indeed there were considered operators of the form $I - tS$ with t a real and positive

number and S a W sublinear operator, and was required the invertability of this operator for small t-s. This last condition has no importance for this paragraph and some other considerations which follow.

8. PROPOSITION. If the superlinear operator G is of the form $G = S_t = I - tS$, with S a W sublinear operator with the property that $S(Y) \subset W$, and t a positive real, then we have $S_{t_2}^{-1}(W) \subset S_{t_1}^{-1}(W)$ whenever $0 \leq t_1 \leq t_2$. The transformation S_t^{-1} leaves invariant the subwedge in W for which S vanish.

Proof. Indeed, let x be in $S_{t_2}^{-1}(W)$. Then $x - t_2S(x) \in W$. Since $t_2S(x) - t_1S(x) \in W$ we conclude by summing the above relations that $x \in S_{t_1}^{-1}(W)$. If $S(x) = 0$ for some x in W, then $x - tS(x) = x \in W$ and hence $x \in S_t^{-1}(W)$ for every t. Q.E.D.

9. COROLLARY. If S satisfies the conditions in Proposition 8 and if $S_{t_0}^{-1}(W)$ is a generating wedge for some positive t_0, then the same is true for $S_t^{-1}(W)$ for every t with the property $0 \leq t \leq t_0$.

2. Simultaneous transformation of the order and of the topology. We shall suppose in what follows that the real vector space ordered by the wedge W is endowed with a locally convex topology T. Denote the ordered topological vector space with (Y,W,T). The continuity and coercivity conditions imposed in (N1) about the sublinear operator S assure in fact that the order transformed by the Fredholm resolvents of S has its topological properties expressed in the original topology of the space Y. Without these restrictions on S it turns out that the topological properties of the transformed order are expressed in a new topology or eventually in a new convergence structure. It seems that all the considered ideas has their natural place also in spaces with convergence structures,

but we will consider here only topological vector spaces. For obtaining technical facilities we shall impose conditions on the original topology and on the W sublinear operator S to obtain the transformed topology in a simple way.

1. PROPOSITION. Let us assume that (Y,W,T) is an ordered topological vector space with a locally convex topology and that

(a) S is a W sublinear operator carrying the space Y into W;

(b) $B(0)$ is a neighbourhood basis of 0 in Y consisting of closed circled and W full sets;

(c) $S(-x) = S(x)$ for every x in Y.

Then

(i) $S_t^{-1}(W)$, where $S_t = I - tS$ and I is the identity operator of Y, is a wedge contained in W for every non-negative real number t;

(ii) the family of sets $B_S(0) = \{S^{-1}(U) \; : U \text{ in } B(0)\}$ forms a neighbourhood basis of 0 in Y which generates a locally convex vector space topology in Y.

Proof. To verify (i) we observe that S_t is W superlinear for every $t \geq 0$, hence $S_t^{-1}(W)$ is a wedge according Proposition 1.1. We are also in the condition of Proposition 1.3 with tS in place of S in this proposition. Hence $S_t^{-1}(W) \subset W$ for every $t \geq 0$.

Let us check (ii). Consider U in $B(0)$. Let v_1 and v_2 be in $S^{-1}(U)$. Then $sS(v_1) + (1-s)S(v_2) \in U$ for every s in $[0,1]$, because U is convex. Since S is W sublinear, we have $sS(v_1) + (1-s)S(v_2) - S(sv_1 + (1-s)v_2) \in W$ for every s in $[0,1]$. Now, since U is a W full set, the above two relations show that $S(sv_1 + (1-s)v_2) \in U$. That is, $S^{-1}(U)$ is convex.

The set $S^{-1}(U)$ is radial. Indeed, for every v in Y there exists a positive s such that $sS(v) = S(sv) \in U$, that is, $sv \in S^{-1}(U)$.

The set $S^{-1}(U)$ is circled. Indeed, for every s with the property $|s| \leq 1$ we have from $v \in S^{-1}(U)$, $sS(v) \in U$ since U is circled. For $s \geq 0$ this implies $S(sv) \in U$ and hence $sv \in S^{-1}(U)$. For $s < 0$ we have from the condition (c), $-v \in S^{-1}(U)$ and since $sv = |s|(-v) \in$ $\in S^{-1}(U)$ it follows also $sv \in S^{-1}(U)$.

Let U be arbitrary in $B(0)$ and let V in $B(0)$ with the property $V + V \subset U$. We shall show that

$$S^{-1}(V) + S^{-1}(V) \subset S^{-1}(U).$$

For v_1 and v_2 in $S^{-1}(V)$ we have $S(v_1) + S(v_2) - S(v_1+v_2) \in W$ from the W sublinearity of S. From this, the fact that $S(v_1+v_2) \in W$ according the condition (a), and the fact that $S(v_1) + S(v_2) \in$ $\in V + V \subset U$ by hypothesis, we have that $S(v_1+v_2)$ is in U since U is W full. Hence v_1+v_2 is in $S^{-1}(U)$. Q.E.D.

The vector space Y endowed with the order relation induced by the wedge $S_t^{-1}(W)$ and the topology defined by $B_S(0)$ for S satisfying the conditions in Proposition 1 will be denoted by $(Y, S_t^{-1}(W), S^{-1}(T))$.

2. Remarks. (i) We observe that the condition c) is imposed for $S^{-1}(U)$ be ready circled if U is circled. Obviously it can be omitted and supplied with an extra construction, but it will be used later in Proposition 4.1.

(ii) If we consider the projective topology on Y defined by $S : Y \to Y$, then in general it isn't invariant with respect to the translations. The topology $S^{-1}(T)$ on Y is in fact the coarses locally convex vector space topology on Y with respect to which S is continuous at 0.

3. PROPOSITION. Let W and S be as in Proposition 1. If S is T continuous at 0 then the ordered topological vector space $(Y, S_1^{-1}(W), S^{-1}(T))$ has the boundendess property.

Proof. Indeed, every $S_t^{-1}(W)$ order bounded set in Y is W order bounded. Since (Y, W, T) is supposed W locally full it follows that every W order bounded set is T bounded. Hence it is also $S^{-1}(T)$ bounded, since the last topology is coarser as T. Q.E.D.

4. COROLLARY. In the conditions of Proposition 3, if Y is complete and metrizable with respect to the topology $S^{-1}(T)$, if $S_t^{-1}(W)$ is an $S^{-1}(T)$ closed wedge, then the space $(Y, S_t^{-1}(W), S^{-1}(T))$ is normal.

Proof. By Proposition 3 the wedge $S_t^{-1}(W)$ has the boundedness property with respect to the topology $S^{-1}(T)$. Since this topology is Hausdorff, $S_t^{-1}(W)$ must be in fact a cone. It suffices now to use Proposition 2.4.6 in (BO) to conclude the result. Q.E.D.

5. PROPOSITION. If K is a normal cone in Y and the K sublinear operator S satisfies the conditions in Proposition 1 with respect to K in place of W, if S is K monotone on K, then the ordered locally convex space $(Y, S_t^{-1}(K), S^{-1}(T))$ is normal.

Proof. Let (x_i) be a net in $S_t^{-1}(K)$ which converges to 0 with respect to the $S^{-1}(T)$ topology. If (y_i) is a net in $S_t^{-1}(K)$ with the property that $x_i-y_i \in S_t^{-1}(K)$, then (y_i) will converge to 0. Indeed, since $S_t^{-1}(K) \subset K$, it follows that x_i, y_i and x_i-y_i are all in K for every i. Since S has its range in K, we have $S(x_i-y_i) \in K$ for every i. From the K monotonity of S we have

$$S(x_i) - S(y_i) \in K.$$

Since (x_i) tends to 0 in the $S^{-1}(T)$ topology, it follows that $(S(x_i))$ tends to 0 in the T topology. But K is a normal cone with respect to this topology, hence $(S(y_i))$ tends to 0 with respect T according the above obtained relation and a well known normality criterion (see e.g. (P), Proposition II.1.3). But then (y_i) tends to 0 in the

$S^{-1}(T)$ topology and applying again the above cited criterion we conclude that $S_t^{-1}(K)$ is a normal cone with respect to the $S^{-1}(T)$ topology.

Q.E.D.

Remark. We have observed at Remark 2 (ii) that the projective topology defined by $S : Y \to Y$ on Y in general is not invariant with respect to the translations, hence we cannot say that $S^{-1}(T)$ is this projective topology. Now, if K is normal with respect to the $S^{-1}(T)$ topology then since S is convex with respect to K and is continuous at 0, it will be continuous everywhere in Y (see e.g. (B)). This means that the projective topology on Y defined by S is coarser as $S^{-1}(T)$. If the projective topology defined by S is translation invariant in Y, then it is equivalent with $S^{-1}(T)$.

6. PROPOSITION. In the conditions of Proposition 1 and $t>0$, the set $S_t^{-1}(W) \cap (-S_t^{-1}(W))$ is the maximal subspace W_0 of W with the property that $-S(W_0) \subset W$. Hence if W is a cone or if S carries no whole subspace of W in W, then $S_t^{-1}(W)$ is a cone for every positive t. If W is contained in the image of S_t, then if W is generator, $S_t^{-1}(W)$ will be generator too.

Proof. We apply the results in Proposition 1.3 and in the Corollary 1.6 for tS in place of S. Q.E.D.

Remark. In general it can happen that $S_t^{-1}(W)$ reduces to $\{0\}$. Conditions which assure that this wedge is rich enough are similar to that in the above proposition concerning the range of S_t. In some special conditions on Y and on S we have that S_t is a bijection (see (N1) Corollary 1). From our point of view is important that for decreasing $t>0$ the cone $S_t^{-1}(W)$ increases (Proposition 1.8). For this in some cases in what follows we shall postulate a good comportment from this point of view of the family of operators S_t for t in a neighbourhood of 0.

3. *Transformation of wedges in quasi regular, respectively in fully quasi regular cones.* We have considered in the preceding paragraph the transformation of the locally W full ordered topological vector space (Y, W, T) into the ordered topological vector space (E, $S_t^{-1}(W)$, $S^{-1}(T)$) with $S_t = I - tS$, where S is a W sublinear operator satisfying the conditions in Proposition 2.1. Let us consider in this paragraph the problem of finding sufficent conditions on S which assure to $S_t^{-1}(W)$ be quasi regular and respectively quasi fully regular in the topology $S^{-1}(T)$. A first result in this direction is contained in

1. PROPOSITION. *Let* W *and* S *have the properties in Proposition* 2.1. *If* co S(W) *is a quasi* W *bound regular subcone of* W *with respect to the topology* T, *then* $S_t^{-1}(W)$ *is a quasi regular cone with respect to the topology* $S^{-1}(T)$ *for every positive* t.

Proof. According to Lemma 1 in (N2) it suffices to prove that $S_t^{-1}(W)$ is sequentially quasi regular, that is, that every sequence (x_n) in $S_t^{-1}(W)$ which is $S_t^{-1}(W)$ decreasing is a Cauchy sequence with respect to $S^{-1}(T)$. Let us assume the contrary : The $S_t^{-1}(W)$ decreasing sequence (x_n) in $S_t^{-1}(W)$ is not Cauchy with respect to $S^{-1}(T)$. Since (x_n) is $S_t^{-1}(W)$ decreasing we have for every natural n and p that x_n-x_{n+p} is in $S_t^{-1}(W)$, which is equivalent with

(1) $$x_n-x_{n+p}-tS(x_n-x_{n+p}) \in W.$$

Since (x_n) is not Cauchy with respect to the topology $S^{-1}(T)$, it follows that there exists a T neighbourhood U of 0 in Y so as to have, passing if necessary to a subsequence of (x_n), the relation

$$x_n - x_{n+1} \notin S^{-1}(U)$$

that is,

(2) $$S(x_n-x_{n+1}) \notin U$$

for each natural n. From the relation (1) wih p = 1 we conclude

$$x_n - x_{n+1} - tS(x_n-x_{n+1}) \in W,$$

wherefrom by summation from n = 1 to n = m we get

$$x_1 - x_{m+1} - t(S(x_1-x_2) + \dots + S(x_m-x_{m+1})) \in W.$$

It follows that the sums

$$\sum_{i=1}^{m} S(x_i-x_{i+1})$$

are W order bounded by $\frac{1}{t}x_1$. But we have (2), and this, via Criterion 5 in (N2) contradicts the hypothesis that the subcone coS(W) in W is quasi W bound regular and proves the proposition. Q.E.D.

In order to prove a similar result for the fully regular case, we have to impose further restrictions about the topology $S^{-1}(T)$. We have the

2. PROPOSITION. *Let* W *and* S *be as in Proposition* 2.1. *If the topology* $S^{-1}(T)$ *is finer as* T *and if the cone* coS(W) *is fully quasi* W *regular with respect to the topology* T, *then* $S_t^{-1}(W)$ *is fully quasi regular cone with respect to the topology* $S^{-1}(T)$ *for every positive number* t.

Proof. We use the method of contradiction. Assume that (x_n) is $S_t^{-1}(W)$ increasing and $S^{-1}(T)$ bounded sequence which is not Cauchy in the topology $S^{-1}(T)$. (We can consider only sequences according Lemma 1 in (N2).) That is, we have the relation

(3) $$x_{n+1}-x_n - tS(x_{n+1}-x_n) \in W$$

for every n, and we can get a T neighbourhood U of 0 such that, passingif necessary to a subsequence of (x_n),

(4) $$S(x_{n+1}-x_n) \notin U$$

for each n. Summing (3) from n=1 to n=m we have

(5) $$x_{n+1} - x_1 - t(S(x_2 - x_1) + \dots + S(x_{n+1} - x_n)) \in W.$$

Since $S^{-1}(T)$ is finer as T and (x_n) is $S^{-1}(T)$ bounded, it is also T bounded. The relation (5) together with the hypothesis that W is T locally full imply that the sums

(6) $$\sum_{i=1}^{n} S(x_{i+1} - x_i)$$

are T bounded, which by (4) via Criterion 5 in (N2) contradicts the assumption that coS(W) is fully quasi W regular with respect to T.

Q.E.D.

The idea of the proof of Proposition 2 can also be used in proving a similar result but with more implicite conditions. We have

3. PROPOSITION. *Let* W *and* S *be as in Proposition* 2.1. *If all the* $S^{-1}(T)$ *bounded subsets in* $S_t^{-1}(W)$ *are* W *order bounded and if the cone* coS(W) *is fully quasi* W *regular with respect to the topology* T, *then* $S_t^{-1}(W)$ *is a fully quasi regular cone with respect to the topology* $S^{-1}(T)$ *for every positive* t.

Proof. The proof runs as that of Proposition 2 with the difference that the formula (5) is interpreted as follows : since (x_n) is $S^{-1}(T)$ bounded in $S_t^{-1}(W)$, it is W order bounded by assumption and hence T bounded because T is W locally full. Now this implies that the sums (6) are T bounded. Q.E.D.

Other sufficient conditions for the quasi regularity, respectively fully quasi regularity of $S_t^{-1}(W)$ in the topology $S^{-1}(T)$ can be obtained when the existence of some functionals with special propeties is postulated. Let us consider some definitions (see (K)). The functional f (defined on T or W) is called W *positive* if $f(x) \geqslant 0$ for every x in W. The W positive functional f is called *uniformly* W *positive* if for every neighbourhood U of 0 there exists a positive number d so as to have $f(x) \geqslant d$ whenever $x \in W \setminus U$. If from the

condition that $x_n \in W \setminus U$ for some neighbourhood U of 0 and every n, it follows that

$$\lim_{n \to \infty} f\left(\sum_{n=1}^{n} x_n\right) = \infty,$$

then f is called *strictly* W *increasing*. The functional f is called W *monotone* if $f(u) \leqslant f(v)$ as soon as $v-u \in W$.

The functional f defined on T or W is called *superlinear* if it is superlinear as an operator with values in R endowed with the usual order.

4. PROPOSITION. *Assume* W *and* S *are as in Proposition* 2.1. *If there exists a superlinear functional* f *defined on* W *which is bounded on the sets which are bounded in the* $S^{-1}(T)$ *topology and is* W *uniformly positive with respect to the topology* T, *then* $S_t^{-1}(W)$ *is a fully quasi regular cone with respect to the topology* $S^{-1}(T)$ *for every positive* t.

Proof. According Lemma 1 in (N2) if $S_t^{-1}(W)$ does not have the property in the proposition, then there exists a sequence (x_n) in $S_t^{-1}(W)$ which is $S_t^{-1}(W)$ increasing and bounded in the $S^{-1}(T)$ topology and which is not Cauchy with respect to this topology. This means that there exists a neighbourhood U of 0 in the topology T such that, passing if necessary to a subsequence of (x_n) , it holds

$$x_{n+1} - x_n \notin S^{-1}(U),$$

that is,

(7) $$S(x_{n+1} - x_n) \notin U,$$

for every n. Since (x_n) is $S_t^{-1}(W)$ increasing, we have

(8) $$x_{n+1} - x_n - tS(x_{n+1} - x_n) \in W,$$

for every n. We apply the functional f to the element (8) and using its superlinearity and W positivity we get

$$f(x_{n+1}-x_n) - tfS(x_{n+1}-x_n) \geqslant f(x_{n+1}-x_n - tS(x_{n+1}-x_n)) \geqslant 0.$$

This relation together with (7) and the W uniform positivity of f with respect to the topology T yield

$$f(x_{n+1}-x_n) \geqslant tfS(x_{n+1}-x_n) \geqslant td,$$

with d an appropriate positive number which appears in the definition of the W uniform positivity of f . Let us use now the method in the proof of Theorem 1.10 in (K) to deduce that

$$f(x_m) = f(x_1 + \sum_{n=1}^{m-1} (x_{n+1}-x_n)) \geqslant f(x_1) + \sum_{n=1}^{m-1} f(x_{n+1}-x_n) \geqslant$$

$$\geqslant f(x_1) + (m-1)td.$$

Hence $f(x_m)$ tends to infinity with m contradicting the assumption that (x_n) is bounded in the topology $S^{-1}(T)$ and that f is bounded on sets which are bounded with respect to this topology. This completes the proof. Q.E.D.

5. PROPOSITION. <u>Let</u> W <u>and</u> S <u>be as in Proposition</u> 2.1. <u>If there exists a</u> W <u>positive superlinear functional</u> f <u>defined on</u> W, <u>if</u> f <u>is bounded on</u> W <u>order bounded sets and if it is strictly increasing on</u> S(W), <u>then</u> $S_t^{-1}(W)$ <u>is a quasi regular cone with respect to the topology</u> $S^{-1}(T)$ <u>for every positive</u> t.

<u>Proof</u>. Assume that $S_t^{-1}(W)$ is not sequentially quasi regular with respect to the topology $S^{-1}(T)$. That is, there exists the sequence (x_n) which is $S_t^{-1}(W)$ increasing and $S_t^{-1}(W)$ order bounded in $S_t^{-1}(W)$ which is not Cauchy. Passing if necessary to a subsequence of (x_n), we can suppose that there exists a neighbourhood U of 0 in the topology T, such that (7) is fulfilled for every n. We have also (8) and summing this relation from n=1 to n=m, it follows that

$$x_{m+1} - x_1 - t(S(x_2-x_1) + \dots + S(x_{m+1}-x_m)) \in W.$$

Apply f to the element in this relation and use the W positivity and superlinearity of its to conclude

$$f(x_{m+1}-x_1) \geqslant tf(S(x_2-x_1) + \dots + S(x_{m+1}-x_m)).$$

The condition (7) and the fact that f is strictly W increasing imply that the left hand side in the above inequality tends to infinity with m. This contradicts the assumption that (x_n) is $S_t^{-1}(W)$ order bounded and that f is bounded on $S_t^{-1}(W)$ order bounded sets. We use Lemma 1 in (N2) to conclude the validity of the propsition. Q.E.D.

<u>Remark.</u> The conditions in this proposition are rather restrictive. Indeed, if the space Y is Hausdorff, then W must be a cone in order to admit a positively homogeneous functional f with the proprietes assumed.

4. <u>Consequences and comments</u>. Let X and Y be two topological vector spaces and denote by C(X,Y) the vector space of continuous operators from X to Y. If Y is endowed with a locally convex topology T and if $\mathcal{M}$ is a family of subsets in X directed with respect to the set theoretic inclusion, then it can be introduced a locally convex topology $\mathcal{T}$ in this space by considering as neighbourhood basis of an element F the sets

$$\mathcal{U}(F,M,V) = \{G \in C(X,Y) : (F-G)(M) \subset V\},$$

where M runs over $\mathcal{M}$ and V runs over a neighbourhood basis of 0 in Y (see e.g. (Sch) III.3). This topology is called <u>the topology of uniform convergence on the members of</u> $\mathcal{M}$. It is obviously translation invariant and converts the additive group of C(X,Y) into a continuous group.

We are interested in getting appropriate conditions on a subset of C(X,Y) in order to assure that for every member F of this family the product tF converges to the zero operator whenever the real number t converges to 0. It is immediate that the necessary and sufficent condition for this is to F(M) be bounded for every member M of $\mathcal{M}$. From this it follows also that the maximal subset H(X,Y) in C(X,Y) having this property is a vector subspace. It turns out that the condition on elements of H(X,Y) is exactly the same which assures to H(X,Y) be a topological vector space in the topology induced from $\mathcal{T}$ (see (Sch) III.3.1).

Let us suppose now that $\mathcal{M}$ is the family of the bounded sets with respect to the topology of X.

From our point of view is of a special importance the following result :

1. PROPOSITION. If X is a bornological space and if Y is a W locally full ordered vector space with a locally convex topology, then a W sublinear operator S from X to Y with the property that $S(x) = S(-x)$ for every x in X, is a member of H(X,Y) if and only if it is continuous at 0.

Proof. We have to prove in fact that in the conditions of the proposition a W sublinear operator $S : X \to Y$ is continuous at 0 if and only if it maps bounded subsets in bounded ones.

Let S be continuous and let B be a bounded set in X. Assume that S(B) is not bounded. Then there exists a neighbourhood V of 0 in Y and a sequence (x_n) in B such that $S(x_n) \notin nV$ for every n. Since S is positively homogeneous we have $S(n^{-1}x_n) \notin V$ for every n. But $n^{-1}x_n$ converges to 0 according to the criterion I. 5.3 in (Sch), and the obtained relation contradicts the continuity of the operator S at 0.

Suppose now that S(B) is bounded for every bounded set B in X and prove that S is continuous at 0. Let V be an arbitrary convex and circled neighbourhood of 0 in Y. It absorbs S(B) for every

bounded set B in X. Hence $S^{-1}(V)$ absorbs every bounded set in X. The set $S^{-1}(V)$ is convex because Y is W locally full and S is W sublinear. It is also circled since $S(x) = S(-x)$ for every x.(See the proof of Proposition 2.1.) Thus $S^{-1}(V)$ is a neighbourhood of 0 in X. This shows that S is continuous at 0. Q.E.D.

Remark. If we suppose that Y is Hausdorff, then W must be a normal cone. Then in the proposition above we can consider in place of continuity of S at 0 its continuity on all the space X. Indeed, in this case from the continuity at 0 of S it follows its continuity throughout (see (BO) Corollary 4.2.3 or (B) Corollary 2.4 b)).

In the paragraphs 2 and 3 we have considered the family $S_t = I - tS$ of operators with I the identity operator, S a W sublinear operator acting in the W locally full ordered topological vector space X, and t was considered a positive real. According to Proposition 1 we can assert that if X is bornological and if S is continuous then S_t tends to the identity map I of X with respect to the topology of the uniform convergence on T bounded sets in X. Hence all the results in the paragraph 3 can be interpreted in the sense that the cone W can be "approximated" from the inside by cones $S_t^{-1}(W)$ which are regular respectively fully regular in the topology $S^{-1}(T)$. In this general setting the term approximated is justified only by the fact that $S_t^{-1}(W)$ for $t>0$ are counter images of W by a net of operators converging in the above sense to the identity operator. This formulation is rather rough since : (a) the topologies are different and (b) the approximation is not by images of W by a net of operators converging to I, but by counter images of a such net.

In (N1) we have imposed special contraints on S to assure that the considered topologies coincide and in order to S_t have for each t sufficently near to 0 continuous inverse and to S_t^{-1} converge to I when t tends to 0. Following the cited paper we consider the ordered topological vector space (Y,W,T) with T a locally convex topology and shall say that the operator S acting in Y is W correct if it satisfies the following conditions :

(a) S is a continuous W sublinear operator with the property that $S(Y) \subset W$;

(b) there exists a positive ε such that $S_t = I - tS$ is surjective and one to one for every t in $[0, \varepsilon)$;

(c) the inverse S_t^{-1} of S_t is for every t in $[0, \varepsilon)$ continuous ;

(d) the family S_t^{-1}, t in $[0, \varepsilon)$ depends continuously on t if we consider the set of continuous operators acting in Y endowed with the topology of uniform convergence on bounded sets.

For Y a normed space there were given various examples of W correct operators in (N1). We can use reasonings similar with that in the proofs of Lemmas 1 and 2 of the cited paper in order to deliver sufficent conditions for correctness. To this end we have to use in place of Banach's contraction principle a generalisation of its similar to those in (M).

We shall say that the W sublinear operator S acting in Y is coercive if the filter basis $\{S^{-1}(U) : U \in B(0)\}$ with B(0) a neighbourhood basis of 0 in Y is finer as the neighbourhood filter of 0.

An important consequence of the construction in (N1) was that every wedge W in a normed space Y with the property that its closure $\overline{W}$ is not a subspace of Y can be approximated by quasi regular and quasi fully regular subcones $S_t^{-1}(W)$ of W, where S_t^{-1} is a set of continuous operators tending uniformly on norm bounded sets to the identity operator when t tends to 0. Here $S_t = I - tS$ with S a W correct operator with some special property something similar with

one of the conditions comprised in the propositions of the paragraph 3.

We observe that the W locally fullness of the ordered topological vector space Y was postulated in the paragraphs 2 and 3 in order to define the topology $S^{-1}(T)$ and has in this context a technical character. If it is supposed that S is W correct and coercive then the topologies T and $S^{-1}(T)$ coincide. In fact we can avoid to introduce formally the topology $S^{-1}(T)$ and hence we can obtain results similar to those in the paragraph 3 directly. As an illustration we shall consider the following analogous of Proposition 3.1 :

2. PROPOSITION. Let Y be an ordered topological vector space with the positive wedge W. If S is a W correct coercive operator acting in Y with coS(W) a quasi W bound regular subcone of W, then $S_t^{-1}(W)$ is a quasi regular cone for every positive t.

Proof. We follow the idea of the proof of Proposition 3.1 admitting that $S_t^{-1}(W)$ is not a sequentially quasi regular cone and supposing (x_n) to be an $S_t^{-1}(W)$ decreasing sequence in this cone which is not Cauchy. Since S is coercive we can suppose, passing if necessary to a subsequence, that there exists a neighbourhood U of 0 in Y so as to have

$$x_n - x_{n+1} \notin S^{-1}(U)$$

that is,

$$S(x_n-x_{n+1}) \notin U \tag{1}$$

for every n. On the other hand, from the condition that (x_n) is $S_t^{-1}(W)$ decreasing we have

$$x_1 - x_{n+1} - t(S(x_1-x_2) + \dots + S(x_n-x_{n+1})) \in W.$$

It follows that the sums

$$\sum_{i=1}^{n} S(x_i-x_{i+1})$$

are W order bounded by $t^{-1}x_1$. This together with (1) and the

Criterion 5 in (N2) contradict the hypothesis that coS(W) is a quasi W bound regular cone. Q.E.D.

We shall show that a construction like that in (N1) which permits the approximation of an arbitrary wedge with W not a subspace by quasi regular cones characterizes the normable spaces. More precisely, the existence of W correct operators which permit the construction comprised in Proposition 2 is limited. We have the

3. PROPOSITION. Let Y be a topological vector space ordered by the wedge W, W≠Y. If int W ≠ ∅, then there exists a W correct coercive operator S on Y with co S(W) a quasi W bound regular subcone if and only if the topology of Y is normable.

We use in the proof the following

4. LEMMA. Every sequentially quasi regular ordered topological vector space Y has the boundendess property.

Proof. Assume the contrary : there exists an order bounded set M in Y which is not topologically bounded. We can suppose $0 \leq M \leq a$. Let U be a neighbourhood of 0 in Y which does not absorb M. Then for every n there exists x_n in M such that $x_n \notin 2^n U$. Consider the sequence $y_m = \sum_{k=1}^{m} 2^{-k} x_k$, $m \in N$. This sequence is increasing and bounded by a. Since Y is sequentially quasi regular, (y_m) must be a Cauchy sequence, which is impossible since $y_{m+1} - y_m = 2^{-(m+1)} x_{m+1}$ is outside U for every m. Q.E.D.

Proof of Proposition 3. If there exists an operator S with the stated properties, then $S_t^{-1}(W)$ will be a quasi regular cone according Proposition 2. The cone $S_t^{-1}(W)$ has nonempty interior since S is correct. It has the boundendess property by Lemma 4. But then according Proposition 2.4.7 in (BO), the topology of Y is normable.

Conversely, if Y is normable, then a W sublinear operator with the properties in the proposition can be constructed in the way was

done in (N1) Proposition 9. Q.E.D.

We observe that in Lemma 4 and Proposition 3 was implicitely proved also the

5. PROPOSITION. The topological vector space Y admits a sequentially quasi regular cone with nonempty interior if and only if it is normable.

Even in the presence of restrictions on the feature of cones which are approximable by quasi regular cones in topological vector spaces, the Proposition 2 gives a method of how an approximation of this kind can be done. If the space Y is bornological then we can assert in plus that the approximation can be done by cones of form $S_t^{-1}(W)$, where S_t is a net converging uniformly to I on bounded sets when t tends to 0. If S is correct then the net S_t^{-1} of operators has itself the last property.

R E F E R E N C E S

(B) BORWEIN, J.M., Continuity and differentiability properties of convex operators,
Proc. London Math. Soc. 44 (1982), 420-444.

(BO) BRECKNER, W.W., ORBÁN, G., Continuity Properties of Rationally s- convex Mappings with Values in an Ordered Topological Linear Space
Univ. "Babeş-Bolyai", Faculty of Mathematics, Cluj-Napoca, 1978.

(EL) EISENFELD, J., LAKSHMIKANTHAM, V., Fixed point theorems through abstract cones,
J. Math. Anal. Appl. 52 (1975), 25-35.

(I) ISAC, G. Cônes localement bornés et cônes complètement réguliers. Application à l'analyse non-linéaire, Department de Mathématiques Universite de Sherbrooke, Sherbrooke, Quebec, Canada 1980.

(K) KRASNOSEL'SKII, M.A., Positive Solutions of the Operatorial Equations (Russian) Fizmatgiz, Moskow, 1962.

(M) MARINESCU, G., Théorèmes des contractions dans les espaces localement convexes, Rev. Roum. Math. Pures Appl. 14 (1969), 1535-1538.

(Mc) MCARTHUR, C.W., Convergence of monotone nets in ordered topological vector spaces, Studia Math. 34 (1970), 1-16.

(N1) NÉMETH, A.B., Nonlinear operators that transform a wedge, Studia Univ. "Babeş-Bolyai", Math. 25 Nr.4 (1980) 55-69.

(N2) " Summation criteria for regular cones with applications, "Babeş-Bolyai" Univ. Faculty of Math. Research Seminaries, Preprint Nr. 4, 1981, pp. 99-124.

(N3) " Normal cone valued metrics and nonconvex vector minimization principles, "Babeş-Bolyai" Univ. Faculty of Math. Research Seminaries, Preprint nr. 1, 1983, pp. 117-153.

(P) PERESSINI, A.L., Ordered Topological Vector Spaces, Harper & Row, New York - Evanston - London, 1967.

(Sch) SCHAEFER, H.H., Topological Vector Spaces, Macmillan, New York, 1966.

Seminar of Functional Analysis and Numerical Methods, Preprint Nr. 1, 1984, pp. 159 - 168.

STRONGLY PROXIMINAL SETS IN ABSTRACT SPACES

by

IOAN ŞERB

Let X be a Hausdorff locally convex space with the topology generated by a family $(p_\alpha)_{\alpha \in A}$ of semi-norms on X and let M be a non-void subset of X. By the metric projection on M with respect to the family $(p_\alpha)_{\alpha \in A}$ we understand the application $P_M : X \longrightarrow 2^M$ defined by

$$(1) \quad P_M(x) = \left\{ m \in M : p_\alpha(x - m) = d_\alpha(x, M) = \inf_{m' \in M} p_\alpha(x - m'),\ \forall \alpha \in A \right\}$$

(see [6]). If X is a normed space then P_M is the usual metric projection on M. In this case $d(x,M) = \inf \{ \|x - m'\| : m' \in M \}$. If (X, ρ) is a metric space and M is a non-void subset of X the metric projection $P_M : X \longrightarrow 2^M$ is defined similarly by

$$(2) \qquad P_M(x) = \left\{ m \in M : \rho(x,m) = \inf_{m' \in M} \rho(x,m') \right\}.$$

In the definitions given below X denotes a locally convex or a metric space, M is a subset of X and P_M is the corresponding metric projection, given by (1) or (2), respectively.

If for every $x \in X \setminus M$, card $P_M(x) \geq 2$, $2 \leq$ card $P_M(x) < \aleph$ or card $P_M(x) = \aleph_0$, then the metric projection P_M is called totally multivalued, finitely multivalued or countably multivalued, respectively (see [11]).

The set M is called proximinal if $P_M(x) \neq \emptyset$ for all $x \in X$.