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10. MUSTĂȚA, C., Asupra unicității prelungirii funcțiilor Hölder reale , Seminarul itinerant de Ecuații Funcționale, Aproximare și Convexitate, Cluj-Napoca, 1980 .

11. MINTY, G.J., On the extension of Lipschitz, Lipschitz-Hölder continuous, and monotone functions , Bull. Amer. Math. Soc. 76 (1970), 334 - 339 .

12. SCHÖNBECK, S.O., Extension of nonlinear contractions , Bull. Amer. Math . Soc. 72 (1966), 99 - 101 .

13. SCHÖNBECK, S.O., On the extension of Lipschitz maps , Arkiv för Mathematik 7 (1967 - 1969), 201 - 209.

14. McSHANE, E.J., Extension of range of functions , Bull. Amer. Math. Soc., 40 (1934), 837 - 842 .

15. VALENTINE, F.A., A Lipschitz condition preserving extension for a vector function , Amer. J. Math. 67 (1945) , 83 - 93 .

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ON SOME UNIVERSAL SUBDIFFERENTIABILITY PROPERTIES OF ORDERED VECTOR SPACES

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Introduction and definitions. Besides many studies of convex operators with values in order complete vector lattices (see for example (V), (L), (IL), (AK), (KU1), (Z3), (K), (B3) and (P)), ZOWE (Z1), (Z2), FEL'DMAN (F), and recently BORWEIN (B1), (B2) and the author (N1), (N2) have considered problems on subdifferentiability of convex operators with values in more general ordered vector spaces. The main result in (N2), which constitutes the complete characterization of ordered vector spaces admitting strictly monotone functionals in order to every convex operator with values in them have pleasant subdifferentiability properties, gives the idea to consider other less restrictive conditions on subdifferentiability from this point of view. More precisely this approach is the following : to consider ordered vector spaces with some universality property formulated in terms of subdifferentiability and then to characterize these spaces in other terms of the theory of ordered vector spaces, as well as to establish interrelations of various subdifferentiability like properties. Our paper constitutes an attempt in this direction. A program of this kind is very general and since the subdifferentiability theory is not very specific for the type of problems the ordered vector space theory deal with, it is not likely to get characterisation in each case in context of classical terms. This happened and is surprisingly simple in the result cited above of (N2). However, some interesting relations can be reached. Such for instance we shall show that the spaces with the subgradient property considered by BORWEIN in (B2) are quite the spaces with the exact Hahn-Banach approximation property having an essential role in the fundamental result of FEL'DMAN (F). The machinery permitting the proof of this equivalence does not be so efficient in the case of V-subdifferentiability considered by THIBAULT (T) and also for other weakener subdifferentiability like notions which will be also considered in the paper.

After introducing some basic notions we shall deal with various aspects of the above outlined problematic in different paragraphs. Hence notions which are specific for a single paragraph only will be introduced at that paragraph.

The subset W of the real vector space Y is called a wedge if it has the properties : (i) $u, v \in W$ imply $u + v \in W$; (ii) if $t \in R_+$, and if $u \in W$ then $tu \in W$. A wedge K is called cone if it satisfies in plus the condition (iii) $K \cap (-K) = \{0\}$. A cone K induces in Y a reflexive, transitive and antisymmetric order relation if we put $u \leq v$ whenever $v - u \in K$. This relation is translation invariant and invariant with respect to multiplication with scalars in R_+ . Conversely, if an order relation in Y has all the above properties, then the set $\{u \in Y : u \geq 0\}$ will be a cone inducing the original order relation in Y .

An ordered vector space is by definition a real vector space

Y endowed with an order relation induced by a cone. The cone inducing the order relation in Y is called the positive cone of the ordered vector space Y . We shall refer in our terminology concerning ordered vector spaces interchangably to the ordered vector space or to its positive cone.

A subset M in the ordered vector space Y is called order bounded if it is a subset of an order interval, that is, a subset of a set defined as $[u,v] = \{z \in Y : u \leq z \leq v\}$ with some u and v in Y .

An ordered vector space Y or equally its positive cone K is called Archimedian when the relation $b \leq ta$ (or equivalently $tb \geq a$) for every $t > 0$ and some a in K implies $b \leq 0$.

Let M be a subset of the ordered vector space Y . The set

$$\text{wd } M = \{t_1u_1 + \dots + t_nu_n : u_i \in M,\ t_i \in R_+,\ n \in N\}$$

is called the wedge engendered by M .

Let us suppose that Y is an ordered vector space endowed with a vector space topology. Then Y will be called an ordered normal vector space if it possesses a neighbourhood basis of the origin consisting of full sets, i.e., sets V with the property that $u, v \in V$ imply $[u,v] \subset V$.

The core (or algebraic interior) of the subset M in the vector space X is by definition the set of points u in M such that for every h in X there exists a $t_o > 0$ such that $u + th \in M$ whenever $0 \leq t < t_o$.

For X and Y real vector spaces we shall denote by $L(X,Y)$ the set of linear operators from X to Y.

Let Y be an ordered vector space, K its positive cone. We adjoin to Y and K an abstract maximal element, infinity, denoted by ∞, and shall denote the new objects by $Y^\bullet$ and $K^\bullet$ respectively. Infinity satisfies the relations :

$$t\cdot\infty = \infty \text{ if } t > 0 \;;\; 0\cdot\infty = 0 \;;\; u + \infty = \infty$$
$$\text{and } u \leq \infty \text{ for each } u \text{ in } Y \;;\; \infty - K = Y^\bullet .$$

The operator F from the vector space X to $Y^\bullet$ is said to be <u>convex</u> if

$$F(tu + (1-t)v) \leq tF(u) + (1-t)F(v)$$

whenever u and v lie in X and t is in $[0,1]$.

We shall say that $F : X \to Y^\bullet$ is <u>sublinear</u> if it is convex and <u>positively homogeneous</u>, i.e., if $F(tx) = tF(x)$ for each x in X and each t in R_+ . A sublinear operator can be also characterised by positive homogeneity and <u>subadditivity</u> that is, with the condition to have $F(u + v) \leq F(u) + F(v)$ for all u and v in X .

If a convex operator G is only defined on a convex set M of X then it can be extended using the element ∞ to a convex operator $F : X \to Y^\bullet$ defined by putting $F(x) = G(x)$ if x is in M and $F(x) = \infty$ if $x \notin M$. Hence one can consider only operators defined on X with values in $Y^\bullet$.

The <u>domain</u> of an operator $F : X \to Y^\bullet$ is by definition the maximal set of elements x in X with the property that $F(x) \in Y$. It is easy to check that the domain of a convex operator is a convex set, while the domain of a sublinear one is a wedge.

I. <u>Auxiliary result : extending of a convex operator to a sublinear one.</u> The aim of the present paragraph is to give a constructive result often used next in the proofs. The idea of the construction of a sublinear operator which agree with a given convex operator on a subset is the following : to imbed the space on which the convex operator is defined into a linear manifold of codimension one of another vector space and then to extend it to the whole space so as to obtain a sublinear operator. More precisely, we shall prove the assertion :

1°. <u>Let us consider the convex operator</u> $F : X \to Y^\bullet$ <u>and let we define the operator</u> $P : R \times X \to Y^\bullet$ <u>by putting</u>

$$P(t,x) = \begin{cases} tF(t^{-1}x) & \text{if } t > 0 , \\ 0 & \text{if } t = 0 \text{ and } x = 0, \\ \infty & \text{if } t = 0 \text{ and } x \neq 0, \\ & \text{or if } t < 0 . \end{cases}$$

<u>Then</u> P <u>is sublinear and</u> $\operatorname{dom} P = \operatorname{wd}(\{1\} \times \operatorname{dom} F)$.

◁ The operator P is obviously positively homogeneous. We have to show only that it is subadditive. Let (r,u) and (s,v) be arbitrary elements in $R \times X$. Suppose first that $r > 0$ and $s > 0$. Then we have

$$P((r,u) + (s,v)) = P(r+s,u+v) = (r+s)F((r+s)^{-1}(u+v)).$$

Using the convexity of F one has

$$F((r+s)^{-1}(u+v)) = F((r+s)^{-1}(rr^{-1}u + ss^{-1}v) \leq$$
$$\leq r(r+s)^{-1}F(r^{-1}u) + s(r+s)^{-1}F(s^{-1}v) =$$

$$= (r+s)^{-1}(rF(r^{-1}u) + sF(s^{-1}v)) = (r+s)^{-1}(P(r,u) + P(s,v)).$$

The obtained relation implies

$$P((r,u) + (s,v)) \leqslant P(r,u) + P(s,v)$$

Suppose now that $r \leqslant 0$. If $r = 0$, $u = 0$ then we have nothing to prove. In the rest of cases the right hand side of the above relation will contain ∞ as summand and hence it is satisfied.

Let us suppose that $x \in \mathrm{dom}\, F$, i.e., $F(x) \in Y$. Then $P(1,x) = F(x)$ and hence $(1,x) \in \mathrm{dom}\, P$. Since P is positively homogeneous, $t(1,x) \in \mathrm{dom}\, P$ for each $t \geqslant 0$. This shows that $\mathrm{wd}\,(\{1\} \times \mathrm{dom}\, F) \subset \mathrm{dom}\, P$.

Conversely, let $(t,x) \in \mathrm{dom}\, P$. If $t \leqslant 0$ then (t,x) is in $\mathrm{dom}\, P$ if and only if $t = 0$, $x = 0$. If $(t,x) \in \mathrm{dom}\, P$ for some $t > 0$, then $tP(1,t^{-1}x) = P(t,x) = tF(t^{-1}x)$, wherefrom it follows that $t^{-1}x \in \mathrm{dom}\, F$ and therefore $t(1,t^{-1}x) \in \mathrm{wd}\,(\{1\} \times \mathrm{dom}\, F)$. ▶

2°. Remark. Observe that the condition that F and P take values in $Y^{\cdot}$ and not in Y is here essential. Even then $\mathrm{dom}\, F = X$ we cannot imbed F into a sublinear operator P with $\mathrm{dom}\, P = R \times X$ if for some x in X the set $\{tF(t^{-1}x) : t > 0\}$ is not order bounded in Y. But convex operators with the property that all such sets are order bounded are very particular. They must satisfy a sort of generalised Lipschitz condition with respect to the order relation. Even in this case a finite extension implies some technical difficulties.

II. Ordered vector spaces for which every convex operator admits affine minorants. Let F be an operator from the vector space X to the ordered vector space $Y^{\cdot}$. The expression $A(.) + b$ with A in $L(X,Y)$ and b in Y is called an affine minorant of F if

$$Ax + b \leqslant F(x) , \quad \forall x \in X .$$

The linear operator A with the above property will be said to be a bounding operator for F .

If $P : X \to Y^{\cdot}$ is a sublinear operator, then the operator A in $L(X,Y)$ is called a supporting operator for P (see (F)) if

$$Ax \leqslant P(x) , \quad \forall x \in X .$$

1°. If $Y^{\cdot}$ is Archimedian and if $P : X \to Y^{\cdot}$ is sublinear, then $A \in L(X,Y)$ is a bounding operator for P if and only if it is a supporting operator for P . This property characterizes the Archimedian ordered vector spaces.

Let A be a bounding operator for P, i.e., let

$$Ax + b \leqslant P(x) , \quad b \in Y , \quad \forall x \in X .$$

Suppose that x is fixed and that $t > 0$. Put in the above relation tx in place of x. Then it follows

$$Ax + t^{-1}b \leqslant P(x)$$

for each $t > 0$. Since $Y^{\cdot}$ is Archimedian, the obtained relation shows that $Ax \leqslant P(x)$. Since x was arbitrary it follows that

A is a supporting operator for P.

To verify the second part of the assertion we assume that $Y^{\cdot}$ is not Archimedian. Then there exists some a in K, the positive cone of $Y^{\cdot}$, and some b in Y such that $ta \geqslant b$ (or equivalently, $a \geqslant tb$) for each $t > 0$, but $b \nleqslant 0$. Let us define the sublinear operator $P : R \to Y^{\cdot}$ by putting $P(t) = t(a-b)$ for $t \geqslant 0$ and $P(t) = -ta$ if $t < 0$. Let the linear operator $A : R \to Y$ be defined by $At = ta$. We show that

$$P(t) \geqslant At - a$$

for each t in R. It is sufficient to check the relation for $t \neq 0$. We have for $t > 0$:

$$P(t) = t(a-b) = ta - tb \geqslant ta - a = At - a$$

since $-tb \geqslant -a$ for each $t > 0$ by hypothesis.

If $t < 0$, then obviously

$$P(t) = -ta \geqslant ta \geqslant ta - a = At - a.$$

In conclusion, A is a bounding operator for P. Assume that it is a supporting operator too, i.e., that

$$P(t) \geqslant At, \quad \forall t \in R.$$

Then we have for $t = 1$

$$a - b = P(1) \geqslant A(1) = a$$

wherefrom $b \leqslant 0$, relation which contradicts the assumption on the element b. This completes the proof.

2^{0}. Remark. Various sufficient conditions for the existence of supporting operators in the case of continuous sublinear operators $P : C(S) \to C(T)$, where $C(S)$ and $C(T)$ are the spaces of real and continuous functions defined on the compact topological spaces S and T respectively, and the pointwise ordering in $C(T)$, were given by Yu. E. LINKE (see e.g., (LI1) and (LI2)). Starting from a construction of D. AMIR and J. LINDENSTRAUSS (AL) he observed (LI2) that for some compact spaces S and T there exist continuous sublinear operators from $C(S)$ to $C(T)$ without supporting operators. Since $C(T)$ with the pointwise ordering is Archimedian, it follows from the assertion 1^{0} that these sublinear operators admit no affine minorants.

The ordered vector space $Y^{\cdot}$ will be said to have the bounding property if each convex operator with values in $Y^{\cdot}$ has affine minorants. Intuitively the bounding property seems to be very mild. But the example cited at 2^{0} shows that the space $C(T)$ of continuous real valued functions defined on the compact topological space T in which the ordering is the usual one, does not have this property for an appropriate T. The standard positive cone in $C(T)$ is closed, hence Archimedian, it is normal and has non empty interior, that is, it is not very pathological.

3^{0}. The ordered vector space $Y^{\cdot}$ has the bounding property if and only if each sublinear operator with values in this space has affine minorants.

We have to check the if part only. Consider an arbitrary vector space X and an arbitrary convex operator $F : X \to Y^{\cdot}$.

Define $P : R \times X \to Y^*$ starting with F in the mode it was done at I. 1^0. Then P is sublinear and hence it has an affine minorant, say $B(.,.) + b$. That is,

$$B(t,x) + b \leq P(t,x) , \quad \forall (t,x) \in R \times X .$$

Define $A : X \to Y$ by putting $Ax = B(0,x)$. Then A is in $L(X,Y)$ and we have by the above relation

$$Ax + B(1,0) + b = B(1,x) + b \leq P(1,x) = F(x),$$

that is,

$$Ax + a \leq F(x) , \quad \forall x \in X$$

with $a = B(1,0) + b$. ▷

The ordered vector space Y^* will be said to have the <u>supporting property</u> if each sublinear operator with values in Y^* has supporting operators. For ordered vector spaces without this property we cite once again that considered at 2^0.

4^0. <u>An Archimedian ordered vector space has the supporting property if and only if it has the bounding property.</u>

◁ According 3^0 the bounding property implies the existence for each sublinear operator of an affine minorant (in fact, this is a consequence of the definition, since a sublinear operator is convex too). Since Y^* is Archimedian, this implies by 1^0 that Y^* has the supporting property. Conversely, the supporting property implies the bounding property according the assertion 3^0. ▷

III. <u>Subdifferentiability and exact Hahn-Banach approximation property</u>. Let F be an operator from the vector space X to the ordered vector space Y^*. Let $x_0 \in \mathrm{dom}\, F$. The set

$$\partial F(x_0) = \{A \in L(X,Y) : Ax \leq F(x_0+x) - F(x_0), \ \forall x \in X\}$$

is called the <u>subdifferential of F at x_0</u>. If $x_0 \notin \mathrm{dom}\, F$ one considers $\partial F(x_0) = \emptyset$. The elements of $\partial F(x_0)$ are called <u>subgradients of F at x_0</u>. The operator F is called <u>subdifferentiable at x_0</u> if $\partial F(x_0) \neq \emptyset$. The ordered vector space Y^* is said to be a space with the <u>subgradient property</u> (see (B2)), if for each vector space X, each convex operator from X to Y^* is subdifferentiable at every point of the core of its domain.

The ordered vector space Y^* has the <u>exact Hahn-Banach approximation property</u> (see e.g. (F)) if for each vector space X, for each sublinear operator $P : X \to Y^*$ and for each point $x_0 \in \mathrm{core\, dom}\, P$ there exists $A \in \partial P(0)$ such that $Ax_0 = P(x_0)$. One of the principal results of this paper can be formulated now as follows :

1^0 THEOREM. <u>The ordered vector space Y^* has the subgradient property if and only if it has the exact Hahn-Banach approximation property</u>.

◁ Let Y^* possess the subgradient property. Let X be an arbitrary vector space and let $P : X \to Y^*$ be a sublinear operator. Suppose $x_0 \in \mathrm{core\, dom}\, P$, $x_0 \neq 0$. Consider an algebraic complementary space Z of $\mathrm{sp}\{x_0\}$ in X and define $F : Z \to Y^*$ by putting

$$F(z) = P(z+x_0).$$

Then $0 \in$ core dom F and since F is convex, it is subdifferentiable at 0. Suppose that $B \in \partial F(0)$ and define $A \in L(X,Y)$ by putting

$$Ax = Bz + tP(x_0)$$

where x is an arbitrary element in X written in the form $x = z + tx_0$ with $z \in Z$, $t \in R$. Let us check that A is in $\partial P(0)$. If $t = 0$, then we have $x \in Z$ and

$$Ax = Bx \leq F(x) - F(0) = P(x+x_0) - P(x_0) \leq P(x) .$$

If $t > 0$, then

$$Ax = Bz + tP(x_0) = t(B(t^{-1}z) + P(x_0)) \leq t(F(t^{-1}z)-F(0)+P(x_0)) =$$

$$= t(P(t^{-1}z+x_0) - P(x_0) + P(x_0)) = P(z+tx_0) = P(x) .$$

Suppose finally that $t < 0$. We have then

$$Ax = Bz + tP(x_0) \leq F(z) - F(0) + tP(x_0) = P(z+x_0) -$$

$$- P(x_0) + tP(x_0) = P(z+x_0) - (1-t)P(x_0) = P(z+x_0) -$$

$$- P((1-t)x_0) \leq P(z+x_0-(1-t)x_0) = P(z+tx_0) = P(x) .$$

It was shown that $Ax \leq P(x)$ for every x in X and hence $A \in \partial P(0)$. Since $Ax_0 = P(x_0)$ and since X and P were arbitrarily chosen it follows that $Y^\bullet$ has the exact Hahn-Banach approximation property.

Suppose now that $Y^\bullet$ has the exact Hahn-Banach approximation property, let X be an arbitrary vector space and consider the convex operator $F : X \to Y^\bullet$ and $x_0 \in$ core dom F. Let us

construct the sublinear operator $P : R \times X \to Y^\bullet$ starting with F as it was done at I. 1°. Since $(1,x_0) \in$ core dom P (see I. 1°), it follows that there exists $B \in \partial P(0,0)$ with $B(1,x_0) = P(1,x_0) = F(x_0)$. Define $A \in L(X,Y)$ by putting $Ax = B(0,x)$. We have for an arbitrary x in X

$$Ax = B(0,x) = B(1,x+x_0) - B(1,x_0) \leq P(1,x+x_0) - P(1,x_0) =$$

$$= F(x+x_0) - F(x_0) .$$

That is, $A \in \partial F(x_0)$. This shows that $Y^\bullet$ has the subgradient property and completes the proof of the theorem. ▷

IV. <u>Subdifferentiability and fully subdifferentiability.</u> Let F be an operator from the vector space X to the ordered vector space $Y^\bullet$. The <u>directional minorant of F at x_0</u> in X in the direction $h \in X$ is by definition the infimum

$$\nabla F(x_0;h) = \inf \{t^{-1}(F(x_0+th) - F(x_0)) : t > 0\}$$

if this infimum exists.

We shall say that $F : X \to Y^\bullet$ is <u>fully subdifferentiable at x_0</u> in core dom F if : (i) $\nabla F(x_0;\cdot)$ is defined on X ; (ii) $\partial F(x_0) \neq \emptyset$ and (iii) $\nabla F(x_0;h) = \max\{Ah : A \in \partial F(x_0)\}$ (see (N2) and (N3)). F is said to be <u>fully subdifferentiable</u> if it is fully subdifferentiable at each point of core dom F. The space $Y^\bullet$ will be called a <u>fully subdifferentiability space</u> if each convex operator with values in $Y^\bullet$ is fully subdifferentiable.

The space $Y^\bullet$ is said to <u>admit a strictly monotone functional</u> if there exist $f : Y \to R$ with the property that $f(u) < f(v)$

whenever $u, v \in Y$, $u \leq v$ and $u \neq v$.

The space $Y^\cdot$ is said to have the *monotone sequence property* (see (B1)), if each order bounded decreasing sequence in it has an infimum.

In (N2) and (N3) we have given a characterization of ordered vector spaces admitting strictly monotone functionals with the fully subdifferentiability property. We state for the sake of completeness this result also here :

1^o. THEOREM. *An ordered vector space which admits a strictly monotone functional is a fully subdifferentiability space if and only if it has the monotone sequence property.*

The proof of the theorem uses a result of FEL'DMAN (F) (see also (N1) and (B2)) and acharacterization of ordered vector spaces with the property that the convex operators with values in them possess directional minorants.

2^o. *Remark*. Observe that a fully subdifferentiability space has the subgradient property but the converse is not true. Indeed, in Theorem III. 1^o it was shown that spaces with the subgradient property are quite the spaces with the exact Hahn-Banach approximation property. In (F) and (B2) there were constructed examples of ordered vector spaces with not lineally closed positive cones which have the exact Hahn-Banach approximation property. We remind that a set in a vector space is said *lineally closed* if its intersection with all straight lines determine closed subsets in the usual topology of the lines (by identification of them with R). From III. 1^o it

follows then that the cited ordered vector spaces have the subgradient property. On the other hand, it is immediate to see that an ordered vector space with not lineally closed positive cone cannot have the monotone sequence property. Accordingly the cited examples constitute ordered vector spaces with the subgradient property which lack the fully subdifferentiability property.

V. S- *and V-subdifferentiability and the supporting property*. Let F be an operator from the vector space X to the ordered vector space $Y^\cdot$. Denote by S a set in Y containing zero. The *S-subdifferential of* F *at* x_0 is defined as

$$\partial_S F(x_0) = \{A \in L(X,Y) : Ax \in F(x+x_0)-F(x_0)+S-K,\ \forall x \in X\}$$

if $x_0 \in \mathrm{dom}\, F$ and $\partial_S F(x_0) = \emptyset$ otherwise (see (T)). The operator F is said to be *S-subdifferentiable at* x_0, if $\partial_S F(x_0) \neq \emptyset$. If $\varepsilon \in K$ and if S is the order interval $[-\varepsilon, \varepsilon]$, defined by $-\varepsilon$ and ε, then the S-subdifferential is called ε-subdifferential (see (KU2) or (TH)) and is denoted by ∂^ε. The elements of $\partial_S F(x_0)$ ($\partial^\varepsilon F(x_0)$) are called *S-subgradients* (ε-*subgradients*).

1^o. *If* $Y^\cdot$ *is Archimedian, then for every sublinear operator with values in* $Y^\cdot$ *each* ε-*subgradient is a supporting operator. This property characterizes Archimedian ordered vector spaces.*

◄ Suppose that $P : X \to Y^\cdot$ is sublinear and let A be in $\partial^\varepsilon P(x_0)$. Then for each x in X

$$Ax \in P(x+x_0) - P(x_0) + [-\varepsilon,\varepsilon] - K \subset P(x) + [-\varepsilon,\varepsilon] - K,$$

wherefrom

$$Ax - P(x) \in [-\varepsilon,\varepsilon] - K .$$

This relation implies that

$$Ax - P(x) \leq \varepsilon.$$

Fix x and put $t^{-1}x$ with $t > 0$ in place of x in this relation. Then it follows that

$$Ax - P(x) \leq t\,\varepsilon.$$

Since $t > 0$ was arbitrarily choosen and since K is Archimedian one has

$$Ax \leq P(x) , \quad \forall x \in X ,$$

that is, $A \in \partial P(0)$.

To prove the second assertion we observe that if $A(.) + b$ is an affine minorant of P then $b \leq 0$ and since

$$Ax \in P(x) - b - K \subset P(x) + [-\varepsilon,\varepsilon] - K , \quad \forall x \in X ,$$

with $\varepsilon = -b$, it follows that A is in $\partial^{\varepsilon} P(0)$. Suppose now that each ε-subgradient of P is a supporting operator of P . Then it follows, according the above observation, that each bounding operator of P is a supporting operator too, wherefrom it follows by II. 1° that Y^* must be an Archimedian ordered vector space. ▷

Henceforth we shall suppose that Y^* is an ordered Hausdorff topological vector space.

2°. <u>If the positive cone K in Y^* is closed, then if A is an S-subgradient of the sublinear operator $P : X \to Y^*$ with S a topologically bounded set, then $A \in \partial P(0)$.</u>

◁ We have by definition for some x_0 in dom P

$$Ax \in P(x+x_0) - P(x_0) + S - K \quad P(x) + V - K , \quad \forall x \in X .$$

Fix $x \in$ dom P and put tx with $t > 0$ in place of x in this relation. Then

$$Ax \in P(x) + t^{-1}S - K .$$

Letting now $t \to \infty$ and taking into account that K is closed and S is topologically bounded we conclude

$$Ax \leq P(x) , \quad \forall x \in X . \quad ▷$$

3°. <u>Remark</u>. It follows that the sublinear operator P considered by LINKE (see II. 2°) does not have any S-subgradient with S a topologically bounded set. Since P acts between Banach spaces S can be considered to be a neighbourhood of 0.

The ordered vector space Y^* has the V-<u>subgradient property</u> (respectively, the V- <u>subgradient property for sublinear operators</u>) if for arbitrary vector space X and arbitrary convex operators $F : X \to Y^*$ (each sublinear operator $P : X \to Y^*$) one holds $\partial_V F(x_0) \neq \emptyset$ ($\partial_V P(x_0) \neq \emptyset$) whenever $x_0 \in$ dom F ($x_0 \in$ dom P) , for each neighbourhood V of 0 in Y .

4°. <u>The ordered normed space Y^* has the V-subgradient</u>

property for sublinear operators if and only if each sublinear operator with values in $Y^{\cdot}$ is W-subdifferentiable at each point of its domain, for W some given, bounded neighbouhood of 0 in Y.

◀ Let $P : X \to Y^{\cdot}$ be sublinear and let $x_0 \in \operatorname{dom} P$ be given. Suppose that V is an arbitrary neighbourhood of 0 in Y. Consider $s > 0$ such that $sW \subset V$. Since P is W-subdifferentiable at $s^{-1}x_0$, we have some A in $L(X,Y)$ such that

$$Ax \in P(x+s^{-1}x_0) - P(s^{-1}x_0) + W - K, \quad \forall x \in X.$$

Let x be arbitrary and fixed in $\operatorname{dom} P$. Put $s^{-1}x$ in place of x in the above relation. Then

$$Ax \in P(x+x_0) - P(x_0) + sW - K \subset P(x+x_0) - P(x_0) + V - K.$$

From this relation it follows that $A \in \partial_V P(x_0)$. ▶

VI. The Hahn-Banach approximation property and the pointwise approximation of convex operators by affine minorants. The ordered topological vector space $Y^{\cdot}$ is said to have the Hahn-Banach approximation property if for an arbitrary vector space X and an arbitrary sublinear operator $P : X \to Y^{\cdot}$ for any $x_0 \in \operatorname{dom} P$ and for arbitrary neighbourhood V of 0 in Y there exists A in $\partial P(0)$ with the property that

$$P(x_0) - Ax_0 \in V.$$

The operator $G : X \to Y^{\cdot}$ is said to be in $\Lambda(X,Y)$ (see (T), 2.1.) if for each x in $\operatorname{dom} G$ and each neighbourhood

V of 0 in Y there exists an affine minorant, say $A(\cdot) + b$ ($A \in L(X,Y)$, $b \in Y$) of G such that

$$G(x) - Ax - b \in V.$$

We shall say that $Y^{\cdot}$ has the Λ-property if for each vector space X each convex operator $F : X \to Y^{\cdot}$ is in $\Lambda(X,Y)$.

1^0. The Hahn-Banach approximation property implies the Λ-property.

◀ Let X be an arbitrary vector space and let $F : X \to Y^{\cdot}$ be an arbitrary convex operator. Consider the sublinear operator $P : R \times X \to Y^{\cdot}$ constructed using F as it was done in I. 1^0. Let $x_0 \in \operatorname{dom} F$. Then $(1,x_0) \in \operatorname{dom} P$. Let V be an arbitrary neighbourhood of 0 in Y. Suppose that $Y^{\cdot}$ has the Hahn-Banach approximation property. Then there exists B in $\partial P(0,0)$ such that

$$(^{*}) \qquad P(1,x_0) - B(1,x_0) \in V.$$

Put $Ax = B(0,x)$. Then A is in $L(X,Y)$ and one has

$$Ax = B(0,x) = B(1,x) - B(1,0) \leq P(1,x) - B(1,0)$$

wherefrom

$$Ax + B(1,0) \leq P(1,x) = F(x).$$

On the other hand, from $(^{*})$ it follows

$$F(x_0) - Ax_0 - B(1,0) = P(1,x_0) - B(0,x_0) -$$
$$- B(1,0) = P(1,x_0) - B(1,x_0) \in V.$$

If we put now $B(1,0) = b$, then $A(.) + b$ will be an affine minorant of the operator P for which we have

$$P(x_0) - Ax_0 - b \in V .$$

This shows that Y' has the Λ-property. ▷

2°. If Y' has the Hahn-Banach approximation property then it has the V-subgradient property too.

This assertion can be proved directly but it follows also from 1° using Proposition 3.3 in (T) .

3°. If the positive cone K of Y' is closed and normal and if Y' has the Λ-property, then it has the Hahn-Banach approximation property too.

◁ Let X be an arbitrary vector space and let $P : X \to Y'$ be an arbitrary sublinear operator. Suppose that $x_0 \neq 0$ and $x_0 \in \operatorname{dom} P$. Let V be an arbitrary full neighbourhood of 0 in Y. We shall show that if Y' has the Λ-property, then there exists an operator A in $\partial P(0)$ such that $P(x_0) - Ax_0 \in V'$. From the Λ-property of Y' it follows that there exists an A in $L(X,Y)$ and a $b \in Y$ such that

$$Ax + b \leq P(x) , \quad \forall x \in X$$

and

$$P(x_0) - Ax_0 - b \in V .$$

We observe first that $b \leq 0$. Further, since K is Archimedian, it follows that A is in $\partial P(0)$ according assertion II. 1°.

We have then

$$P(x_0) - Ax_0 \in K .$$

Hence

$$P(x_0) - Ax_0 \in (V + b) \cap K .$$

Let $P(x_0) - Ax_0 = v + b \in K$ with $v \in V$. Then since $b \leq 0$ we have $0 \leq v + b \leq v$. Now V being full, this relation implies $v + b \in V$. That is, we conclude

$$P(x_0) - Ax_0 \in V$$

and the proof is complete . ▷

The assertion 1° together with the assertion 3° imply the following result :

4°. Let the positive cone in the ordered topological vector space Y' be closed and normal. Then Y' has the Λ-property if and only if it has the Hahn-Banach approximation property.

R E F E R E N C E S

(AK) AKILOV, G.P.; KUTATELADZE, S.S. , Ordered Vector Spaces (Russian) Novosibirsk, 1978.

(AL) AMIR, D.; LINDENSTRAUSS, J., The structure of weakly compact sets in Banach spaces, Ann. Math. 88 (1968), 35-46.

(B1) BORWEIN, J.M., Continuity and differentiability properties of the convex mappings, Proc. London Math. Soc. 44 (1982), 420-444.

(B2) BORWEIN, J.M., Subgradients of convex operators, Math. Operationsforschung u. Statistic ser. Optimization 15 (1984), 179-191.

(B3) BORWEIN, J.M., A Lagrange multiplier theorem and a sandwich theorem for convex relations, Math. Scand. 48 (1981), 189-204.

(EN) ELSTER, K-. H.; NEHSE, R., Necessary and sufficient conditions for the order-completeness of partially ordered vector spaces, Math. Nachr. 81 (1978),301-311.

(F) FEL'DMAN, M.M., On the sufficient conditions of the existence of supporting operators to sublinear mappings, Sibirsk. Mat. Z., 16 (1975), 132-138.

(IL) IOFFE, A.D.; LEVIN, V.L., Subdifferentials of convex functions (Russian), Trudy Moskov. Mat. Obsc. 26 (1972), 3-73.

(K) KIROV, N.K., Differentiability of convex mappings and generalized monotone mappings, C. R. Acad. Bulg. Sci. 34 (1981)

(KU1) KUTATELADZE, S.S., Convex operators, Russian Math. Surveys 34 (1979), 181-214.

(KU2) KUTATELADZE, S.S., Convex   -programming, Soviet Math. Doklady, 20 (1979), 391-393.

(L) LEVIN, V.L., The subdifferentials of convex mappings and composed functionals (Russian) Sibirsk Mat. Z. 13 (1979), 1295-1303.

(LI1) LINKE, Yu.E., On the support set of sublinear operators, Soviet Math. Doklady 13 (1972), 1561-1568.

(LI2) LINKE, Yu.E., Sublinear operators with values in the spaces of continuous functions, Soviet Math. Doklady 17 (1976), 774-777.

(N1) NÉMETH, A.B., Some differential properties of the convex mappings, Mathematica 22(1980), 107-114.

(N2) NÉMETH, A.B., On the subdifferentiability of convex operators, to appear.

(N3) NÉMETH, A.B., Monotone sequence property and directional minorability of convex operators are equivalent, Proc. Nat. Conf. on Optimization and Approximation Cluj-Napoca, 1984.

(P) PAPAGEORGE, N.S., Nonsmooth analysis on partially ordered vector spaces : part 1 - convex case, Pacific J. Math. 107 (1983), 403-485.

(R) RUBINOV, A.M., Sublinear operators and their applications, Russian Math. Surveys 32 (1977), 115-175.

(TH) THERA, M., Calcul   -sousdifférentiel des applications convexes, C. R. Acad. Sci. Paris 290 (1980), 549-551.

(T) THIBAULT, L., V-subdifferentials of convex operators, to appear.

(V) VALADIER, M., Sous- différentiabilité de fonctions convexes à valeurs dans un espace vectoriel ordonné, Math. Scand. 30 (1972), 65-74.

(Z1) ZOWE, J., Subdifferentiability of convex functions with values in an ordered vector space , Math. Scand. 34 (1974), 69-83.

(Z2) ZOWE, J., Linear maps majorized by a sublinear map, Arch. Math. 26 (1975), 637-645.

(Z3) ZOWE, J., A duality theorem for a convex programming problem in order complete vector lattices, J. Math. Anal. Appl. 50 (1975), 273-285.

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Research Seminaries
Seminar on Functional Analysis and Numerical Methods
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SUR UNE FORMULE DE QUADRATURE

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§ 1. En [1] GOLOMB et OLECH ont prouvé le

THÉORÈME. Il existe un nombre $\theta \in (0,1)$ (et un seul) à la propriété que la formule de quadrature

(0) $$\int_0^1 f(x)\,dx = \lambda_0 f(0) + \lambda_1 f(\theta) + \lambda_2 f(1) + R(f)$$

est exacte pour $f = 1, x^p , x^q , x^r$, c'est-à-dire

(1) $$R(1) = R(x^p) = R(x^q) = R(x^r) = 0 \quad ,$$

où p, q, r sont des nombre positifs donnés, différents entre eux.

Dans la note présente on donne au § 2 une démonstration de ce Théorème, sous une forme différente de celle donnée par ses auteurs et aux §§ 3 et 4 on présente deux des généralisations qu'on peut donner au sus-cité Théorème.

§ 2. On peut bien entendu supposer

(2) $$0 < p < q < r \ .$$

Les relations (1) s'écrivent

(3) $$\lambda_0 + \lambda_1 + \lambda_2 = 1$$

(4) $$\lambda_1 \theta^p + \lambda_2 = \frac{1}{1+p}$$

(5) $$\lambda_1 \theta^q + \lambda_2 = \frac{1}{1+q}$$

(6) $$\lambda_1 \theta^r + \lambda_2 = \frac{1}{1+r}$$