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A NECESSARY CONDITION FOR SUBDIFFERENTIABILITY OF CONVEX OPERATORS

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The existence of the subgradients for a convex operator is intimately related to the properties of the ordering in its range space. The latticially completeness is one of the conditions which guarantees nice subdifferentiability properties of the convex operators. This circumstance explains the interest in these spaces at the beginning of the vector valued convex analysis in the papers of Valadier (V) and Levin (L). But the complete latticiality is a rather restrictive hypothesis for an ordered vector space and it is much more than it is needed in order to assure nice subdifferentiability of the convex operators. This is a direct consequence of a result of Fel'dman (F). We have proved recently (N2) that Fel'dman's condition is also necessary for the so called full subdifferentiability of convex operators.

A less restrictive subdifferentiability condition is the subgradient property considered by Borwein (B). It reduces to the existence of subgradients at every point to each convex operator ranged in the considered ordered vector space. This property is equivalent to the exact Hahn-Banach approximation property considered by Fel'dman (F), as this was shown in (N3).

This equivalence is one of the characterisations of the spaces having the subgradient property.

The present paper exhibit a necessary condition for an ordered vector space to have the subgradient (and hence the exact Hahn-Banach approximation) property. It is presented in context of the monotonic sequences. For ordered normal Banach spaces a strengthened form of the condition is achieved, which is the main result of the paper.

The basic tool in the proofs is the constructive technique developed in (N1) and simplified in (N2).

1. Prerequisites

Let Y be a vector space over the reals and let K be a cone in Y , i. e., a nonempty set having the properties :
(i) $K + K \subset K$; (ii) $tK \subset K$ whenever $t \in R_+ = [0,\infty)$; and (iii) $K \cap (-K) = \{0\}$. We shall assume throughout the paper that $K \neq \{0\}$. If we put $u \leq v$ whenever $v - u \in K$, then $\leq$ is a reflexive, transitive and antisymmetrical relation on Y , which is translation invariant and invariant with respect to the multiplication with scalars in R_+ . The vector space Y endowed with this order relation will be called an ordered vector space and will be denoted by $(Y;K)$. The cone K is said to be the positive cone of this ordered vector space. Upper and lower bounds of sets, supremum and infimum for a set are defined in a natural way.

The order interval $[u,v]$ in $(Y;K)$ is by definition the set $\{z \in Y : u \leq z \leq v\}$.

The ordered vector space $(Y;K)$ is called *Archimedian* if $u \leq tv$ for $v \in K$ and every $t > 0$ implies $u \leq 0$.

The space $(Y;K)$ is said to have the *monotone sequence property* (shortened MSP) if every increasing sequence in Y which has an upper bound, has a supremum.

We shall introduce now a new notion which is weaker as the MSP. Let us denote first by $\{(v_m);(u_n)\}$ an object consisting of two sequences, (v_m) and (u_n) in $(Y;K)$ such that $v_m \leq v_{m+1} \leq u_{n+1} \leq u_n$ for any m and n in N.

D e f i n i t i o n . The ordered vector space $(Y;K)$ will be said to have the property (b) if for every $\{(v_m);(u_n)\}$ in Y one has

$$(*) \qquad \bigcap_{m,n \in N} [v_m, u_n] \neq \emptyset .$$

The space $(Y;K)$ will be said to have property (b') if for every $\{(v_m);(u_n)\}$ with the property that there exist the positive numbers α_n and β_m such that

$$u_{n-1} - u_n \geq \alpha_n (u_n - u_{n+1}) \quad \text{and} \quad v_m - v_{m-1} \geq \beta_m (v_{m+1} - v_m)$$

for each m and n, the condition $(*)$ holds.

Example. The vector space c of all the convergent sequences of real numbers endowed with the order relation induced by the cone of sequences with non-negative terms in c does not have property (b). Indeed, consider $(v_m);(u_n)$ defined by

$$v_{2k} = v_{2k-1} = (\underbrace{1,0,1,0,\dots,0,1}_{2k-1 \text{ terms}},0,0,\dots,0,\dots)$$

and

$$u_{2j+1} = u_{2j} = (\underbrace{1,0,1,0,\dots,1,0}_{2j \text{ terms}},1,1,\dots,1,\dots) .$$

Then we have $v_m \leq v_{m+1} \leq u_{n+1} \leq u_n$ for every m and n, while obviously

$$\bigcap_{m,n \in N} [v_m, u_n] = \emptyset .$$

From Lemma 3 it will follow that the space c endowed with the above order relation has not property (b') too.

Suppose now that Y is a normed space. Then $(Y;K)$ is called *normal* if there exists a positive number β such that from $0 \leq u \leq v$ it follows that $\|u\| \leq \beta \|v\|$. We shall use the term normal also for the positive cone K.

If the space c of oll convergent real sequences is equipped with the usual norm, then it is a Banach space and the cone in c of all sequences with non-negative terms is closed and normal.

The ordered normed space $(Y;K)$ is called *sequentially regular* if every increasing sequence in it which is bounded from above, is convergent. It can be seen (see e.g. (P), II.3.1) that a sequentially regular ordered normed space $(Y;K)$ has the MSP.

Let U be a convex subset of a vector space X. The operator

$$F : U \to (Y;K)$$

is said to be *convex* if

$$F(tx_1 + (1-t)x_2) \leq tF(x_1) + (1-t)F(x_2)$$

whenever $x_1, x_2 \in U$ and $t \in [0,1]$.

The directional minorant of F at $x \in \operatorname{core} U$ in the direction $h \in X$ is the infimum

$$\nabla F(x;h) = \inf_{t>0} t^{-1}(F(x+th) - F(x)) ,$$

when it exists.

The linear operator $A : X \to Y$ is said to be a subgradient of the operator F at $x \in U$ if

$$Ah \leq F(x+h) - F(x)$$

for every h for which the right hand side of this relation is defined. The set of the subgradients of F at x is denoted by $\partial F(x)$ and is called the subdifferential of F at x .

The ordered vector space $(Y;K)$ is said to have the subgradient property if for every convex set U and for every convex operator $F : U \to (Y;K)$ one has $\partial F(x) \neq \emptyset$, whenever $x \in \operatorname{core} U$.

The ordered vector space $(Y;K)$ is said to have the fully subdifferentiability property if it has the subgradient property, $\nabla F(x;.)$ is defined on X for each $x \in \operatorname{core} U$, and

$$\nabla F(x;h) = \sup \{ Ah : A \in \partial F(x) \} .$$

for each h in X .

In (N2) it was shown that the ordered vector space $(Y;K)$ which admits a functional g with the property that $g(u) < g(v)$ whenever $u \leq v$ and $u \neq v$, has the full subdifferentiability property if and only if it possesses the MSP .

2. A necessary condition for the subgradient property of an ordered vector space

PROPOSITION 1. If an ordered vector space $(Y;K)$ possesses the subgradient property, then it has the property (b') .

We prove first two auxiliary lemmas :

LEMMA 1. Let $\{(v_m);(u_n)\}$ and the strictly decreasing to zero real sequences (a_n) and (b_m) be given and define $f : \bigcup \{-b_m , a_n : m, n \in N\} \to Y$ by putting

$$f(-b_m) = -b_m v_m \quad \text{and} \quad f(a_n) = a_n u_n , \quad m, n \in N .$$

Then f has a convex extension to $[-b_1, a_1]$ if and only if

$$\Delta''_{m-1} \leq \Delta''_m \leq \Delta_{mn} \leq \Delta'_n \leq \Delta'_{n-1} \tag{1}$$

for any $m, n \geq 2$, where

$$\Delta''_m = \frac{b_m v_m - b_{m+1} v_{m+1}}{b_m - b_{m+1}} , \quad \Delta_{mn} = \frac{a_{n+1} u_{n+1} + b_{m+1} v_{m+1}}{a_{n+1} + b_{m+1}} , \tag{2}$$

$$\Delta'_n = \frac{a_n u_n - a_{n+1} u_{n+1}}{a_n - a_{n+1}} .$$

Proof . The necessity of the condition (1) is in fact the monotonicity property of the slope of a convex operator. It can be shown directly or by the technique using the operator

$$\varphi(x,h;t) = t^{-1}(f(x+th) - f(x))$$

$R \setminus \{0\}$ for

which is defined and increasing with t for t in

x and h fixed whenever $f : R \to (Y;X)$ is convex (see e.g.

(V) or (L)). Indeed, by putting $x = -b_m$, $h = 1$, $t_1 = b_m - b_{m-1} <$

< 0 and $t_2 = b_m - b_{m+1} > 0$ and using the monotonicity of

in t, we obtain the relation

$$\varphi(-b_m, 1;\ b_m - b_{m-1}) \leqslant \varphi(-b_m, 1 ;\ b_m - b_{m+1})$$

which is quite $\Delta''_{m-1} \leqslant \Delta''_m$. Then we consider $x = -b_{m+1}$,

$h = 1$, $t_1 = b_{m+1} - b_m < 0$ and $t_2 = a_{n+1} + b_{m+1} > 0$ to deduce

the relation

$$\varphi(-b_{m+1}, 1 ;\ b_{m+1} - b_m) \leqslant \varphi(-b_{m+1}, 1 ;\ a_{n+1} + b_{m+1})$$

which is in fact $\Delta''_m \leqslant \Delta_{mn}$. Similarly, the relations

$$\varphi(a_{n+1}, 1 ;\ -a_{n+1} - b_{m+1}) \leqslant \varphi(a_{n+1}, 1 ;\ a_n - a_{n+1})$$

and

$$\varphi(a_n, 1 ;\ a_{n+1} - a_n) \leqslant \varphi(a_n, 1 ;\ a_{n-1} - a_n)$$

are $\Delta_{mn} \leqslant \Delta'_n$ and $\Delta'_n \leqslant \Delta'_{n-1}$, respectively.

To verify the sufficience of the condition we consider the

affine operators

$$f_n(t) = (t - a_{n+1})\Delta'_n + a_{n+1}u_{n+1}, \quad n \in N$$

and

$$g_m(t) = (t + b_{m+1})\Delta''_m - b_{m+1}v_{m+1}, \quad m \in N .$$

We have for arbitrary m and n in N the relations

$$f_n(t) - g_m(t) = (t - a_{n+1})\Delta'_n - (t + b_{m+1})\Delta''_m + a_{n+1}u_{n+1} + b_{m+1}v_{m+1} =$$

$$= (t - a_{n+1})(\Delta'_n - \Delta''_m) - (a_{n+1} + b_{m+1})\Delta''_m + a_{n+1}u_{n+1} + b_{m+1}v_{m+1} \geqslant$$
$$\geqslant (t - a_{n+1})(\Delta'_n - \Delta''_m) - (a_{n+1} + b_{m+1})\Delta_{mn} + a_{n+1}u_{n+1} + b_{m+1}v_{m+1} =$$
$$= (t - a_{n+1})(\Delta'_n - \Delta''_m)$$

since $-\Delta''_m \geqslant -\Delta_{mn}$ by (1) . From (1) it follows also that

(3) $$f_n(t) - g_m(t) \geqslant 0 \quad \text{if} \quad t \geqslant a_{n+1} .$$

By the same way,

$$f_n(t) - g_m(t) = (t + b_{m+1})(\Delta'_n - \Delta''_m) - (b_{m+1} + a_{n+1})\Delta'_n +$$
$$+ a_{n+1}u_{n+1} + b_{m+1}v_{m+1} \leqslant (t + b_{m+1})(\Delta'_n - \Delta''_m) -$$
$$- (b_{m+1} + a_{n+1})\Delta_{mn} + a_{n+1}u_{n+1} + b_{m+1}v_{m+1} =$$
$$= (t + b_{m+1})(\Delta'_n - \Delta''_m)$$

since $-\Delta'_n \leqslant -\Delta_{mn}$ by (1) , and using again this relation we conclude that

(4) $$f_n(t) - g_m(t) \leqslant 0 \quad \text{if} \quad t \leqslant -b_{m+1} .$$

Consider now the difference

$$f_{n-1}(t) - f_n(t) = (t - a_n)\Delta'_{n-1} - (t - a_{n+1})\Delta'_n + a_n u_n - a_{n+1}u_{n+1} =$$
$$= (t - a_n)(\Delta'_{n-1} - \Delta'_n) - (a_n - a_{n+1})\Delta'_n + a_n u_n - a_{n+1}u_{n+1} =$$
$$= (t - a_n)(\Delta'_{n-1} - \Delta'_n)$$

wherefrom

$$f_{n-1}(t) - f_n(t) \geqslant 0 \quad \text{for} \quad t \geqslant a_n ,$$

(5)

$$f_{n-1}(t) - f_n(t) \leqslant 0 \quad \text{for} \quad t \leqslant a_n .$$

Finally we have

$$g_{n-1}(t) - g_n(t) = (t+b_n)\,\Delta''_{n-1} - (t+b_{n+1})\,\Delta''_n - b_n v_n + b_{n+1} v_{n+1} =$$

$$= (t+b_n)\,(\Delta''_{n-1} - \Delta''_n) - (-b_n + b_{n+1})\,\Delta''_n - b_n v_n + b_{n+1} v_{n+1} =$$

$$= (t+b_n)\,(\Delta''_{n-1} - \Delta''_n) \;,$$

wherefrom

$$(6) \qquad \begin{aligned} g_{n-1}(t) - g_n(t) &\geq 0 \quad \text{if} \quad t \leq -b_n \;, \\ g_{n-1}(t) - g_n(t) &\leq 0 \quad \text{if} \quad t \geq -b_n \;. \end{aligned}$$

We define now the operator $f : [-b_1, a_1] \to (Y;K)$ by

$$f(t) = \begin{cases} \sup \{ g_n(t) \,,\, f_n(t) \;:\; n \,,\, n \in N \} & \text{if} \quad t \neq 0 \,, \\ 0 & \text{if} \quad t = 0. \end{cases}$$

For $t \in [a_{n+1}, a_n]$ it holds $f_n(t) \geq g_n(t)$ for every n by (3) and $f_n(t) \geq f_k(t)$ for every k by the relations (5). If $t \in [-b_n, -b_{n+1}]$ there hold $g_n(t) \geq f_n(t)$ for every n by (4) and $g_n(t) \geq g_k(t)$ for every k by the relations (6). Hence f is well defined and $f(t) = f_n(t)$ if $t \in [a_{n+1}, a_n]$ and $f(t) = g_n(t)$ if $t \in [-b_n, -b_{n+1}]$.

The operator f is convex on $[-b_1, 0)$ and on $(0, a_1]$ since it is the supremum of affine operators on these intervals. We shall show that f is convex on the whole interval $[-b_1, a_1]$.

Suppose first that $t_1 \in [-b_n, -b_{n+1}]$, $t_2 \in [a_{n+1}, a_n]$. We have to check the relation

$$(7) \qquad f(\lambda t_1 + (1-\lambda) t_2) \leq \lambda f(t_1) + (1-\lambda) f(t_2) \;.$$

From our above observation we have $f(t_1) = g_n(t_1)$ and $f(t_2) = f_n(t_2)$ and hence we have to verify in fact the relation

$$(8) \qquad f(\lambda t_1 + (1-\lambda) t_2) \leq \lambda g_n(t_1) + (1-\lambda) f_n(t_2) \;.$$

If $\lambda t_1 + (1-\lambda) t_2 \in [-b_n, 0)$, then $\lambda t_1 + (1-\lambda) t_2 \in [-b_k, -b_{k+1}]$ for some $k \geq n$. Hence

$$(9) \qquad f(\lambda t_1 + (1-\lambda) t_2) = g_k(\lambda t_1 + (1-\lambda) t_2) = \lambda g_k(t_1) + (1-\lambda) g_k(t_2)$$

since g_k is affine. Because $g_k(t_2) \leq f_n(t_2)$ and $g_k(t_1) \leq g_n(t_1)$ it follows that

$$\lambda g_k(t_1) + (1-\lambda) g_k(t_2) \leq \lambda g_n(t_1) + (1-\lambda) f_n(t_2)$$

relation which together with (9) and (8) shows that in this case (7) is verified.

Similarly, if $\lambda t_1 + (1-\lambda) t_2 \in (0, a_n]$, then $\lambda t_1 + (1-\lambda) t_2 \in [a_{h+1}, a_h]$ for some $h \geq n$. Hence

$$(10) \quad f(\lambda t_1 + (1-\lambda) t_2) = f_h(\lambda t_1 + (1-\lambda) t_2) = \lambda f_h(t_1) + (1-\lambda) f_h(t_2)$$

by the definition of f and by the fact that f_h is affine. But $f_h(t_1) \leq g_n(t_1)$ and $f_h(t_2) \leq f_n(t_2)$ and accordingly

$$\lambda f_h(t_1) + (1-\lambda) f_h(t_2) \leq \lambda g_n(t_1) + (1-\lambda) f_n(t_2) \;.$$

This relation together with (10) and (8) concludes (7) in this case too.

Suppose now that $\lambda t_1 + (1-\lambda) t_2 = 0$. Then we have $f(\lambda t_1 + (1-\lambda) t_2) = 0$ and we have to check the relation

$$0 \leq \lambda f(t_1) + (1-\lambda) f(t_2)$$

which reduces to

$$0 \leq \lambda g_m(t_1) + (1-\lambda) f_n(t_2) .$$

We have $g_m(-b_m) = -b_m v_m$, $g_m(-b_{m+1}) = -b_{m+1} v_{m+1}$, $f_n(a_n) = a_n u_n$ and $f_n(a_{n+1}) = a_{n+1} u_{n+1}$. Since $t_1 \in [-b_m, -b_{m+1}]$ and $t_2 \in [a_{n+1}, a_n]$ there exist some r, s in $[0,1]$ such that

$$t_1 = r(-b_m) + (1-r)(-b_{m+1}) \quad \text{and} \quad t_2 = s a_n + (1-s) a_{n+1} .$$

The operators g_m and f_n being affine, one has

$$g_m(t_1) = r(-b_m v_m) + (1-r)(-b_{m+1} v_{m+1})$$

and

$$f_n(t_2) = s a_n u_n + (1-s) a_{n+1} u_{n+1} .$$

Let us determine the elements u^s and v^r from the relations

$$s a_n u_n + (1-s) a_{n+1} u_{n+1} = (s a_n + (1-s) a_{n+1})\, u^s$$

and

$$r(-b_m v_m) + (1-r)(-b_{m+1} v_{m+1}) = (r(-b_m) + (1-r)(-b_{m+1}))\, v^r .$$

Since $u_n \geq u_{n+1}$ and $v_m \leq v_{m+1}$ we have $u_{n+1} \leq u^s \leq u_n$ and $v_m \leq v^r \leq v_{m+1}$ and accordingly $v^r \leq u^s$. One has further $g_m(t_1) = t_1 v^r$ and $f_n(t_2) = t_2 u^s$ and since $\lambda t_1 + (1-\lambda) t_2 = 0$ we obtain

$$\lambda g_m(t_1) + (1-\lambda) f_n(t_2) = \lambda t_1 v^r + (1-\lambda) t_2 u^s =$$

$$= (1-\lambda) t_2 (u^s - v^r) \geq 0$$

that is, we have got the needed relation.

Finally, we have to verify the relation (7) when one of the endpoints t_1 or t_2 is zero. Suppose first that $t_1 = 0$, $t_2 \in [a_{n+1}, a_n]$. Then $\lambda 0 + (1-\lambda) t_2 \in [a_{h+1}, a_h]$ for some $h \geq n$. Therefore

(11) $f(\lambda 0 + (1-\lambda) t_2) = f_h((1-\lambda) t_2) = \lambda f_h(0) + (1-\lambda) f_h(t_2)$.

We have from the other hand

(12) $\lambda f(0) + (1-\lambda) f(t_2) = (1-\lambda) f_n(t_2)$.

We shall evaluate $f_h(0)$ as follows

$$f_h(0) = (0 - a_{h+1}) \frac{a_h u_h - a_{h+1} u_{h+1}}{a_h - a_{h+1}} + a_{h+1} u_{h+1} =$$

$$= \frac{a_{h+1} a_h (u_{h+1} - u_h)}{a_h - a_{h+1}} \leq 0$$

since $u_{h+1} - u_h \leq 0$ and $a_h - a_{h+1} > 0$. Using this relation and the relations (5), by (11) and (12) it holds

$$f(\lambda 0 + (1-\lambda) t_2) \leq (1-\lambda) f_h(t_2) \leq (1-\lambda) f_n(t_2) = \lambda f(0) + (1-\lambda) f(t_2).$$

Similarly, if $t_1 \in [-b_m, -b_{m+1}]$, $t_2 = 0$, then $\lambda t_1 \in [-b_k, -b_{k+1}]$ for some $k \geq m$ and hence

(13) $f(\lambda t_1 + (1-\lambda) 0) = g_k(\lambda t_1 + (1-\lambda) 0) = \lambda g_k(t_1) + (1-\lambda) g_k(0)$,

and from the other hand

$$(14) \qquad \lambda f(t_1) + (1-\lambda)f(0) = \lambda g_n(t_1) \ .$$

Evaluate now $g_k(0)$. We have

$$g_k(0) = (0+b_{k+1}) \frac{b_k v_k - b_{k+1} v_{k+1}}{b_k - b_{k+1}} - b_{k+1} v_{k+1} =$$

$$= \frac{b_{k+1} b_k (v_k - v_{k+1})}{b_k - b_{k+1}} \leqslant 0$$

since $v_k - v_{k+1} \leqslant 0$ and $b_k - b_{k+1} > 0$. The obtained relation together with (6) gives via (14) and (13) :

$$f(\lambda t_1 + (1-\lambda)0) \leqslant \lambda g_k(t_1) \leqslant \lambda g_n(t_1) = \lambda f(t_1) + (1-\lambda)f(0) \ .$$

The proved relations show that the operator f is convex on the interval $[-b_1, a_1]$ and completes the proof.

Q. E. D.

REMARK . If we suppose that $(Y;K)$ is Archimedian then we can avoid the laborious verification of the convexity of f in the last part of the proof of Lemma 1 . Indeed, as we have shown in (N4) , Proposition 5, an Archimedian space $(Y;K)$ can be characterized by the fact that for every decreasing lower bounded sequence (w_k) it exists $\inf \{w_k - w_{k+1} : k \in N\} = 0$.

We have seen in the proof of Lemma 1 that

$$g_k(0) = \frac{b_{k+1} b_k (v_k - v_{k+1})}{b_k - b_{k+1}} \leqslant 0$$

and

$$f_h(0) = \frac{a_{h+1} a_h (u_{h+1} - u_h)}{a_h - a_{h+1}} \leqslant 0$$

and since from the Archimedian property of $(Y;K)$ one has $\inf \{u_h - u_{h+1} : h \in N\} = 0$ and $\inf \{v_{k+1} - v_k : k \in N\} = 0$ and since $\left(\frac{b_{k+1} b_k}{b_k - b_{k+1}} \right)$ and $\left(\frac{a_{h+1} a_h}{a_h - a_{h+1}} \right)$ are positive sequences converging to 0 , it follows that $\sup\limits_k g_k(0) = 0$ and $\sup\limits_h f_h(0) = 0$. The operator $f : [-b_1, a_1] \to (Y;K)$ defined by the relation

$$f(t) = \sup \{ g_m(t) , f_n(t) : n , m \in N \}$$

is then well defined and convex being a supremum of affine operators.

It was shown in (F) and (B) that the lineally closedness of the positive cone in an ordered vector space is not necessary to this space possess the subgradient property. For cones in the sense of our paper (as well as in the examples in (F) and (B)) lineally closedness and the Archimedian property are equivalent (see Proposition 4 in (N4)). Hence we can assert that the Archimedian property is not necessary in order to an ordered vector space have the subgradient property. This is a reason to avoid the condition of Archimedianity in our considerations.

LEMMA 2. *Let $\{(v_m);(u_n)\}$ satisfy the condition in (b') and suppose that $v_m , u_n \in K$ for any m and n . Then there can be defined the sequences (a_n) and (b_n) in R_+ both strictly decreasing to 0 such that Δ''_m , Δ_{mn} and Δ'_n defined with the above v_m , u_n , b_m , a_n by (2) satisfy the relations (1) .*

Proof. According to the condition (b') there exist $\alpha_n > 0$ and $\beta_m > 0$ such that

(15) $$u_{n-1} - u_n \geq \alpha_n(u_n - u_{n+1})$$

and

(16) $$v_m - v_{m-1} \geq \beta_m(v_{m+1} - v_m)$$

for any m and n. Put $a_1 = b_1 = 1$, $a_2 = b_2 = 2^{-1}$. Let us verify directly that $\Delta_1'' \leq \Delta_{1,1} \leq \Delta_1'$. These relations are in our case of form

$$\frac{v_1 - 2^{-1}v_2}{1 - 2^{-1}} \leq \frac{2^{-1}v_2 + 2^{-1}u_2}{2^{-1} + 2^{-1}} \leq \frac{u_1 - 2^{-1}u_2}{1 - 2^{-1}} .$$

The first relation is in fact $4v_1 - 2v_2 \leq v_2 + u_2$ which is obvious since it is equivalent with $3(v_1 - v_2) \leq u_2 - v_1$. The second relation reduces to $v_2 + u_2 \leq 4u_1 - 2u_2$ which is equivalent with the obvious relation $v_2 - u_1 \leq 3(u_1 - u_2)$.

Suppose that $a_1, \dots, a_n$ and $b_1, \dots, b_m$ were constructed so as to have (1) for k in place of m and h in place of n with $k \leq m-1$ and $h \leq n-1$. Then

$$\Delta''_{m-1} \leq \Delta_{m-1,n-1} \leq \Delta'_{n-1}$$

and we shall determine a_{n+1} and b_{m+1} such that $0 < a_{n+1} \leq 2^{-1}a_n$ and $0 < b_{m+1} \leq 2^{-1}b_m$ and to have (1).

Let us consider

$$\Delta_m(s) = \frac{b_m v_m - s v_{m+1}}{b_m - s}$$

and shall show that there exists some s with the property $0 < s \leq 2^{-1}b_m$ such that $\Delta''_{m-1} \leq \Delta''_m(s)$. Since

$$\Delta''_{m-1} = \frac{b_{m-1}v_{m-1} - b_m v_m}{b_{m-1} - b_m} = \frac{b_{m-1}}{b_{m-1} - b_m}(v_{m-1} - v_m) + v_m$$

and

$$\Delta''_m(s) = \frac{b_m v_m - s v_{m+1}}{b_m - s} = v_m + \frac{s}{b_m - s}(v_m - v_{m+1})$$

it suffices to show that there exist some s, $0 < s \leq 2^{-1}b_m$ such that

$$\frac{b_{m-1}}{b_{m-1} - b_m}(v_m - v_{m-1}) \geq \frac{s}{b_m - s}(v_{m+1} - v_m) .$$

According (16) this relation holds if

$$\frac{s}{b_m - s} \, \frac{b_{m-1} - b_m}{b_{m-1}} \leq \beta_m$$

relation which can be realized for sufficiently small s. Choose a such s satisfying $0 < s \leq 2^{-1}b_m$ and put $b_{m+1} = s$.

Consider now

$$\Delta'_n(r) = \frac{a_n u_n - r u_{n+1}}{a_n - r} .$$

We shall show that there exists an r, $0 < r \leq 2^{-1}a_n$ such that $\Delta'_n(r) \leq \Delta'_{n-1}$. Since

$$\Delta'_{n-1} = \frac{a_{n-1}u_{n-1} - a_n u_n}{a_{n-1} - a_n} = \frac{a_{n-1}}{a_{n-1} - a_n}(u_{n-1} - u_n) + u_n$$

and

$$\Delta_n'(r) = u_n + \frac{r}{a_n - r}(u_n - u_{n+1})$$

it is sufficient to get some r, $0 < r \leq 2^{-1}a_n$ such that

$$\frac{a_{n-1}}{a_{n-1} - a_n}(u_{n-1} - u_n) > \frac{r}{a_n - r}(u_n - u_{n+1}) .$$

According (15) this can be realized for sufficiently small positive r which satisfies the relation

$$\frac{r}{a_n - r}\,\frac{a_{n-1} - a_n}{a_{n-1}} < \alpha_n .$$

Take an $r > 0$ satisfying this relation such that $r \leq 2^{-1}a_n$ and put $a_{n+1} = r$.

Let us verify finally the relations

(17) $$\Delta_m'' \leq \Delta_{mn} \leq \Delta_n' .$$

We have

$$\Delta_m'' = \frac{b_m}{b_m - b_{m+1}}(v_m - v_{m+1}) + v_{m+1}$$

and

$$\Delta_{mn} = \frac{a_{n+1}}{a_{n+1}+b_{m+1}}(u_{n+1} - v_{m+1}) + v_{m+1} .$$

The first relation in (17) reduces then to

$$\frac{b_m}{b_m - b_{m+1}}(v_m - v_{m+1}) < \frac{a_{n+1}}{a_{n+1}+b_{m+1}}(u_{n+1} - v_{m+1})$$

which is obvious.

Similarly, using the representations

$$\Delta_n' = \frac{a_n}{a_n - a_{n+1}}(u_n - u_{n+1}) + u_{n+1}$$

and

$$\Delta_{mn} = u_{n+1} + \frac{b_{m+1}}{a_{n+1}+b_{m+1}}(v_{m+1} - u_{n+1})$$

the second relation in (17) reduces to

$$\frac{b_{m+1}}{a_{n+1}+b_{m+1}}(v_{m+1} - u_{n+1}) \leq \frac{a_n}{a_n - a_{n+1}}(u_n - u_{n+1})$$

which is obvious too. This completes the proof of the lemma.

Q. E. D.

<u>The proof of Proposition 1.</u> Consider an ordered vector space $(Y;K)$ which lacks property (b') . Then there is some $\{(v_m);(u_n)\}$ such that it holds the relation

$$\bigcap_{m,n \in N} [v_m, u_n] = \emptyset .$$

We can assume that v_m , $u_n \in K$ for any m and n . According to Lemma 2 we can construct the real sequences (a_n) and (b_m) decreasing strictly to 0 so as to have (1) with Δ_m'', Δ_{mn} and Δ_n' defined by (2) with the above entries. Then we can construct a convex operator $f : [-b_1, a_1] \to (Y;K)$ by Lemma 1 such that $f(0) = 0$, $f(-b_m) = -b_m v_m$ and $f(a_n) = a_n u_n$ for any m and n . Suppose that $A \in \partial f(0)$. Then we have

$$v_n = -b_m^{-1}f(-b_m) = -b_m^{-1}(f(-b_m) - f(0)) \leqslant A(1) \leqslant$$
$$\leqslant a_n^{-1}(f(a_n) - f(0)) = a_n^{-1}f(a_n) = u_n$$

for any m and n. This means that $A(1)$ is an element in $(Y;K)$ such that $v_m \leqslant A(1) \leqslant u_n$ for any m and n, wherefrom

$$A(1) \in \bigcap_{m,n\in N} [v_m,u_n]$$

in contradiction with the hypothesis. The obtained contradiction shows that $\partial f(0) = \emptyset$.

In conclusion, if $(Y;K)$ possesses the subgradient property it must have property (b').

Q. E. D.

3. Application to ordered normal Banach spaces

PROPOSITION 2. Let $(Y;K)$ be a Banach space ordered by the closed normal cone K. If $(Y;K)$ has the subgradient property then it possesses property (b).

We prove first the following lemma:

LEMMA 3. If $(Y;K)$ is a Banach space ordered by the closed normal cone K then it has the property (b) if and only if it has the property (b').

Proof. Property (b) implies obviously the property (b'). To see the converse we consider an arbitrary $\{(v_m);(u_n)\}$ with v_m and u_n in K for any m and n in N. We shall show that there can be constructed $\{(v_m');(u_n')\}$ which satisfies the

conditions in (b') and for which $v_m \leqslant v_m' \leqslant u_n$ and $v_m \leqslant$ $\leqslant u_n' \leqslant u_n$ for any m,n in N. Hence

$$\bigcap_{m,n\in N} [v_m',u_n'] \subset \bigcap_{m,n\in N} [v_m,u_n]$$

and if the first set is nonempty it follows that the second set is nonempty too. But if $(Y;K)$ has the property (b') this is the case and accordingly property (b') implies property (b) and we are done.

Let we construct now (v_m') and (u_n') by putting

$$v_m' = \sum_{k=0}^{\infty} 2^{-(k+1)} v_{m+k} \quad \text{and} \quad u_n' = \sum_{k=0}^{\infty} 2^{-(k+1)} u_{n+k} .$$

From the normality of K it follows that the sequences (v_m) and (u_n) are norm bounded and hence v_m' and u_n' are well defined since the series in their definitions are convergent. Since (v_m) is increasing, $v_{m+k} \geqslant v_m$ for $k \geqslant 0$ and since the cone K is closed we have

$$v_m' \geqslant v_m \sum_{k=0}^{\infty} 2^{-(k+1)} = v_m .$$

We can show similarly that $u_n' \leqslant u_n$, $v_m' \leqslant v_{m+1}'$ and $u_{n+1}' \leqslant u_n'$. Since $v_m \leqslant u_n$ for any m and n we have also $v_n' \leqslant u_n'$ for any m and n, that is, we can write $\{(v_m');(u_n')\}$. Let us check finally the condition in the definition of (b') for $\{(v_m');(u_n')\}$. We have

$$v_m' = \sum_{k=0}^{\infty} 2^{-(k+1)} v_{m+k}$$

and hence $2v_m' = v_m + v_{m+1}'$, thist relation yielding

$$v'_n - v'_{n-1} = \tfrac{1}{2}(v'_{n+1} - v'_n) + \tfrac{1}{2}(v_n - v_{n-1}) \geq \tfrac{1}{2}(v'_{n+1} - v'_n) .$$

We have similarly $2u'_n = u_n + u'_{n+1}$ wherefrom it follows

$$u'_n - u'_{n-1} = \tfrac{1}{2}(u'_{n+1} - u'_n) + \tfrac{1}{2}(u_n - u_{n-1}) \leq \tfrac{1}{2}(u'_{n+1} - u'_n) .$$

It follows that the conditions in the definition of the property (b') hold for $\alpha_n = \beta_n = 1/2$ for any m and n and the proof of the lemma is complete.

Q. E. D.

The proof of Proposition 2 follows now directly from Lemma 3 and Proposition 1 since if $(Y;K)$ has the subgradient property then it must have the property (b') by Proposition 1 and property (b') implies property (b) by Lemma 3.

Q.E. D.

PROPOSITION 3. Let $(Y;K)$ be a Banach space ordered by the closed normal cone K. Every convex operator $f : I \to (Y;K)$ with I a convex set in R, possesses nonempty subdifferential at every interior point of I if and only if $(Y;K)$ has the property (b).

Proof. The necessity of the condition (b) follows from the lemmas 3, 1 and 2. To show the sufficiency we assume that $0 \in \operatorname{int} I$, and define φ by $\varphi(t) = t^{-1}(f(t) - f(0))$, $t \in I \setminus \{0\}$. For any sequences (a_n) and (b_m) decreasing strictly to 0 one has

$$\varphi(-b_{m-1}) \leq \varphi(-b_m) \leq \varphi(a_n) \leq \varphi(a_{n-1})$$

for every m and n since φ is an increasing operator on $I \setminus \{0\}$. Hence if we consider $\{(\varphi(-b_m));(\varphi(a_n))\}$ the property (b) of

$(Y;K)$ assures the existence of an element y with the property that $\varphi(-b_m) \leq y \leq \varphi(a_n)$ for every m and n. From the monotonicity of φ and from the fact that (a_n) and (b_m) converge to 0, we have $\varphi(t_1) \leq y \leq \varphi(t_2)$ for $t_1 < 0$ and $t_2 > 0$ in I. Define the linear operator $A : R \to Y$ by putting $At = ty$. Then we have for every t in R

$$At = ty \leq t\varphi(t) = f(t) - f(0)$$

which shows that A is in $\partial f(0)$. The general case can be reduced by translations to the above one.

Q. E. D.

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LA CONVERGENCE DE CERTAINES MÉTHODES ITÉRATIVES POUR RÉSOUDRE CERTAINES ÉQUATIONS OPÉRATORIELLES

Ion Păvăloiu

Désignons par E un espace de Banach et considérons une équation opératorielle

$$x = \lambda D(x) + y \tag{1}$$

où $D : E \to E$ est une application nonlinéaire.

Dans ce qui suit et pour resoudre l'équation (1), nous considérons le procédé itératif suivant:

$$x_{n+1} = x_n - A(x_n)\left[x_n - \lambda D(x_n) - y\right], \quad n = 0,1, \ldots, \quad x_0 \in E \tag{2}$$

où $A(x) : E \to E$ est une application lineaire pour chaque $x \in E$.

Nous désignerons, pour fixer les idées, par $F : E \to E$ l'application $F(x) = x - \lambda D(x) - y$.

Si nous supposons que l'application D admet des dérivées jusqu'au second ordre inclusivement sur l'espace E, alors $F'(x) = I - \lambda D'(x)$ et $F''(x) = - \lambda D''(x)$. Si $r \in \mathbb{R}$ et $r > 0$, nous