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KNOWN AND NEW EQUIVALENT FORMS OF THE ARCHIMEDIAN PROPERTY OF ORDERED VECTOR SPACES

A. B. Németh

The Archimedian property of the ordered vector spaces, its equivalent or weaken forms become important in some recent results on vectorial optimization (see e.g., (F) , (N1) (B) (N2) etc.). This suggests us to gather the various equivalent forms, to estabilish new equivalences and to present them together in order to facilite further references.

The Archimedian property involves in its statement two elements of the space, hence it is typically finite dimensional: it holds for the whole space if and only if it holds for its finite dimensional subspaces. It is in fact a geometrical property concerning straight lines. The Archimedian property is equivalent with the lineally closedness of the positive cone of the ordered vector space. The two different approaches : Archimedian property and the lineally closedness subzist in the literature perhaps since the first concerns with vector spaces ordered by cones while the second term is used merely in the case when the vector spaces are ordered by wedges (see e.g. (SY), (D), (F) etc.).

We shall explicitely state and prove the above equivalence, which was used implicitely by us in comments in (N3). A new sequential characterization of Archimedian ordered vector spaces is then given which will be used in (N5). A weak subdifferentiability-like property is also characteristic for Archimedian ordered vector spaces which was also considered implicitely in (N4) and will be explicitely presented here.

The non-empty subset W in the real vector space Y is called a <u>wedge</u> if (i) $W + W \subset W$ and (ii) $tW \subset W$ whenever $t \in R_+ = [0, \infty)$. The wedge W induces a reflexive and transitive relation in Y if we put $u \leqslant v$ when $v - u \in W$. The relation is traslation invariant and invariant with respect to the multiplication with non-negative reals.

The wedge K in Y is called a <u>cone</u> if in plus (iii) $K \cap (-K) = \{0\}$. The order relation in Y induced by K is then also antisymmetrical. The vector space Y ordered by a cone is called an <u>ordered vector space</u> and it will be denoted by $(Y;K)$. In some problems concerning order structures on vector spaces it is natural to consider only orderings induces by wedges (see e.g. the generalized Hahn-Banach theorem and its equivalences (SY), (D), (EN), the directional minorability and support operators to sublinear operators (F) and (B), etc.).

A set M in Y is called <u>lineally closed</u> if its intersection with an arbitrary straight line in Y determines a subset of this line which is closed in the usual topology of this line identified with the real axis. As an immediate consequence of this definition we observe that if M is lineally closed then

$t_0u + (1-t_0)v \in M$ whenever $tu + (1-t)v \in M$ for every $t < t_0$ (or every $t > t_0$) in a neighbourhood of t_0.

The ordered vector space $(Y;K)$ is called <u>Archimedian</u> if for any v in K and u in Y from the relation $tv \geqslant u$ for every $t > 0$ it follows that $u \leqslant 0$. It is immediate that we have the following equivalent forms of the property: $v \geqslant tu$ for every $t > 0$ implies $u \leqslant 0$, or, $v \geqslant nu$ for every $n \in N$ implies $u \leqslant 0$.

1. PROPOSITION. <u>The ordered vector space</u> $(Y;K)$ <u>is Archimedian if and only if for each subspace</u> Y_0 <u>of</u> Y <u>with</u> $\dim Y_0 = 2$ <u>the ordered vector space</u> $(Y_0;K\cap Y_0)$ <u>is Archimedian.</u>

<u>Proof</u>. Suppose that $(Y;K)$ is Archimedian and take x in $K \cap Y_0$ and y in Y_0 for which it holds $y \leqslant_{K\cap Y_0} tx$ for every $t > 0$. This means that $tx - y \in K \cap Y_0 \subset K$ for every $t > 0$. Thus $y \leqslant 0$. But $y \in Y_0$ and $y \in -K$, hence $y \in Y_0 \cap (-K) = -(Y_0 \cap K)$, that is $y \leqslant_{K\cap Y_0} 0$ which shows that $(Y_0;K\cap Y_0)$ is Archimedian.

Suppose now every subspace in Y of dimension 2 is Archimedian with respect to the induced order. Suppose that $y \leqslant tx$ for some x in K and y in Y and for every $t > 0$. Consider the subspace $Y_0 = sp\{x,y\}$ of Y. Then we have $tx - y \in Y_0$ for every $t > 0$ and since $tx - y \in K$ for every $t > 0$, we have $tx - y \in K \cap Y_0$ for every $t > 0$. The space $(Y_0;K\cap Y_0)$ being Archimedian, it follows that $y \in -K \cap Y_0 \subset -K$, that is, $y \leqslant 0$, which shows that $(Y;K)$ is Archimedian.

Q. E. D.

2. LEMMA. <u>If a set of the form</u> $\{tx : t > 0\}$ <u>with</u> $x \geqslant 0$ <u>in an arbitrary ordered vector space has an infimum, this infimum must be</u> 0 .

<u>Proof</u>. Suppose that $y = \inf \{tx : t > 0\}$ for some $x \geqslant 0$. Then $y \geqslant 0$ and since $y \leqslant tx$ for arbitrary $t > 0$, it holds also $sy \leqslant rx$ for any $r > 0$ and $s > 0$. Put $s = 2$. Then $2y$ is a minorant of the set $\{rx : r > 0\}$ and hence $2y \leqslant y$, wherefrom $y \leqslant 0$. This shows that $y = 0$.

Q. E. D.

3. PROPOSITION. <u>The ordered vector space</u> $(Y;K)$ <u>is Archimedian if and only if for every</u> x <u>in</u> K <u>there exists</u> $\inf \{tx : t > 0\}$.

<u>Proof</u>. If $(Y;K)$ is Archimedian and $y \leqslant tx$ for some x in K , some y in Y and for every $t > 0$, then $y \leqslant 0$. This shows in fact that $0 = \inf \{tx : t > 0\}$.

Suppose now that for each x in K the infimum $\inf \{tx : t > 0\}$ exists. From Lemma 2 it follows that this infimum must be 0 . Now if we have $y \leqslant tx$ for some y in Y and every $t > 0$, it follows that y is a minorant of the above considered set and hence it must hold $y \leqslant 0$. This shows that $(Y;K)$ is Archimedian.

Q. E. D.

The result in Proposition 3 was used in (N3) and (N4) and was observed by the referee of (N3) , by J.M. Borwein and independently by the author in (N2) .

4. PROPOSITION. <u>The ordered vector space</u> $(Y;K)$ <u>is Archimedian if and only if</u> K <u>is lineally closed</u>.

<u>Proof</u>. Consider a straight line Δ and suppose that $(Y;K)$ is Archimedian. Consider the set $\Delta \cap K$. If it is empty or if coincides with Δ then the intersec tion is a closed set in the topology of Δ . Suppose that $\Delta \cap K$ is not empty and does not coincide with Δ . Let $x \in \Delta \cap K$. If x is the single element of the intersection, then it is a point in Δ and hence is closed. If not, it is an interval in $K \cap \Delta$. Let y be an endpoint of this interval. We have to show that $y \in \Delta \cap K$. If $y = x$ we are done. If not, from the convexity of $\Delta \cap K$ it follows that for every s in $(0,1)$ one has $sy + (1-s)x \in \Delta \cap K \subset K$, that is, $sy + (1-s)\,x \geqslant 0$ and hence $-y \leqslant \frac{1-s}{s}\,x$ for every s in $(0,1)$. But then $-y \leqslant tx$ for each $t > 0$ and since $(Y;K)$ is Archimedian, it follows that $-y \leqslant 0$, that is y is in K . This shows that K is lineally closed.

Suppose now that K is lineally closed and let x in K and y in Y have the property that $y \leqslant tx$ for every $t > 0$. Consider the line $\{sx + (1-s)(-y) : s \in R\}$. Then the set

$$\left\{ \frac{t}{1+t}\,x + \frac{1}{1+t}\,(-y) : t > -1 \right\}$$

will be an open semi-line of it and we have by hypothesis

$$u(t) := \frac{t}{1+t}\,x + \frac{1}{1+t}\,(-y) \in K$$

for every $t > 0$. Since K is lineally closed and the involved coefficients are continuous functions of t at $t = 0$, it follows that $u(0) \in K$. This means that $-y \in K$, that is , $y \leqslant 0$. This shows that $(Y;K)$ is Archimedian.

Q. E. D.

The assertion in Proposition 4 was implicitely used in the comments in (N3) and will be used in the remark after Lemma 1 in (N5).

5. PROPOSITION. The ordered vector space $(Y;K)$ is Archimedian if and only if for every increasing sequence (x_n) which is bounded from above there exists $\inf \{x_{n+1} - x_n : n \in N\}$. If the above type infima exist, then they must be 0.

Proof. Suppose that $(Y;K)$ is Archimedian and let (x_n) be an increasing sequence having as upper bound the element u. We have obviously $x_{n+1} - x_n \geqslant 0$ for every n and consider an arbitrary lower bound v of the set $\{x_{n+1} - x_n : n \in N\}$. Then

$$x_2 - x_1 \geqslant v$$
$$x_3 - x_2 \geqslant v$$
$$\cdots\cdots\cdots\cdots$$
$$x_{n+1} - x_n \geqslant v$$

and adding these relations we get

$$u - x_1 \geqslant x_{n+1} - x_1 \geqslant nv .$$

That is, we have $nv \leqslant u - x_1$ with $u - x_1$ in K and for every n in N. Then since $(Y;K)$ is Archimedian we get $v \leqslant 0$. This shows that

$$0 = \inf \{x_{n+1} - x_n : n \in N\}.$$

To prove the converse implication suppose that x is in K. Consider the sequence $(-\frac{1}{n}x)$. It is increasing and bounded from above by 0. Hence the set

$$\{ (-\tfrac{1}{n+1} + \tfrac{1}{n})\, x = \tfrac{1}{n(n+1)}\, x : n \in N \}$$

has an infimum as far as the condition in the proposition holds. Since $\frac{1}{n(n+1)}$ decreases to 0 with $n \to \infty$ and since x is in K, it follows that the set $\{tx : t > 0\}$ has the same infimum. Using now Proposition 3 it follows that the ordered vector space $(Y;K)$ is Archimedian and the proof is complete.

Q. E. D.

Let X be a real vector space. The operator $P : X \to (Y;K)$ is called sublinear if it is positively homogeneous, that is, if $P(tx) = tP(x)$ for every x in X and every $t \geqslant 0$, and if it is subadditive, that is, if $P(x_1 + x_2) \leqslant P(x_1) + P(x_2)$ for any x_1 and x_2 in X.

The linear operator $A : X \to Y$ is called a bounding operator for P if there exists some y in Y such that

$$Ax + y \leqslant P(x)$$

for each x in X. The linear operator $A : X \to Y$ is called a supporting operator of P if

$$Ax \leqslant P(x)$$

for each x in X. Each supporting operator of P is obviously also a bounding operator for P. The converse implication characterizes the Archimedian ordered vector spaces. More precisely we have :

6. PROPOSITION. The ordered vector space $(Y;K)$ is Archimedian if and only if each bounding operator to each

sublinear operator with values in $(Y;K)$ is also a supporting operator of this sublinear operator.

Proof. Suppose that $(Y;K)$ is Archimedian and let $P : X \to (Y;K)$ be a sublinear operator with a bounding operator A. That is, we have for some y in Y the relation

$$Ax + y \leqslant P(x)$$

for every x in X. Let we fix x for the moment and put tx in place of x in the above relation with an arbitrary $t > 0$. Then it holds

$$t(P(x) - Ax) - y \in K$$

and

$$\left\{ \frac{t}{1+t}(P(x) - Ax) + \frac{1}{1+t}(-y) \ : \ t > 0 \right\}$$

is a line segment in K. Since $(Y;K)$ is Archimedian, it follows by Proposition 4 that K is lineally closed. Hence the limit point of the above line segment for $t \to \infty$ must be in K. This means that

$$P(x) - Ax \in K.$$

Since x was arbitrarily chosen, the obtained relation shows that A is a supporting operator for P.

To verify the converse implication we shall show that if $(Y;K)$ is not Archimedian, then it can be constructed a sublinear operator which has a bounding operator which is

not a supporting operator too.

Assume that $(Y;K)$ is not Archimedian. Then there exists some a in K and some b in Y such that $ta \geqslant b$ (or equivalently, $a \geqslant tb$) for each $t > 0$, but $b \not\leqslant 0$. Let us define the operator $P : R \to (Y;K)$ by putting $P(t) = t(a-b)$ for $t \geqslant 0$ and $P(t) = -ta$ if $t < 0$. Let we see that P is sublinear. It is obviously positively definite. To verify its subadditivity it suffices to show that

$$P(t_1 + t_2) \leqslant P(t_1) + P(t_2)$$

for $t_1 < 0$ and $t_2 > 0$. If $t_1 + t_2 \geqslant 0$ then $P(t_1 + t_2) = (t_1 + t_2)(a - b)$, $P(t_1) = -t_1 a$, $P(t_2) = t_2(a - b)$ and hence it holds

$$P(t_1 + t_2) = t_1(a - b) + t_2(a - b) \leqslant -t_1 a + t_2(a - b) = P(t_1) + P(t_2).$$

If $t_1 + t_2 < 0$, then we have $P(t_1 + t_2) = -(t_1 + t_2)a$ and then

$$P(t_1 + t_2) = -t_1 a - t_2 a \leqslant -t_1 a \leqslant -t_1 a + t_2(a - b) = P(t_1) + P(t_2).$$

Let the linear operator $A : R \to Y$ be defined by $At = ta$. We shall show that

$$P(t) \geqslant At - a$$

for each t in R. It is sufficient to verify the relation for $t \neq 0$. We have for $t > 0$ the relations

$$P(t) = t(a - b) = ta - tb \geqslant ta - a = At - a$$

since $-tb \geq -a$ for each $t > 0$ by hypothesis.

If $t < 0$ then obviously

$$P(t) = -ta \geq ta \geq ta - a = At - a .$$

In conclusion, A is a bounding operator for P . Assume that it is also a supporting operator of P , i. e. , that

$$P(t) \geq At$$

for every t in R . We have then for $t = 1$ the relations

$$a - b = P(1) \geq A(1) = a$$

wherefrom it follows that $b \leq 0$. The obtained contradiction completes the proof.

Q. E. D.

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