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C O N T E N T S

$$-\varepsilon \leq \|F_1 - s\| - \|s - G_1\| \leq \varepsilon$$

$$\|s - G_1\| - \varepsilon \leq \|F_1 - s\| \leq \|s - G_1\| + \varepsilon .$$

Similarly

$$\|s - G_2\| - \varepsilon \leq \|F_2 - s\| \leq \|s - G_2\| + \varepsilon$$

These inequalities give

$$\max\{\|F_1 - s\|, \|F_2 - s\|\} \leq \max\{\|s - G_1\|, \|s - G_2\|\} + \varepsilon$$

and

$$\min\{\|s - G_1\|, \|s - G_2\|\} - \varepsilon \leq \min\{\|F_1 - s\|, \|F_2 - s\|\},$$

and the Corollary is proved .

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DOMINATION OF CONES AND SUBDIFFERENTIABILITY OF CONVEX OPERATORS [*)]

by A. B. Németh

Vector spaces with two cones or two different orderings were already considered by M. A. Krasnosel'skii in his monograph ((K)). They play an important role in many problems concerning positive operators, nonlinear operators and vectorial optimization. In these applications appeared the notion of domination of cones.

Domination is also important in some constructions due to M. M. Fel'dman ((F)) and recenly developed by J. M. Borwein ((B)) in order to exhibit examples of ordered vector spaces in which every convex operator is subdifferentiable, but which lack the chain completeness property. This technique is also used in the present note for the investigation of the subdifferentiability of a convex operator when the positive cone of the space dominates a cone with good properties. The obtained results furnish examples where subdifferentiability takes place in some restricted sense and give indications on the properties of a positive cone for which the subdifferentiability of convex operators fails.

In order to simplify the exposition we shall consider only spaces with pointed (proper) positive cones. This context is rich enough to include the most important situations which can appear.

*) This note is in final form and no version of it is or will be, submitted for publication elsewhere.

0. Definitions

Throughout this note the vector spaces will be considered over the reals.

The subset Q in the vector space Y is called a <u>cone</u> if (i) $Q + Q \subset Q$, (ii) $tQ \subset Q$ whenever $t \in R_+ := [0,+\infty)$, and (iii) $Q \cap (-Q) = \{0\}$. We shall suppose that cones are nontrivial ($\neq 0$) but we shall admit $\{0\}$ as a <u>subcone</u> of a cone. By putting

$$u \leqslant v \quad \text{if} \quad v - u \in Q ,$$

the cone Q <u>induces</u> a reflexive, transitive and antisymmetrical order relation in Y related to the linear structure by the properties

$u \leqslant v$ implies $u+z \leqslant v+z \;\forall z \in Y$ and $u \leqslant v$ impies $tu \leqslant tv \;\forall t \in R_+$.

The obtained object : the space Y with this order relation is called an <u>ordered vector space</u>, while Q is termed as its <u>positive cone</u>. The set M in Y has a <u>lower bound</u> z if $z \leqslant x, \forall x \in M$. Similarly, w is an <u>infimum</u> of M if it is a lower bound of M and $z \leqslant w$ for any other lower bound z . Infima are unique.

An ordered vector space is said to have the <u>chain completeness property</u> if every decreasing transfinite sequence which has a lower bound admits an infimum. If the ordered vector space is endowed with a locally convex Hausdorff vector space topology and if every decreasing transfinite sequence with a lower bound is convergent, then the respective space is called <u>regular</u>. If the positive cone of a regular ordered vector space is closed, then it is chain complete (see e.g. ((P)), Corollary II. 3. 2) .

An ordered vector space is called <u>latticially complete</u> (conditionally order complete) if all the subsets in the positive cone admit infima.

If Y is an ordered vector space we shall adjoin an abstract maximal element ∞ (infinity) to Y and shall denote the resulting object by $Y^{\cdot} = Y \cup \{\infty\}$. Infinity satisfies $t\infty = \infty$ if $t > 0$, $y + \infty = \infty$, $0\,\infty = 0$ and $y \leqslant \infty, \forall y \in Y$.

We shall next frequenly deal with two cones in Y which induce two different orderings but always one of them contains the other, hence we can consider the adjoint infinities to be identical. When necessary, we distinguish the order relation induced by Q by writing $\leqslant_Q$. The simple notation $Y^{\cdot}$ means that the positive cone is indifferent.

An operator F from the vector space X to $Y^{\cdot}$ is called <u>convex</u> if

$$F(tx_1 + (1-t)x_2) \leqslant tF(x_1) + (1-t)F(x_2)$$

whenever x_1, x_2 lie in X and $t \in [0,1]$. We shall use the term <u>Q-convex</u> when this distinction is necessary.

Let $L(X,Y)$ stand the set of linear operators from X to Y . The <u>subdifferential</u> of F at x_0 X is defined by

$$\partial F(x_0) := \{A \in L(X,Y) : F(x) - F(x_0) \leqslant A(x - x_0) , \forall x \in X\} .$$

The elements of $\partial F(x_0)$ are the so-called <u>subgradients</u> of F at x_0 . We use also the notation $\partial_Q F(x_0)$ and the term Q-subgradient when necessary.

If P and Q are cones in Y with $P \subset Q$, then P-convex operators are obviously also Q-convex and P-subgradients are Q-subgradients too.

We define the <u>domain</u> of an operator $F : X \to Y^{\cdot}$ by

$$\operatorname{dom} F := \{ x \in X : F(x) \in Y \} .$$

The *intrinsic core*, $\operatorname{icr} M$, of a set M in X is the algebraic interior of M relatively to the linear manifold it spans.

The intrinsic core of the domain of $F : X \to Y^{\cdot}$ will be denoted $\operatorname{icr} F$. The operator F is said *subdifferentiable at* $x_0 \in X$ if $\partial F(x_0) \neq \emptyset$. It is called simply *subdifferentiable* if $\partial F(x) \neq \emptyset$ for any x in $\operatorname{icr} F$.

1. Domination and facial structure

Let us say that a convex cone P *dominates* the convex cone Q or Q is *dominated* by P if

$$(P \setminus \{0\}) + Q \subset P .$$

(Remember our agreement to consider only cones different from $\{0\}$.) In the important applications of dominating cones one has also $P \subset Q$. If P is the (nonempty) interior, the (nonempty) algebraic core or the (nonempty) quasi-interior ((Sch)) of Q then we are in this last situation, that is we have

$$(\mathrm{PdQ}) \qquad (P \setminus \{0\}) + Q \subset P \subset Q .$$

It turns out that the condition (PdQ) is intimately related to the facial structure of the cone Q. Hence we shall give some notions and some basic facts related to this structure.

The subcone R of Q is called its *face* if it verifies the relation $R = (R-Q) \cap Q$. The cone Q is itself a face. A face of Q different from Q is called *proper*. The family of faces of Q forms a complete lattice of sets if the lattice operations are defined by $\bigwedge_{\alpha} R_{\alpha} = \bigcap_{\alpha} R_{\alpha}$ and $\bigvee_{\alpha} R_{\alpha} = \langle \bigcup_{\alpha} R \rangle$ where $\langle M \rangle$ denotes the intersection of all the faces of Q containing

M. The face $\langle x \rangle$ (the minimal face containing x or the face engendered by x) can be represented in the form

$$\langle x \rangle = \{ y \in Q : \exists\, \mu > 0 ,\ y \leq \mu x \} .$$

The *open face* of $x \in Q$ is the set defined by

$$\Phi(x) := \{ y \in Q : \exists\, \lambda > 0,\ \mu > 0,\ \lambda x \leq y \leq \mu x \} .$$

From this definition it follows that if $y \in \Phi(x)$ then $\Phi(y) = \Phi(x)$, wherefrom it follows that two different open faces are disjoint. The only open face being a singleton is $\Phi(0) = \{0\}$. Since obviously $x \in \Phi(x)$, Q is the union of disjoint open faces.

From the definition of $\langle x \rangle$ it follows that

$$(1) \qquad \Phi(x) \subset \langle x \rangle , \quad \forall\, x \in Q$$

and the equality may hold if and only if $x = 0$.

Open faces in finite dimensional case are the interiors of faces with respect to the linear manifolds they span.

We shall say that the open face Φ of Q *bounds* the open face Ψ of Q if $\langle \Phi \rangle \neq Q$ and $\Psi \subset \langle \Phi \rangle$. Since $\Phi = \Phi(x)$ for every $x \in \Phi$, by (1) one has that $\Phi \subset \langle \Phi \rangle$ and hence Φ bounds itself whenever $\langle \Phi \rangle \neq Q$.

In the following assertion which characterizes the cones in the relation (PdQ), and in its proof, faces will mean throughout faces with respect to the cone Q.

1. PROPOSITION. *Let P and Q be cones in the vector space Y, $P \subset Q$. In order to have (PdQ) it is sufficient, and under the additional hypothesis that Q is finite dimensional, it is also necessary that the following conditions hold true :*

(i) $Q \setminus P$ contains only points in proper faces of Q.

(ii) If $x \in P$ and $\langle \Phi(x) \rangle \neq Q$, then P contains all the open faces that bound (x).

Proof. To show the sufficiency of the conditions we have only to check that if $x \in P \setminus \{0\}$ and $y \in Q$, then $x + y \in P$.

If $x \in P$ is in no proper face of Q, then if $x + y$, $y \in Q$ would be in some proper face, this face would contain $\langle x + y \rangle$. But $x \in \langle x + y \rangle$ since $x \leqslant_Q x + y$. This contradiction, together with $P \subset Q$ and the condition (i) show that $x + y \in P$.

Let $\langle x \rangle$ be a proper face of Q. If $x + y$ does not lie in any proper face of Q it must be an element of P by (i).

Let $x + y$ be in a proper face of Q. Then $\langle \Phi(x+y) \rangle = \langle x + y \rangle \neq Q$. From the obvious inclusions

$$\Phi(x) \subset \langle x \rangle \subset \langle x + y \rangle \subset \langle \Phi(x+y) \rangle$$

it follows that $\Phi(x+y)$ bounds $\Phi(x)$ and hence by (ii) it holds $x + y \in \Phi(x+y) \subset P$. This proves the sufficiency.

Let Q be finite dimensional; we show the necessity of (i) and (ii).

Assume that (i) does not hold. Then there exists $x \in Q \setminus P$ which is in no proper face of Q and hence it is in the interior of Q with respect to the unique locally convex Hausdorff topology of $Q - Q$. Let $y \in P \setminus \{0\}$. Then $z := x - ty \in Q$ if $t > 0$ is small enough. Hence $x = ty + z \notin P$ with $ty \in P \setminus \{0\}$ and $z \in Q$, so we cannot have (PdQ).

Assume now that (ii) does not hold. Let $y \in P \setminus \{0\}$ with the property that P does not contain the open face Φ which bounds $\Phi(y)$. Let $x \in \Phi \setminus P$. Then since Φ is convex and open in $\Phi - \Phi$ and since y is in the boundary of Φ in this space, we have $z := x - ty \in \Phi$ for $t > 0$ small enough. But then

we have $x = z + ty$ with $ty \in P \setminus \{0\}$ and $z \in \Phi \subset Q$, and hence (PdQ) cannot be true.

Q. E. D.

2. COROLLARY. *If* Q *is finite dimensional and it is a projecting cone from* 0 *of a strictly convex set* M *with* $0 \notin M$, *then* (PdQ) *holds for every subcone* P *of* Q *such that* $\overline{P} = \overline{Q}$, *where the closure is taken with respect to the locally convex Hausdorff vector space topology of* $Q - Q$. *In particular, if* $\dim P = 2$ *and* $\overline{P}$ *is cone, then we have* $(\mathrm{Pd}\overline{P})$.

2. Subgradients at linearity sets of convex operators

Let $F : X \to Y^{\cdot}$ be a convex operator. The *linearity set* $L(x_o)$ of F at x_o $\in$ dom F is a maximal subset of dom F which contains x_o and has the property that

(2) $\forall n \in N, \forall x_1, \ldots, x_n \in L(x_o), \forall t_1, \ldots, t_n \in R_+, \sum_{i=1}^{n} t_i = 1$ it holds the relation $F(t_1x_1 + \ldots + t_nx_n) = t_1F(x_1) + \ldots + t_nF(x_n)$.

A maximal set containing x_o and having the property (2) exists since any family of sets with the property (2) totally ordered by inclusion has a maximal element: it is simply the union of the members of this family. Though $L(x_o)$ is not uniquely determined by x_o. This can be seen considering $Y^{\cdot} = R^{\cdot}$ with the usual ordering and $F : R^2 \to R^{\cdot}$ having the graph the lower half of a circular cone's shell determined by a plane parallel to R^2 through the vertex of the cone. Then the projection in R^2 of an arbitrary generator of this surface is a linearity set of F and all these linearity sets have a common point: the projection of the vertex.

4. LEMMA. The linearity sets of a convex operator are convex.

Proof. (See also the first part of the proof of Theorem 2.2 in ((B)).) Let F be a convex operator and let $L(x_o)$ be a linearity set of F at $x_o \in \operatorname{dom} F$. Suppose $y_1, y_2 \in L(x_o)$. We have to prove that $y := sy_1 + (1-s)y_2$ is in $L(x_o)$ as soon as $s \in]0,1[$. For any $x_1, \dots, x_n$ $L(x_o)$ and any $t_o, t_1, \dots, t_n \in R_+$ with $\sum_{i=0}^{n} t_i = 1$ one has

$$F(t_o y + \sum_{i=1}^{n} t_i x_i) = F(t_o s y_1 + t_o(1-s)y_2 + \sum_{i=1}^{n} t_i x_i) =$$

$$= t_o sF(y_1) + t_o(1-s)F(y_2) + \sum_{i=1}^{n} t_i F(x_i) \geqslant t_o F(sy_1 + (1-s)y_2) + \sum_{i=1}^{n} t_i F(x_i) = t_o F(y) + \sum_{i=1}^{n} t_i F(x_i),$$

since $y_1, y_2, x_1, \dots, x_n \in L(x_o)$ and $t_o s, t_o(1-s), t_1, \dots, t_n$ are non-negative and add to one. In consequence

$$F(t_o y + \sum_{i=1}^{n} t_i x_i) \geqslant t_o F(y) + \sum_{i=1}^{n} t_i F(x_i).$$

The converse of this relation follows from the convexity of F. Hence we must have equality in the above relation which shows, by the maximality of $L(x_o)$, that $y \in L(x_o)$.

Q. E. D.

5. PROPOSITION. Let us suppose that P and Q are cones in Y with the property (PdQ). If the P-convex operator $F: X \to Y^\cdot$ has the property that for $x_o \in \operatorname{icr} F$ there exists a Q-subgradient A of F at x_o such that

(3) $$Ax - Ax_o = F(x) - F(x_o), \quad \forall x \in L(x_o),$$

then A is also a P-subgradient of F at x_o.

In particular, if F is Q-subdifferentiable and (3) holds for some $A \in \partial_Q F(x_o)$ and each $x_o \in \operatorname{icr} F$, then F is P-subdifferentiable too.

Proof. Let us show first that if $A \in \partial_Q Fx_o)$ has the property (3), then $A \in \partial_Q F(x)$, $\forall x \in L(x_o)$. To this end consider an $x \in L(x_o)$ and add the relation $F(x_o) - F(x) - Ax_o + Ax = 0$ to $F(y) - F(x_o) - Ay + Ax_o \in Q$, $\forall y \in X$, which express the hypothesis that $A \in \partial_Q F(x_o)$.

Suppose that $y \notin L(x_o)$. Then there exist $x_1, \dots, x_n \in L(x_o)$ and $t_o, t_1, \dots, t_n \in R_+$ with $\sum_{i=0}^{n} t_i = 1$ such that

(4) $$t_o F(y) + \sum_{i=1}^{n} t_i F(x_i) - F(t_o y + \sum_{i=1}^{n} t_i x_i) \in P \setminus \{0\}$$

since F is P-convex by hypothesis. It follows in particular that $t_o \in]0,1[$, and hence $1 - t_o = \sum_{i=1}^{n} t_i \neq 0$. Since $\sum_{i=1}^{n} \frac{t_i}{1-t_o} = 1$ and since $L(x_o)$ is convex, then for the element of $L(x_o)$ defined by $x := \sum_{i=1}^{n} \frac{t_i}{1-t_o} x_i$ we have

$$\sum_{i=1}^{n} t_i F(x_i) = (1-t_o) \sum_{i=1}^{n} \frac{t_i}{1-t_o} F(x_i) = (1-t_o) F(\sum_{i=1}^{n} \frac{t_i}{1-t_o} x_i) = (1-t_o)F(x).$$

Using this relation (4) becomes

(5) $$t_o F(y) + (1-t_o)F(x) - F(t_o y + (1-t_o)x) \in P \setminus \{0\}.$$

We have shown above that $A \in \partial_Q F(x)$, hence

$$F(t_o y + (1-t_o)x) - F(x) - A(t_o y + (1-t_o)x) + Ax \in Q.$$

By adding this relation to (5) we obtain

$$F(y) - F(x) - Ay + Ax \in (P \setminus \{0\}) + Q \subset P.$$

Add to this relation the variant

$$F(x) - F(x_o) - Ax + Ax_o = 0$$

of (3) to conclude that

$$F(y) - F(x_o) - Ay + Ax_o \in P .$$

According to the arbitraryness of $y \notin L(x_o)$ and to the relation (3) we conclude that $A \in \partial_P F(x_o)$.

Q. E. D.

6. Remark. If Y ordered by Q is latticially complete then every convex operator from an arbitrary vector space to Y' has subgradients at every point of its intrinsic core which fulfil the condition in Proposition 5. This follows from Theorem 1.4 in ((B)). Hence every P-convex operator where P satisfies (PdQ) is P-subdifferentiable by Proposition 5. This is the content of Theorem 2.2 in ((B)). Originated in ((F)), this method furnishes examples of spaces having the so called <u>subgradient property</u> (ordered vector spaces in which every convex operator is subdifferentiable) without the chain completeness property.

7. PROPOSITION. <u>Let the cones P and Q in Y satisfy (PdQ). If for the P-convex operator $F : X \to Y'$ it holds $\partial_Q F(y) \neq \emptyset$ for some $y \in \operatorname{icr} L(x_o)$ and $x_o \in \operatorname{icr} F$ with $L(x_o)$ a linearity set of F containing x_o, then it follows that $\partial_P F(x_o) \neq \emptyset$.</u>

<u>In particular, if F is Q-subdifferentiable and every linearity set of F has non-empty intrinsic core, then F is also P-subdifferentiable.</u>

<u>Proof</u>. Let A be a Q-subgradient at $y \in \operatorname{icr} L(x_o)$. We shall see first that

$$(6) \qquad A(x - y) = F(x) - F(y) , \quad \forall\, y \in L(x_o) .$$

Since $y \in \operatorname{icr} L(x_o)$ and $x \in L(x_o)$ there exists $t \in]0,1[$ such that $y \pm t(x-y) \in L(x_o)$. From the relation $y = ½(y + t(x-y)) + ½(y - t(x-y))$ we deduce using the convexity of $L(x_o)$ that $F(y) = ½F(y + t(x-y)) + ½F(y - t(x-y))$ or its equivalent form

$$(7) \qquad F(y + t(x-y)) - F(y) = -(F(y + t(y-x)) - F(y)) .$$

From the condition $A \in \partial_Q F(y)$ one has $A(t(x-y)) \leq F(y+t(x-y)) - F(y)$ and $A(t(y-x)) \leq F(y-t(x-y)) - F(y)$ which together with (7) imply

$$(8) \qquad tA(x - y) = F(y + t(x-y)) - F(y) .$$

Since from the convexity of $L(x_o)$ it follows $F(y + t(x-y)) = F((1-t)y + tx) = (1-t)F(y) + tF(x)$, (8) yields (6).

Put $x = x_o$ in (6) and subtract the obtained relation from (6) to obtain

$$(9) \qquad A(x - x_o) = F(x) - F(x_o) \quad \forall\, x \in L(x_o) .$$

By the same way as it was done in the proof of Proposition 6, it can be seen that $A \in \partial_Q F(y)$, $\forall\, y \in L(x_o)$ and hence, in particular, $A \in \partial_Q F(x_o)$. This, together with (9), implies that the conditions of Proposition 6 are fulfilled for A, and hence according this proposition, $A \in \partial_P F(x_o)$.

Q. E. D.

8. COROLLARY. <u>Let us consider the cones P and Q possessing the property (PdQ). Then every Q-subdifferentiable P-convex operator from a finite dimensional space to Y' is also P-subdifferentiable.</u>

<u>In particular, if P is finite dimensional and $P \subset Y$ dominates $\bar{P}$ (the closure of P in the locally convex Hausdorff topology of $P - P$) and if Ω is a cone, then every P-convex</u>

operator from a finite dimensional vector space to Y' is P-subdifferentiable.

Proof. The first part of the assertion follows directly from Proposition 7 since every convex set in a finite dimensional space has nonempty intrinsic core.

If P is finite dimensional and $\overline{P}$ is a cone, then Y ordered by $\overline{P}$ has the chain completeness property since $\overline{P}$ is regular. Since P dominates $\overline{P}$ by hypothesis, it holds $(Pd\overline{P})$. Thus every P-convex operator (which is also $\overline{P}$-convex since $P \subset \overline{P}$) is $\overline{P}$-subdifferentiable by the theorem of Fel'dman ((F)) , hence it is also P-subdifferentiable by the first part of our proof.

Q. E. D.

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