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C O N T E N T S

operator from a finite dimensional vector space to Y' is P-subdifferentiable.

Proof. The first part of the assertion follows directly from Proposition 7 since every convex set in a finite dimensional space has nonempty intrinsic core.

If P is finite dimensional and $\bar{P}$ is a cone, then Y ordered by $\bar{P}$ has the chain completeness property since $\bar{P}$ is regular. Since P dominates $\bar{P}$ by hypothesis, it holds $(Pd\bar{P})$. Thus every P-convex operator (which is also $\bar{P}$-convex since $P \subset \bar{P}$) is $\bar{P}$-subdifferentiable by the theorem of Fel'dman ((F)), hence it is also P-subdifferentiable by the first part of our proof.

Q. E. D.

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((B)) BORWEIN, J.M., Subgradients of convex operators, Math. Operationsforsch. Statistik Ser. Optimization 15 (1984), 179-191.

((F)) FEL'DMAN, M.M., About the sufficient conditions of the existence of supporting operators to sublinear mappings, Sibirsk. Mat. Z. 16 (1975), 132-138 (Russian).

((K)) KRASNOSEL'SKII, M.A., Positive Solutions of Operatorial Equations, Fizmatgiz, Moskow, 1962 (Russian).

((P)) PERESSINI, A.L., Ordered Topological Vector Spaces, Harper&Row, New York, 1967.

((SCH)) SCHAEFER, H.H., Banach Lattices and Positive Operators, Springer, Berlin, 1974.

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SUBDIFFERENTIABILITY OF CONVEX OPERATORS RANGING IN LATTICIALLY ORDERED BANACH SPACES [*]

by A. B. Németh

The Present note depends on and completes our recent paper ((N3)). In ((N2)) it was given a simple characterization in classical terms of ordered vector spaces which, when considered as range spaces for convex operators, assure well behaving subdifferentials of them. It was shown that this nice subdifferentiability property, called there fully subdifferentiability property (abbreviated, FSP) for ordered vector spaces which admit a strictly isotone functional is equivalent with the monotone sequence property (abbreviated, MSP) of the space.

A weaker condition, the so called subgradient property ((B)) (abbreviated SGP) of an ordered vector space requires only the existence of nonempty subdifferentials for convex operators ranging in the respective space. In ((N1)) it was given the characterization of these spaces by a Hahn-Banach like property. But if this last property is regarded as a subdifferentiability property too, then such a characterization is in fact intrinsic. It appears hence the question to characterize the spaces with the SGP in

[*] This note is in final form and no version of it is, or will be, submitted for publication elsewhere.

terms comparable to those of the spaces with the FSP given in the paper ((N2)).

In ((N3)) necessary conditions for SGP wer considered in ordered vector spaces, respectively in ordered normal Banach spaces. These conditions in terms of monotone sequences are for Banach spaces ordered by closed normal cones seemingly very close to the MSP which assures FSP. However, the characterization of the spaces in which these two conditions are equivalent presents unexpected difficulties. Our note aims to show this equivalence for lattically ordered Banach spaces. If such a space has a closed normal positive cone and if it admits a strictly isotone functional, then the FSP and the SGP are equivalent.

We shall use throughout this note the terms ordered vector space, convex operator, normal cone, subgradient, subdifferential, directional minorant in the sense of ((N1)), ((N2)) and ((N3)).

Let $Y^{\cdot}$ denote an ordered vector space completed with a maximal element, infinity, for which we define the operations as in ((N1)). Let us remember that $Y^{\cdot}$ is said to have the <u>subgradient property</u> (SGP) if for each vector space X and every convex operator $F : X \rightarrow Y^{\cdot}$ it holds $\partial F(x_0) \neq \emptyset$ whenever x_0 is in the intrinsic core of dom F. The space $Y^{\cdot}$ is said to have the <u>full subdifferentiability property</u> (FSP) if it has the SGP and for each x_0 in the intrinsic core of dom F one has the relation:

$$\nabla F(x_0;h) = \max \{Ah : A \in \partial F(x_0)\} ,$$

where $\nabla F(x_0;h)$ stands for the directional minorant of F at x_0 in the direction h.

The ordered vector space is said to <u>admit a strictly isotone functional</u> if there exists $f : Y \rightarrow R$ with the property that

$u \leqslant v$, $u \neq v$ imply $f(u) < f(v)$. The space $Y^{\cdot}$ is said to have the <u>monotone sequence (net) property</u> (MSP and MNP respectively) if every lower bounded decreasing sequence (net) has an infimum. For spaces which admit strictly isotone functionals, MSP and MNP are equivalent (see Proposition II. 4. 9 in ((SCH))).

The main result in ((N2)) shows that in spaces having strictly isotone functionals FSP and MSP are equivalent. This result can be regarded a characterization in simple classical terms of the spaces with the FSP. Counter-examples constructed by Fel'dman ((F)) and Borwein ((B)) show that MSP is not necessary in order to a space have the weaker SGP.

Let $\{(v_m);(u_n)\}$ be a pair of sequences in $Y^{\cdot}$ with the property that $v_m \leqslant v_{m+1} \leqslant u_{n+1} \leqslant u_n$ for any $m,n \in N$. The space $Y^{\cdot}$ was said in ((N3)) to have <u>property</u> (b) if for every couple of sequences as above one has

$$\bigcap_{m,n} [v_m,u_n] \neq \emptyset .$$

where $[v,u]$ denotes the order interval defined by v and u.

One of the principal results in ((N3)) is the assertion :

<u>If</u> Y <u>is a Banach space ordered by a closed normal cone, then property</u> (b) <u>is necessary in order to</u> $Y^{\cdot}$ <u>have the</u> SGP.

Taking into account this assertion and the above cited result in ((N2)) it becomes clear that the characterization of the spaces in which property (b) implies the MSP (and in consequence, they are equivalent, since MSP implies obviously property (b)), would be of a real interest. The single result at this moment in this direction is the following

PROPOSITION. <u>For vector lattices which admit strictly isotone functionals the property (b) and the MSP are equivalent.</u>

<u>Proof</u>. It is clear that the MSP implies the property (b) for arbitrary ordered vector spaces.

Let Y^* be a vector lattice admitting a strictly isotone functional and having the property (b). Consider the decreasing lower bounded sequence (u_n) in Y^*. Let M_o be a maximal totally ordered subset of

$$M = \{ y \in Y : y \leqslant u_n , \forall n \in N \} .$$

Then M_o can be considered a totally ordered net, directed upward by the order relation of the space. Thus M_o becomes an increasing net which we shall denote by $(v_i)_{i \in I}$. Since Y^* admits a strictly isotone functional, every net of this kind admits a cofinal increasing subsequence. Let (v_m) denote such a subsequence for $(v_i)_{i \in I}$. According to property (b) of Y^* we have then

$$Q := \bigcap_{m,n \in N} [v_m, u_n] \neq \emptyset .$$

Suppose that $v \in Q$. Then $v_i \leqslant v$ for each $i \in I$ since (v_m) is cofinal with $(v_i)_{i \in I}$. Obviously, $v \leqslant u_n$, $\forall n \in N$, and hence $v \in M$. Since $(v_i)_{i \in I}$ is a maximal totally ordered set in M, we must have $v \in (v_i)_{i \in I}$. Let w be another element of M. Then $v' := v \vee w$ is again an element of M with the property that $v_i \leqslant v \leqslant v'$, $\forall i \in I$. But then $v' \in (v_i)_{i \in I}$ since this set is maximal by hypothesis. This shows that we must have $v = v' = v \vee w \geqslant w$. The obtained relation shows that $v = \inf \{ u_n : n \in N \}$. Hence Y^* has the MSP.

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THEOREM. <u>Let Y be a Banach space ordered by a closed normal cone and admitting a strictly isotone functional. If Y^* is a lattice then it has the FSP if and only if it has the SGP.</u>

<u>Proof</u>. According to the result in ((N3)) cited above, the SGP of Y^* implies property (b) of Y^*. By our above Proposition, it follows that Y^* has the MSP too, which according to Theorem 1 in ((N2)) implies that Y^* has the FSP. Since the FSP contains in its formulation SGP, the proof is complete.

Q. E. D.

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((B)) BORWEIN, J.M., Subgradients of convex operators, Math. Operationsforschung and Statistik, Ser. Optimization 15 (1984), 179-191.

((F)) FEL'DMAN, M.M., About the sufficient conditions of the existence of supporting operators to sublinear mappings, Sibirsk. Mat. Z. 16 (1975), 132-138 (Russian).

((N1)) NEMETH, A.B., On some universal subdifferentiability properties of ordered vector spaces, "Babeş-Bolyai" Univ. Faculty of Math. Research Seminars, Preprint Nr. 1, 1985.

((N2)) NEMETH, A.B., On the subdifferentiability of convex operators, J. London Math. Soc. (2) 34 (1986), 552-558.

((N3)) NEMETH, A.B., A necessary condition for subdifferentiability of convex operators, "Babeş-Bolyai" Univ. Faculty of Math. Research Seminars, Preprint Nr. 1, 1986.

((SCH)) SCHAEFER, H.H., Banach Lattices and Positive Operators, Springer-Verlag, Berlin, 1974.