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THE DINI THEOREM AND NORMAL CONES IN BANACH SPACES

By A.B. Németh

Abstract. It is shown that the Dini property does not imply the normality of cones in Banach spaces. A necessary and sufficient condition in order to a cone in a Banach space be normal is to weak convergence of series with terms in the cone imply their unconditional convergence. The criterion does not work in not complete normed spaces.

1. Introduction and the main result. The Dini theorem on monotone convergence of continuous functions [3] has the following abstract form in normed spaces :

THEOREM. Let K be a normal cone in the normed space E and $x_i \in K$, $i = 1,2,\ldots$. If the series $\sum_{i=1}^{\infty} x_i$ is weakly convergent, then it is norm convergent.

In fact this formulation follows from a more general statement in locally convex spaces (Theorem V.4.3 in [7]) which in turn is equivalent with a classical Dini type theorem (see V.4.4 in [7]).

DEFINITION. The cone K in the normed space E is said to have the Dini property if from the weak convergence of a series with terms in K it follows its norm convergence.

Thus normal cones in normed spaces have the Dini property.

But what about the converse : does the Dini property imply the normality of the cone ? It is rather simple to see that the answer is negative even for closed cones in Banach spaces (see section 3).

The main result of our note is the following

THEOREM 1. *Let K be a cone in the Banach space E . Then the following assertions are equivalent :*

(i) *K is normal ;*

(ii) *For each series* $\sum_{i=1}^{\infty} x_i$ *with* $x_i \in K$, $i = 1,2,\dots$ *weak convergence implies its unconditional convergence.*

The completeness of E in this theorem is essential. This will be shown in the final section of the note.

The idea of this note occurred when discussing on the paper [5] in this volume of I. Muntean, whom the author express his gratitude.

2. *Terminology*. The set K in the vector space E is called a *cone* if it is convex, it is invariant with respect to multiplication with positive reals and if $K \cap (-K) = \{0\}$. The cone K induces an order relation $\leqslant$ in E considering $u \leqslant v$ when $v-u \in K$.

If E is a normed space then the cone $K \subset E$ is called *normal* if there exists a universal positive constant b such that whenever one holds $0 \leqslant u \leqslant v$, then it follows $\|u\| \leqslant b\|v\|$. An equivalent definition (see [4] § 2) says : K is normal if if there exists a universal constant $d > 0$ such that from $u, v \in K$, $\|u\| = \|v\| = 1$, it follows $\|u+v\| \geqslant d$. We shall also use the following slight modification of this definition : K is normal if for any given positive numbers a_1 and a_2 , $a_1 < a_2$, there exists a positive constant c , such that from $u, v \in K$, $a_1 < \|u\| < a_2$, $a_1 \leqslant \|v\| \leqslant a_2$ it follows $\|u+v\| > c$.

The *dual* K' of the cone K is the set in the space E' of continuous linear functionals on E given by

$$K' = \{f \in E' : f(x) \geqslant 0 , \forall x \in K\} .$$

A classical lemma of M.G. Krein asserts that :

If K is a normal cone in the normed space E , then $E' = K' - K'$. (See Lemma 1 in Chapt. V , § 3 of [7].)

Let E be a normed space and (x_n) be a sequence therein. The formal sum

$$(1) \qquad \sum_{n=1}^{\infty} x_n$$

is called a *series*. The series (1) is said to be (norm) *convergent*, when $\lim_{n \to \infty} \sum_{i=1}^{n} x_i$ exists. It is called *unconditionally convergent* if every series $\sum_{i=1}^{\infty} x_{\sigma(i)}$ with σ a permutation of indices, is convergent. The series (1) is called *subseries convergent* if every subseries $\sum_{i=1}^{\infty} x_{k_i}$ of its is convergent. The similar notions for the case when the convergence is in the sense of the $\sigma(E,E')$ topology of E are : *weakly convergent* , *weakly unconditionally convergent* and *weakly subseries convergent* respectively.

3. *The Dini property does not characterize the closed normal cones in Banach spaces*. The assertion in this subtitle becomes likely if we remind that there exist infinite dimensional Banach spaces in which weak and norm convergence of a sequence to an element are equivalent. According to a result of Phillips the space ℓ^1 of absolute summable real sequences has this property

(see Corollary II. 2.2 in [2]). In such a space every cone possesses obviously the Dini property. The only thing we have to do is to show that some such space contains closed not normal cones. This follows from the following general result :

LEMMA 1. *In every infinite dimensional Banach space there exist closed not normal cones.*

Proof. A classical result due to Banach asserts that every complete normed space contains a normed basic sequence, i.e., a sequence (x_n) with $\|x_n\| = 1$, $n = 1,2,\dots$ such that every element x in the linear span $[x_n]$ of the sequence has a unique representation in the form of a convergent series

$$x = \sum_{i=1}^{\infty} a_i x_i , \quad a_i \in R , \quad i = 1,2,\dots . \tag{2}$$

(see e.g. [6] , Theorem 1.2, p. 49). Let we consider the vectors

$$u_n = x_{2n-1} \quad \text{and} \quad v_n = -x_{2n-1} + \frac{1}{n+1} x_{2n} , \quad n = 1,2,\dots .$$

From the linear independence of the terms in a basic sequence it follows that all the vectors u_n and v_n are linearly independent and hence

$$K_1 = \text{cone}\, \{u_1, v_1, u_2, v_2, \dots\}$$

(where cone M denotes the set of all linear combinations with non-negative scalars of elements in M) is a cone. Let we see that $K = \overline{K}_1$ (the closure of K_1) is a cone too. Assume the contrary : there exists an $x \in K$, $x \neq 0$, such that $-x \in K$. Consider the representation (2) of x . We have $a_{i_0} \neq 0$ for some i_0 . Denote by P the projection onto the space $[x_1, x_2, \dots, x_{2i_0}]$. Then P projects K_1 onto the cone

$$K_0 = \text{cone}\, \{u_1, v_1, \dots, u_{i_0}, v_{i_0}\} ,$$

and $P(x) \neq 0$. Denote by (x^n) and (y^n) respectively sequences in K_1 which converge to x , respectively to $-x$. Since $P(x^n) \in K_0$, and $P(y^n) \in K_0$ and since P is continuous and K_0 is closed, it follows that $P(x) \in K_0$ and $-P(x) \in K_0$, which is impossible since $P(x) \neq 0$.

Thus K is a closed cone. It is not normal since $u_n , v_n \in K$, $\|u_n\| = 1$, $1/2 < \|v_n\| \leq 1$ and $\|u_n + v_n\| = \frac{1}{n+1} \|x_{2n}\| = \frac{1}{n+1}$, $n = 1,2,\dots$.

4. A summation criterion for normal cones in Banach spaces.

THEOREM 2. *Let K be a cone in the Banach space E . Then the following conditions are equivalent :*

(i) The cone K is normal ;

(ii) For the series

$$\sum_{i=1}^{\infty} x_i \tag{3}$$

with $x_i \in K$, $i = 1,2,\dots$, every one of the following conditions implies each other :

(a) The series (3) is weakly convergent ;

(b) (3) is weakly unconditionally convergent ;

(c) (3) is weakly subseries convergent ;

(d) (3) is convergent ;

(e) (3) is unconditionally convergent ;

(f) (3) is subseries convergent .

Proof. To verify that (i) implies (ii) it is sufficient to see that if K is normal, then condition (a) implies (f).

Denote by x the weak sum of (3). Since K is normal, by Krein's lemma every continuous linear functional f can be represented in the form $f = g - h$ with $g, h \in K^*$. From the hypothesis on (3) we have $\sum_{i=1}^{\infty} g(x_i) = g(x)$ and $\sum_{i=1}^{\infty} h(x_i) = h(x)$. Since the last two series are numerical series with non-negative terms, we have obviously $\sum_{i=1}^{\infty} g(x_{\sigma(i)}) = g(x)$ and $\sum_{i=1}^{\infty} h(x_{\sigma(i)}) = h(x)$ for every permutation σ of indices. Hence

$$f(x) = g(x) - h(x) = \lim_{n\to\infty} \sum_{i=1}^{n} g(x_{\sigma(i)}) - \lim_{n\to\infty} \sum_{i=1}^{n} h(x_{\sigma(i)}) =$$

$$= \lim_{n\to\infty} \left(\sum_{i=1}^{n} (g(x_{\sigma(i)}) - h(x_{\sigma(i)}))\right) = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{\sigma(i)})$$

which shows that (3) is weakly unconditionally convergent. From the Dini theorem it follows that (3) is unconditionally convergent.

By the completeness of E, unconditionally convergence implies subseries convergence (see IV. § 1 p. 78 in [2]). Thus we get (f) fulfilled and this condition implies each other.

To show that (ii) implies (i) it suffices to see that if K is not normal, then some condition (a) - (f) does not imply some other.

Assume that K is not normal. Then for any n there exist the elements x_n, y_n in K with the properties

$$\|x_n\| = \|y_n\| = 1 \quad \text{and} \quad \|x_n + y_n\| < \frac{1}{2^n} .$$

Let we consider the elements in K defined by

$$z_{2k-1} = \frac{1}{2^n} x_n$$

$$2^n - 2 < k \leqslant 2^{n+1} - 2$$

$$z_{2k} = \frac{1}{2^n} y_n$$

$$n = 1, 2, \dots .$$

Let we show that $\sum_{i=1}^{\infty} z_i$ converges. Consider the sum of the form

$$\sum_{i=p}^{q} z_i \ , \quad p \leqslant q \ .$$

There exist the non-negative integers k, r, s, t such that

$$p = 2^{k+1} - 2^2 + r \ ,$$

$$q = 2^{k+s} - 2^2 + t$$

with $0 \leqslant r < 2^{k+1}$ and $0 \leqslant t < 2^{k+s}$. If $s = 1$ then $t \geqslant r$ and

$$(4) \qquad \sum_{i=p}^{q} z_i = \sum_{i=2^{k+1}-2^2+r}^{2^{k+1}-2^2+t} z_i \ .$$

If $s = 2$ and $t \geqslant 1$, then

$$(5) \qquad \sum_{i=p}^{q} z_i = \sum_{i=2^{k+1}-2^2+r}^{2^{k+2}-2^2} z_i + \sum_{i=2^{k+2}-2^2+1}^{2^{k+2}-2^2+t} z_i \ ,$$

while if $t = 0$ the second sum lacks. If $s \geqslant 3$ and $t \geqslant 1$, then

$$(6) \quad \sum_{i=p}^{q} z_i = \sum_{i=2^{k+1}-2^2+r}^{2^{k+2}-2^2} z_i + \sum_{j=k}^{k+s-3} \sum_{i=2^{j+2}-2^2+1}^{2^{j+3}-2^2} z_i + \sum_{i=2^{k+s}-2^2+1}^{2^{k+s}-2^2+t} z_i$$

while when $t = 0$ the last sum lacks.

Let we do first an estimation :

$$\Big\| \sum_{i=2^{j+2}-2^2+1}^{2^{j+3}-2^2} z_i \Big\| = \Big\| \underbrace{\frac{1}{2^{j+1}} x_{j+1} + \frac{1}{2^{j+1}} y_{j+1} + \dots + \frac{1}{2^{j+1}} x_{j+1} + \frac{1}{2^{j+1}} y_{j+1}}_{2^{j+2} \text{ terms}} \Big\|$$

$$= \|x_{j+1} + y_{j+1}\| < \frac{1}{2^{j+1}} \ .$$

We have similarly the estimations :

$$\Big\| \sum_{i=2^{k+1}-2^2+r}^{2^{k+2}-2^2} z_i \Big\| < \frac{1}{2^k} + \frac{1}{2^k}$$

$$\Big\| \sum_{i=2^{k+s}-2^2+1}^{2^{k+s}-2^2+t} z_i \Big\| < \frac{1}{2^{k+s-1}} + \frac{1}{2^{k+s-1}} ,$$

$$\Big\| \sum_{i=2^{k+1}-2^2+r}^{2^{k+1}-2^2+t} z_i \Big\| < \frac{3}{2^k} \quad .$$

Putting together these estimates, we have in case (4) :

$$\Big\| \sum_{i=p}^{q} z_i \Big\| < \frac{3}{2^k} \ ;$$

in case (5) the estimate

$$\Big\|\sum_{i=p}^{q} z_i\Big\| \leqslant \Big\| \sum_{i=2^{k+1}-2^2+r}^{2^{k+2}-2^2} z_i \Big\| + \Big\| \sum_{i=2^{k+2}-2^2+1}^{2^{k+2}-2^2+t} z_i \Big\| < \frac{2}{2^k} + \frac{2}{2^{k+1}} = \frac{3}{2^k}$$

and finally, in case (6) :

$$\Big\| \sum_{i=p}^{q} z_i \Big\| \leqslant \Big\| \sum_{i=2^{k+1}-2^2+r}^{2^{k+2}-2^2} z_i \Big\| + \sum_{j=k}^{k+s-3} \Big\| \sum_{i=2^{j+2}-2^2+1}^{2^{j+3}-2^2} z_i \Big\| +$$

$$+ \Big\| \sum_{i=2^{k+s}-2^2+1}^{2^{k+s}-2^2+t} z_i \Big\| < \frac{2}{2^k} + \sum_{j=k}^{k+s-3} \frac{1}{2^{j+1}} + \frac{2}{2^{k+s-1}} < \frac{5}{2^k} .$$

The obtained estimations show that if $p \to \infty$ (and hence $k \to \infty$) the norm of the sum $\sum_{i=p}^{q} z_i$ tends to 0 . That is, the series $\sum_{i=1}^{\infty} z_i$ converges.

By an appropriate rearrangement of terms of the series

$$\sum_{i=1}^{\infty} z_i$$

it becomes a series of the form

$$\frac{1}{2}x_1 + \frac{1}{2}x_1 + \frac{1}{2}y_1 + \frac{1}{2}y_1 + \dots + \underbrace{\frac{1}{2^n}x_n + \dots + \frac{1}{2^n}x_n}_{2^n \text{ terms}} +$$

$$+ \underbrace{\frac{1}{2^n}y_n + \dots + \frac{1}{2^n}y_n}_{2^n \text{ terms}} + \dots$$

which is obviously nonconvergent since

$$\Big\| \underbrace{\frac{1}{2^n}x_n + \dots + \frac{1}{2^n}x_n}_{2^n \text{ terms}} \Big\| = \|x_n\| = 1 \quad .$$

Thus we have constructed a series

$$\sum_{i=1}^{\infty} z_i$$

with the terms z_i in K which is convergent but it is not unconditionally convergent. This shows that if K is not normal, then condition (d) does not imply condition (e) and the proof is complete.

Proof of Theorem 1. From the proof of Theorem 2 follows directly the proof of Theorem 1. Indeed, the implication (i) $\Rightarrow$ (ii) in Theorem 1 is a consequence of the implications (a) $\Rightarrow$ (b) $\Rightarrow$ (e) in Theorem 2, while (ii) $\Rightarrow$ (i) of Theorem 1 can be proved exactly in the way of the proof of implication (ii) $\Rightarrow$ (i) in Theorem 2, with the remark that convergence of a series implies its weak convergence.

Remark. From the above proofs we see that the Theorems 1 and 2 are in fact equivalent. We decided to include the most detailed formulation of Theorem 2 in order to exhibit also other aspects of the problem.

Remark. Going back to the original formulation of Dini [3] of his theorem, it says that if we consider the series

$$(7) \qquad \sum_{n=1}^{\infty} u_n(t)$$

where (i) $u_i : [a,b] \to R$ are continuous and non-negative functions defined on the interval $[a,b]$ in R ; (ii) (7) converges pointwise to a function $u(t)$; (iii) $u : [a,b] \to R$ is continuous, then it follows that (7) converges uniformly to u. Theorem 2 asserts among others that this statement can be slightly strengthened : we can assert that every subseries $\sum_{n=1}^{\infty} u_{k_n}(t)$ of (7) converges uniformly to some continuous function. In this classical context the obtained improvement is immediate and hence quite inessential. The interest of Theorem 2 is that it shows that this formulation is in some sense the best possible which we can expect : it is equivalent with the normality of the cone of the non-negative functions in the Banach space $C[a,b]$ of continuous functions defined on the interval $[a,b]$.

5. Theorems 1 and 2 do not hold in noncomplete normed spaces. To prove this we remind that the cone K in the normed space E is called regular if every monotone order bounded sequence in E is convergent.

LEMMA 2. *If K is a regular cone in the normed space E, then the conditions (a) - (f) in (ii) of Theorem 2 are equivalent.*

Proof. We have to show that (a) implies (f). Consider the series $\sum_{i=1}^{\infty} x_i$, $x_i \in K$, $i = 1,2,\ldots$ having the weak limit x. Then by the proposition V.4.2 in [7] x is the supremum of the set $\{\sum_{i=1}^{n} x_i : n \in N\}$. It follows then that for each subseries

$$\sum_{i=1}^{\infty} x_{k_i}$$

the sequence

$$\left(\sum_{i=1}^{n} x_{k_i}\right)_{n=1}^{\infty}$$

is monotonically increasing and bounded above by x. Since K is regular this sequence converges, wherefrom the convergence of subseries. That is, we get (f) fulfilled and the lemma is proved.

We have given in [6] an example of a noncomplete normed space E with a closed regular and not normal cone K. (Our construction exploited an idea of Breckner and Orbán in [1].) For that space the condition (ii) of Theorem 2 holds by Lemma 2 while condition (i) is obviously false.

Remark. From Theorem 2 and Lemma 2 follows in particular that every regular cone in a Banach space is normal, which is a result due to Krasnosel'skiĭ (see [4], Theorem 1.6 p. 34).

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MONOTONE ENCLOSURE FOR CONVEX OPERATORS

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1. Introduction. In order to solve approximately the equation of form

(1) $P(x) = 0$,

where $P : X \longrightarrow Y$, X and Y being locally convex ordered spaces (PLC spaces), J. Vandergraft [9] adapted Newton's well known method [5], obtaining an increasing sequence which converges to a solution of (1). By using the divided difference [6], we construct a decreasing sequence, and in this manner, the approximated solution is enclosed in a "decreasing" sequence of intervals. For our considerations we use the definitions and properties of the PLC spaces given in [9] and improved in [3], and the propertis of the divided differences given in [1].

We note that for equations in Banach spaces there are several results concerning the monotone enclosure (see e.g. [2], [7], [8], etc.).

The next lemma will be used in the proof of our result.

1.1. LEMMA. Let X and Y be PLC spaces, $D \subseteq X$ a convex subset and the mapping $P : D \longrightarrow Y$ with $[x', x'', x''';P] \geqslant 0$ for all x', x'', x''' in D. Then:

(i) the mapping P is (o)-convex;

(ii) for all x_o, x_1, x_2 in D with $x_1 \leqslant x_2$ we have the inequality $[x_o, x_1;P] \leqslant [x_o, x_2;P]$.