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ORDERED FRÉCHET SPACES WITH UNIVERSAL SUBDIFFERENTIABILITY PROPERTIES

by A. B. Németh

In this note it is shown that if a separable Fréchet space ordered by an arbitrary closed normal cone possesses the subgradient property, then it possesses also the fully subdifferentiability property. If every closed lattice ordered subspace of a Fréchet space with continuous lattice operations possesses the subgradient property then it possesses also the fully subdifferentiability property.

The vector valued convex analysis is primarily concerned with the study of subdifferentials for convex operators taking values in ordered vector spaces. When the range spaces are order complete vector lattices results of this kind go back to Valadier ((V)) , Raffin ((R)) , Levin ((L)) and Rubinov ((Ru)) . But order completeness is a rather restrictive requirement. In the first results due to Zowe (((Z1)) and ((Z2))) where it was avoided, some structure conditions were imposed on the domain space. This line was continued and extended by Borwein in ((B1)). With respect to the existence of subgradients Fel'dman ((F)) proved that nice subdifferentiability properties can be obtained relaxing considerably the conditions on the range space and with no assumption on the domain space. It turned out that Fel'dman's condition is also necessary for the above cited nice subdifferentiability property ((N2)). If no special conditions are imposed on subdifferentials except their non-emptity, then the conditions on the range space can be further relaxed. But the examples of ordered vector spaces having this weak subdifferentiability property, but not the stronger one considered above, are rather pathologic (see ((F)) and ((B2))). There exist important classes of ordered locally convex spaces for which the two properties coincide. A first result of this kind was obtained for Banach lattices in ((N4)). In the present note other results are obtained for separable Fréchet spaces ordered by closed normal cones and also for Fréchet spaces for which all closed lattice ordered subspaces possess special structural properties.

Let Y be a vector space over the reals and let K be a cone in Y, i.e., a nonempty set having the properties (i) $K + K \subset K$; (ii) $tK \subset K$ for each non-negative real number t; and (iii) $K \cap (-K) = 0$. If we put $u \leqslant v$ whenever $v-u$ is in K, then $\leqslant$ is a partial ordering in Y induced by K. Then Y is called an ordered vector space with the positive cone K.

The order interval $[u,v]$ in the ordered vector space Y is by definition the set $\{z \in Y : u \leqslant z \leqslant v\}$. The space Y is said to have property (b) if for every pair of sequences (v_m) and (u_n) with $v_m \leq v_{m+1} \leq u_{n+1} \leq u_n$ for any $m, n \in N$, one has

$$\bigcap_{m,n \in N} [v_m, u_n] \neq \emptyset .$$

The ordered vector space Y is said to admit an isotone (real) functional if there exists $f : Y \to R$ with $f(u) < f(v)$ whenever $u \leqslant v$ and $u \neq v$.

If Y is an ordered vector space endowed with a locally convex topology, then Y (and its positive cone) K) is called normal (or locally full) if there is a base of neighbourhoods V of 0 with

$$V = (V - K) \cap (K - V) .$$

The ordered locally convex space Y (and its positive cone K) is regular if each decreasing sequence in K is convergent.

An ordered vector space Y is said lattice ordered or a vector lattice if every pair of its elements x, y possesses a supremum (or equivalently, an infimum) denoted by $x \vee y$ (and respectively, by $x \wedge y$). The operations $(x,y) \mapsto x \vee y$ and $(x,y) \mapsto x \wedge y$ are called lattice operations.

The vector lattice Y is order complete if every upper bounded (lower bounded) set possesses a supremum (an infimum) in Y.

We shall denote by $Y^{\cdot}$ the ordered vector space completed with an abstract element ∞ (infinity) for which, as usual, we define the operations:

$$\infty + \infty = \infty , \quad 0 \cdot \infty = 0 , \quad y + \infty = \infty \text{ for each } y.$$

We shall suppose that $y \leqslant \infty$ for every y in Y.

The operator F defined on the vector space X with values in $Y^{\cdot}$ is called convex if for any u_1, u_2 in X and every t in $[0,1]$ the following relation holds

$$F(tu_1 + (1-t)u_2) \leqslant tF(u_1) + (1-t)F(u_2) .$$

The domain of F is the subset of the elements x in X having the property that $F(x)$ is in Y. We denote this set by dom F. By the convexity of F it follows that dom F is a convex set.

If x is in core dom F, then we say that F admits a directional minorant at x in the direction h, if

$$\nabla F(x;h) = \inf_{t>0} t^{-1}(F(x+th) - F(x))$$

exists. The space $Y^{\cdot}$ is said to have the directional minorability property if every convex operator F with values in $Y^{\cdot}$ has directional minorants at every point in core dom F and in every direction.

If $Y^{\cdot}$ is an ordered locally convex space, then if the limit

$$F'(x;h) = \lim_{t\searrow 0} t^{-1}(F(x+th) - F(x))$$

exists, where $x \in$ core dom F, then it is called the directional derivative of F at x in the direction h. If the positive cone of $Y^{\cdot}$ is closed, then from the convexity of F it follows that a directional derivative is also a directional minorant.

The ordered locally convex space $Y^{\cdot}$ is said to have the directional differentiability property if for every convex operator F and every x in core dom F there exist directional derivatives in every direction.

The linear operator $A : X \to Y$ is called a subgradient of $F : X \to Y^{\cdot}$ at x in dom F if

$$Ah \leq F(x+h) - F(x) \quad , \forall\, h \in X .$$

The set of subgradients of F at x is called the subdifferential of F at x and is denoted by $\partial F(x)$.

The ordered vector space $Y^{\cdot}$ is said to have the subgradient property if for every convex operator F and every x in core dom F it holds $\partial F(x) \neq \emptyset$.

The ordered vector space $Y^{\cdot}$ is said to have the fully subdifferentiability property if it has the subgradient property and the directional minorability property and if it holds the relat on

$$(1) \qquad \nabla F(x;h) = \inf \{ Ah : A \in \partial F(x) \}, \quad \forall\, h$$

for every convex operator F and every x in core dom F.

The ordered locally convex space $Y^{\cdot}$ possesses the fully subdifferentiability property if it has the subgradient property and the directional differentiability one and if (1) takes place for the directional derivative $F'(x;h)$ instead of the directional minorant $\nabla F(x;h)$, for every convex operator F and every x in core dom F.

In ((V)), ((L)), ((Z1)), ((F)), ((N1)), ((B1)), ((B2)) various conditions were given to assure various types of subdifferentiability of convex operators. Since ((F)) it seemed that the principal role in the existence of subgradients belongs to the range space $Y^{\cdot}$. This becomes a certitude by proving in ((N2)) that the necessary and sufficient condition for an ordered vector space (respectively, for an ordered locally convex space) which admits an isotone functional to have the fully subdifferentiability property is that every decreasing lower bounded sequence has an infimum (respectively, to be regular).

The fully subdifferentiability property seems to be a rather strong one. In ((F)) and ((B2)) were considered ordered vector spaces with the subgradient property and without the fully subdifferentiability property. But these spaces are rather particular (for instance, they are not Archimedian ordered vector spaces). The problem of a complete characterization of spaces of this kind is still open.

In the recent note ((N4)) it was shown that for lattice ordered Banach spaces with closed normal positive cones which admit isotone functionals, subgradient property and fully subdifferentiability property coincide.

Starting with the paper ((N3)) which was the basic reference for ((N4)) too, we shall conclude results for Fréchet spaces. The basic tool in obtaining them is the characterization of the Fréchet spaces in which every closed normal cone is regular due to McArthur ((M)), and some related questions in Banach lattice theory due to Mayer-Nieberg ((MN)) and Schaefer ((Sch)) which were joined in ((N5)).

THEOREM 1. For a separable Fréchet space Y the following conditions are equivalent :

(i) Every closed subspace of Y ordered by a closed normal cone possesses the fully subdifferentiability property.

(ii) Every closed subspace of Y ordered by a closed normal cone possesses the subgradient property.

(iii) Every convex operator defined on R with values in an arbitrary closed subspace of Y ordered by a closed normal cone possesses subgradients in every core point of its domain.

(iv) The space Y does not contain any subspace isomorphic with c_0, the Banach space of the real sequences converging to 0.

Proof. The implications (i)$\Rightarrow$(ii)$\Rightarrow$(iii) follow by the definitions.

(iii)$\Rightarrow$(iv). Assume that (iv) does not hold. Then Y contains a subspace isomorphic to the Banach space c of the convergent real sequences, since c and c_0 are isomorphic. Let us consider c ordered by the cone of sequences with non-negative terms. This cone is closed and normal. We have shown in ((N3)) that c does not have property (b) which, by Proposition 3 in the same paper is necessary in order to every convex operator from R to c have subgradients in every point in R. Thus we conclude that (iii) does not hold for Y.

(iv)$\Rightarrow$(i). From Theorem 1 in ((M)) every closed normal cone in Y is regular. Since Y is separable, by Theorem 2.1 in ((KR)) it admits an isotone real functional whenever it is ordered by a closed cone. The similar statement holds for each closed subspace of Y. Then by Theorem 3 in ((N2)) every such subspace ordered by a closed normal cone possesses the fully subdifferentiability property, since its positive cone is regular and it admits an isotone functional.

THEOREM 2. Let Y be a Fréchet space. Then the following conditions are equivalent :

(i) Every closed lattice ordered subspace of Y with continuous lattice operations possesses the fully subdifferentiability property.

(ii) Every closed lattice ordered subspace of Y with continuous lattice operations possesses the subgradient property.

(iii) Every convex operator defined on R with values in an arbitrary closed lattice ordered subspace of Y with continuous lattice operations possesses subgradients in every core point of its domain.

(iv) The space Y does not contain any subspace isomorphic (as a topological vector space) to c_0.

Proof. The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are trivial.

(iii) $\Rightarrow$ (iv). Let us assume that (iv) does not hold. Then Y contains a subspace isomorphic to c. Now, as in the proof of Theorem 1, we see via ((N3)) that if we consider c to be ordered by the cone of non-negative sequences, then there exist convex operators from R to c without subgradients in some point in R. Taking into account that c with this canonical ordering is a lattice with continuous lattice operations (this follows from Theorems 8.1 and 8.2 of ((Na)) since the positive cone is normal), we arrive to the conclusion that (iii) cannot hold.

(iv) $\Rightarrow$ (i). If Y does not contain any subspace isomorphic with c_0, then by ((N5)) every closed lattice ordered subspace in it with continuous lattice operations is order complete. Hence it possesses the fully subdifferentiability property by a classical result in ((V)) and ((L)) (see also ((B1))).

Remark. The obtained results are characterizations of a structural property of Fréchet spaces by subdifferentiability properties of their ordered subspaces. The equivalent properties listed in the above theorems can be completed with other ones as the so called exact Hahn-Banach approximation property ((F)) which is equivalent with the subgradient property, or with some lattice theoretic conditions in the spirit of ((MN)).

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SUR L'APPROXIMATION DES RACINES DES ÉQUATIONS DANS UN ESPACE MÉTRIQUE

par

Ion Păvăloiu

Dans cette note nous étudions l'évaluation des erreurs qui surgissent pendant la résolution numérique des équations en espaces métriques à l'aide de certaines méthodes d'itération à plusieurs pas .

Considerons un espace métrique (X, ρ) complet et l'équation suivante :

$$x = f(x) \quad , \tag{1}$$

où $f: X \to X$ est un opérateur quelconque.

Désignons par $X^{k+1} = X \times X \times \ldots \times X$, c'est-à-dire le produit cartèsien de l'ensemble X avec lui même $k+1$ fois.

Pour la résolution de l'équation (1) nous considérons l'application $G: X^{k+1} \longrightarrow X$ dont nous supposons que sa restriction à la diagonale de l'espace X^{k+1} coïncide avec l'opérateur f , c'est-à-dire :

$$G(x,x,\ldots,x) = f(x) \quad , \tag{2}$$

pour chaque $x \in X$.

Considérons la suite $(x_n)_{n=0}^{\infty}$ fournie par le procédé d'itération suivant :