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A PHELPS TYPE THEOREM FOR SPACES WITH ASYMMETRIC NORMS

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Abstract. If $(X, \|\cdot|)$ is a linear space with asymmetric norm and Y is a subspace of X, for every $f \in Y_+^*$ (the cone of linear bounded functional on Y) there exists at most one functional $F \in X_+^*$ extending f and preserving the asymmetric norm of f. The problem of uniqueness of the extension in terms of uniqueness of elements of best approximation of $F \in X_+^*$ by elements of $Y_+^\perp = \{G \in X_+^* : G|_Y = 0,\ F \geq G\}$ is discussed.
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1. Asymmetric norms

Let X be a real linear space and $\|\cdot| : X \to [0, \infty)$ a function with the following properties:

1) $\|x| > 0$ for all $x \neq \theta$; 2) $\|\lambda x| = \lambda \|x|$ for all $\lambda \geq 0$ and all $x \in X$; 3) $\|x + y| \leq \|x| + \|y|$ for all $x, y, \in X$. Then the function $\|\cdot|$ is called an *asymmetric norm* on X and the pair $(X, \|\cdot|)$ is called a *space with asymmetric norm* (see [5]). In such a space, in general $\|-x| \neq \|x|$.

Example ([1]) Consider the real linear space

$$C_0([0,1], 1, 0) = \left\{ x : [0,1] \to \mathbb{R}, x \text{ is continuous and } \int_0^1 x(t)\, dt = 0 \right\}.$$

The function $\|\cdot| : C_0([0,1],1,0) \to [0,\infty), \|x| = \max\{x(t) : t \in [0,1]\}$ satisfies the properties 1) - 3) of asymmetric norm. The functions $x_\alpha(t) = \alpha\left(t - \frac{1}{2}\right), \alpha \in \mathbb{R}$ are in $C_0([0,1],1,0)$ and $\|x_\alpha| = \frac{|\alpha|}{2} = \|-x_\alpha|$, but the functions $y_n(t) = 1 - nt^{n-1}, n > 2$ $(n \in \mathbb{N})$, which also belong to $C([0,1],1,0)$ satisfy $\|y_n| = 1$ and $\|-y_n| = n - 1 > 1$, i.e. $\|y_n| \neq \|-y_n|$.

By definition, the balls $B(x,r) = \{y \in X : \|y - x| < r\}$ $\quad x \in X$ and $r > 0$ form a base of the topology of the space $(X, \|\cdot|)$. The space $(X, \|\cdot|)$ equipped with this topology need not be a topological linear space, since the multiplication by scalars is not continuous. In the preceding example, for $x = 0$ and $\lambda = -1$, $(-1)\,0 = 0$ and for all $r > 0$, $-B(0,r) \not\subseteq B(0,1)$ i.e. the multiplication by scalars is not continuous.

For each asymmetric norm $\|\cdot|$ on X one defines $\|x\| = \max\{\|x|, \|-x|\}$. Then $\|x| \leq \|x\|$, $x \in X$. If there exists $c > 0$ such that $\|x\| \leq c\|x|$, i.e. the norm $\|\cdot\|$ and asymmetric norm $\|\cdot|$ are equivalent, then $(X, \|\cdot|)$ is a topological linear space. Such a situation occurs when $\dim X < \infty$. In this case all the norms and asymmetric norms are equivalent ([5], I.2.1. pp.21-23). If $\|\cdot\|$ and $\|\cdot|$ are equivalent then $\|\cdot|$ is continuous on X.

An example of an asymmetric norm on the normed space $(X, \|\cdot\|)$ is given by $\|x| = \|x\| + \varphi(x), x \in X$ where $\varphi \in X^*$, $\varphi \neq 0$, (a linear and continuous functional on X).

2. Linear and bounded functional on a linear space with asymmetric norm. Let $(X, \|\ |)$ be a space with asymmetric norm and $f : X \to \mathbb{R}$ a linear functional.

The linear functional f is called *bounded* if

$$\|f| := \sup\left\{\frac{f(x)}{\|x|} < \infty : x \neq 0\right\} < \infty. \tag{1}$$

(see [5], Ch.9, Sec.5, p.483). If f is a linear and bounded functional, then

$$f(x) \leq \|f| \cdot \|x|, \ x \in X, \tag{2}$$

and, changing x with $-x$, one obtains $-f(x) = f(-x) \leq \|f| \cdot \|-x|$. Consequently

$$-\|f| \cdot \|-x| \leq f(x) \leq \|f| \cdot \|x|, \ x \in X$$

and in general, $\|-f| \neq \|f|$. Denote by $X^{\#}$ the algebraic dual of the linear space X and by X_+^* the set of all linear and bounded functional on the space X with an asymmetric norm $\||$.

For $f, g \in X_+^*$ one obtains $f + g \in X_+^*$ and $\lambda f \in X_+^*$ $(\lambda \geq 0)$ $(\lambda f + \mu g \in X_+^*$, for all $f, g, \in X_+^*$ and all $\lambda, \mu \geq 0)$. Consequently X_+^* is a convex cone in $X^{\#}$.

The functional $\|\cdot| : X_+^* \to [0, \infty)$ defined by formula (1) satisfies the axioms 1) - 3) of an asymmetric norm. Indeed, if $f \neq 0$ then there exists $x \in X$, $x \neq 0$ such that $f(x) > 0$ or $f(-x) > 0$. It follows that $\|f| = \sup(f(x) / \|x|) > 0$. If $\lambda \geq 0$ then $\|\lambda f| = \lambda \|f|$ and $\|f + g| \leq \|f| + \|g|$ are evidently fullfielled.

Finally, observe that the function $d : X \times X \to [0, \infty)$ defined by

$$d(x, y) = \|x - y|, \quad x, y \in X, \tag{3}$$

where X is a space with asymmetric norm $\|\cdot|$, is a quasi-metric on X, i.e. d satisfies the conditions:

a) $d(x, y) = 0 \Longleftrightarrow x = y$; b) $d(x, y) \leq d(x, z) + d(z, y), \quad x, y, z \in X$ (see [6])

For $f \in X_+^*$ and all $x, y \in X$, we have $f(x - y) \leq \|f| \cdot \|x - y|$, so that

$$f(x) - f(y) \leq \|f| \cdot \|x - y|, \quad x, y \in X. \tag{4}$$

The last inequality means that every bounded linear functional f on $(X, \|\cdot|)$ is semi-Lipschitz (see [11]) i.e. $X_+^* \subset S\,Lip_0\,X$ where

$S\,Lip_0\,X = \left\{f : X \to \mathbb{R},\ f(0) = 0,\ \sup \frac{(f(x) - f(y)) \vee 0}{\|x - y|} < \infty\right\}$

is the semi-linear space of semi-Lipschitz real functions defined on $(X, \|\cdot|)$ (see [11]). Because $X_+^* \subset S\,Lip_0\,X$, for every $f \in X_+^*$, we have

$$\sup_{x\neq 0} \frac{f(x) \vee 0}{\|x|} = \sup_{x\neq 0} \frac{f(x)}{\|x|} \quad \text{and} \quad \sup_{x-y\neq 0} \frac{f(x)-f(y)}{\|x-y|} = \|f| \tag{5}$$

i.e. the asymmetric norm of $f \in X_+^*$ is the smallest semi-Lipschitz constant of f.

Let Y be a subspace of the linear space X with asymmetric norm $\|\cdot|$ and let $f \in Y_+^*$. Then $f \in S\,Lip_0\,Y$ and, by an analogue of an extension theorem of McShane ([7]), there exists at least one function $F \in S\,Lip_0\,X$ such that $F|_Y = f$ and $\|F| = \|f|$ (see [8], Th.2). In our case the following result holds:

Theorem 1. *([5]). Let X be a real linear space with the asymmetric norm $\|\cdot|$ and Y be a subspace of X. Then for every $f \in Y_+^*$ there exists $F \in X_+^*$ such that*

$$a)\ F|_Y = f, \qquad b)\ \|F| = \|f|.$$

Proof. If $f \in Y_+^*$ let $p : X \to \mathbb{R}$ be defined by $p(x) = \|f| \cdot \|x|$. Then $f(y) \leq \|f| \cdot \|y| = p(y),\ y \leq Y$, and by Hahn-Banach theorem, there exists $F \in X^*$ such that

$$F\,|_Y = f \text{ and } F(x) \leq \|f| \cdot \|x|,\ \ x \in X.$$

Then

$$\frac{F(x)}{\|x|} \leq \|f|, x \in X, x \neq 0$$

and taking the supremum with respect to $x \in X$ one obtains $\|F| \leq \|f|$. On the other hand

$$\begin{aligned} \|F| &= \sup\left\{\frac{F(x)}{\|x|}, x \in X, x \neq 0\right\} \geq \sup\left\{\frac{F(y)}{\|y|}, y \in Y, y \neq 0\right\} = \\ &= \sup\left\{\frac{f(y)}{\|y|}, y \in Y, y \neq 0\right\} = \|f| \end{aligned} \tag{6}$$

and, consequently $\|F| = \|f|$.

By Theorem 1 it follows that if Y is a subspace of $(X, \|\cdot|)$ then for every $f \in Y_+^*$ the set

$$\mathcal{E}(f) = \left\{F \in X_+^* : F|_Y = f \text{ and } \|F| = \|f|\right\} \tag{6}$$

is nonvoid.

Observe that, for every $f \in Y_+^*$, the set $\mathcal{E}(f)$ of all extensions of f, is included in $S(0) := \left\{F \in X_+^* : \|F| = \|f|\right\}$ and $\mathcal{E}(f)$ is convex.

Indeed, if $F_1, F_2 \in \mathcal{E}(f)$ and $\lambda \in [0,1]$ then $f = \lambda F_1|_Y + (1-\lambda)\,F_2|_Y$ and

$$\|f| = \|\lambda F_1|_Y + (1-\lambda) F_2| \le \|\lambda F_1 + (1-\lambda) F_2| \le \lambda \|F_1| + (1-\lambda) \|F_2| =$$
$$= \lambda \|f| + (1-\lambda) \|f| = \|f| \text{ so that } \lambda F_1 + (1-\lambda) F_2 \in \mathcal{E}(f).$$

3. Extension and approximation

In [10] R.R. Phelps made a connection between the set of the extensions of a linear and continuous functional $f \in Y^*$(Y^* is the algebraic - topological dual of the subspace Y of a normed space $(X, \|\cdot\|)$) and the set of elements of best approximation of a functional $F \in X^*$ by the elements of the annihilator $Y^{\perp} = \{G \in X^* : G|_Y = 0\}$.

If $F \in X^*$ then the set of elements of best approximation of F in $Y^{\perp}$ is $P_{Y^{\perp}}(F) = F - \mathcal{E}(F|_Y)$ where $\mathcal{E}(F|_Y) = \{H \in X^* : H|_Y = F|_Y \text{ and } \|H\| = \|F|_Y\|\}$. The extension of a functional $f \in Y^*$ is unique if and only if $Y^{\perp}$ is a Chebyshevian subspace of X^*.

In the proof of R.R. Phelps'result one uses an essential fact: together with $F \in X^*$ the functional $F-G$ belongs to X^*, for every $G \in \mathcal{E}(F|_Y)$, i.e. the fact that X^* has a structure of linear space.

Because X^*_+ has only a structure of a convex cone, it could exist a linear and bounded functional $F \in X^*_+$, such that for certain extensions G from $\mathcal{E}(F|_Y)$, or for all of them, we could have $F-G$ unbounded, i.e. $F - G \notin X^*_+$. Some additional definitions are necessary. For a cone $\mathcal{K}$ in a linear space $\mathcal{V}$ and $x, y \in \mathcal{V}$, we will write $x \le y$ if and only if $y - x \in \mathcal{K}$.

Let $\mathcal{M}$ be a non-empty subset of the cone X^*_+ and $F \in X^*_+$. We say that F *admits minorants* in $\mathcal{M}$ if there exists $G \in \mathcal{M}$ such that $F \ge G$ (i.e. $F - G \in X^*_+$) and we say that F *majorizes* the set $\mathcal{M}$ if $F \ge G$ for every $G \in \mathcal{M}$. (i.e. $F - \mathcal{M} \subset X^*_+$). Obviously, if $F \in X^*_+$ and majorizes $\mathcal{M}$, then F admits minorants in $\mathcal{M}$.

For a subspace Y of the space X with asymmetric norm, we denote by $Y^{\perp}_+$ the annihilator of Y in X^*_+ i.e., the set

$$Y^{\perp}_+ = \{G \in X^*_+ : G|_Y = 0\}. \tag{7}$$

We state the following problem of best approximation:

For $F \in X^*_+$ find $G_0 \in Y^{\perp}_+$ such that $\|F - G_0| = d_+(F, Y^{\perp}_+)$ where

$$d_+(F, Y^*_+) = \inf\{\|F - G| : G \in Y^{\perp}_+, F \ge G\}. \tag{8}$$

Let

$$P_{Y^{\perp}_+}(F) := \{G \in Y^*_+ : F \ge G, \|F - G| = d_+(F, Y^{\perp}_+)\}. \tag{9}$$

We say that $Y^{\perp}_+$ is F- *proximinal* if $P_{Y^{\perp}_+}(F) \ne \emptyset$. If, in addition, *card* $P_{Y^{\perp}_+}(F) = 1$ then $Y^{\perp}_+$ is called F- *Chebyshevian*.

The following result is similar to Phelps'result([10]).

Theorem 2. *Let X be a space with asymmetric norm, Y a subspace of X, and $F \in X^*_+$. Let*

$$\mathcal{E}(F|_Y) = \{H \in X^*_+ : H|_Y = F|_Y \text{ and } \|H| = \|F|\} \tag{10}$$

and

$$\mathcal{E}_+ (F|_Y) = \{H \in \mathcal{E} (F|_Y) : H \leq F\} \tag{11}$$

a) If $\mathcal{E}_+ (F|_Y) \neq \emptyset$ *then* $Y_+^\perp$ *is* F *- proximinal and the following equality holds:*

$$d_+ \left(F, Y_+^\perp\right) = \|F|_Y| . \tag{12}$$

b) If $G_0 \in P_{Y_+^\perp} (F)$ *then* $F - G_0 \in \mathcal{E}_+ (F|_Y)$.

c) We have $\mathcal{E}_+ (F|_Y) \neq \emptyset$ *if and only if* $P_{Y_+^\perp} (F) \neq \emptyset$ *and the following equality holds:*

$$F - \mathcal{E}_+ (F|_Y) = P_{Y_+^\perp} (F) . \tag{13}$$

d) $Y_+^\perp$ *is* F *- Chebyshevian if and only* $card\, \mathcal{E}_+ (F|_Y) = 1$.

e) $F \in \mathcal{E}_+ (F|_Y)$ *if and only if* $0 \in P_{Y_+^\perp} (F)$.

Proof. Let G_0 be a minorant of F in $\mathcal{E} (F|_Y)$ (G_0 exists, because $\mathcal{E}_+ (F|_Y \neq \emptyset)$. Then, $F - G_0 \in X_+^*$ and

$$\|F|_Y| = \|G_0| = \|F - (F - G_0)| \geq d_+ \left(F, Y_+^\perp\right) .$$

On the other hand, for every $G \in Y_+^\perp$ $(F \geq G)$ we have

$$\|F|_Y| = \|F|_Y - G|_Y| \leq \|F - G| .$$

Taking the infimum with respect to $G \in Y_+^\perp$ $(F \geq G)$ we find

$$\|F|_Y| \leq d_+ \left(F, Y_+^\perp\right) .$$

Therefore, the formula (12) holds, and $Y_+^\perp$ is F - proximinal.

b) Let $G_0 \in P_{Y_+^\perp} (F)$. Then $F \geq G_0$ (according to the definition of $P_{Y_+^\perp} (F)$), $(F - G_0)\,|_Y = F|_Y$ and

$$\|F - G_0| = \inf \left\{\|F - G| : G \in Y_+^\perp, F \geq G\right\} = d_+ \left(F, Y_+^\perp\right) = \|F|_Y|$$

(according to a)). Thus $F - G_0 \in \mathcal{E}_+ (F|_Y)$.

c) Follows from a) and b).

If $H \in \mathcal{E}_+ (F|_Y)$ then $F \geq H, (F - H)\,|_Y = 0$ and

$$\|F - (F - H)| = \|H| = \|F|_Y| = d_+ \left(F, Y_+^\perp\right) ,$$

and then $F - H \in P_{Y_+^\perp} (F)$.

Conversely, $G \in P_{Y_+^\perp} (F)$ implies $F \geq G$, so that $F - G \in X_+^*$, $(F - G)\,|_Y = F|_Y$, and

$$\|F - G| = \|F|_Y| = d_+ \left(F, P_{Y_+^\perp}\right) .$$

It follows that $F-G\in\mathcal{E}_+\left(F|_Y\right)$, i.e. $G\in F-\mathcal{E}_+\left(F|_Y\right)$.

d) If $Y_+^\perp$ is F - Chebyshevian, it results that there exists only one element $G\in P_{Y_+^\perp}(F)$ such that $F\geq G$, so that $F-G\in X_+^*$, $(F-G)|_Y=F|_Y$ and

$$\|F-G|=d_+\left(F,Y_+^\perp\right)=\|F|_Y|,$$

i.e. $\mathcal{E}_+\left(F|_Y\right)$ contains only one element, namely $F-G$.

e) If $F\in\mathcal{E}_+\left(F|_Y\right)$ then there exists $H\in\mathcal{E}_+\left(F|_Y\right)$ such that $F=H$. Thus, according to c) $F-H=F-F=0\in P_{Y_+^\perp}(F)$.

If $0\in P_{Y_+^\perp}(F)$ then $\|F|=d_+\left(F,Y_+^\perp\right)=\|F|_Y|$, so $F\in\mathcal{E}_+\left(F|_Y\right)$. ■

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