PROPOSED PROBLEMS

R. Askey

1. Let $P_{n-1}(x)$ be a polynomial of degree $n-1$, $x_{k,n}$ the zeros of the polynomials orthogonal with respect to $d\alpha(x)$, a positive measure on $[-1, 1]$, and λ_k the corresponding Christoffel numbers. When is

$$[\sum_{k=1}^{n} |P_{n-1}(x_{k,n})|^p \lambda_k]^{\frac{1}{p}} \leqq A [\int_{-1}^{1} |P_{n-1}(x)|^p d\alpha(x)]^{\frac{1}{p}},$$

where A is independent of n and the polynomial $P_{n-1}(x)$. For $p = \infty$ and $p = 2$ it holds for all positive measures with $A = 1$. It holds for $1 \leqq p \leqq \infty$ if $d\alpha(x) = (1-x)^\alpha (1+x)^\beta dx$ for some values of (α, β). For $\alpha = \beta = -\frac{1}{2}$ this was proven by Marcinkiewicz and I can prove it for $\alpha = \beta > -\frac{1}{2}$; $\alpha > \beta = -\frac{1}{2}$; $\alpha = k$; $\beta = 0$; $\alpha = 2k+1$, $\beta = 1$; $\alpha = >$, $\beta = 3$, k a positive integer.

2. Let A and B be two Banach spaces of functions on $[-1, 1]$ with $A \subset B$. Let $P_n(x)$ be a polynomial of degree n. Assume that A contains all polynomials of degree n. Then

$$\| Q_n \|_A \leqq a(n, A, B) \| P_n \|_B.$$

Find the correct order of growth of $a(n, A, B)$. My main interest is the case $A = L^q(d\alpha)$, $B = L^p(d\beta)$, but the problem is also of interest for Lipschitz spaces. (Markov's and Bernstein's inequalities are of this type.) A much harder question is to find the best constant. This is unknown even in the case $A = L^4$, $B = L^2$ for trigonometric polynomials. See A. Garsin, E. Rodemich and H. Rumsey, "On some extremal positive definite functions", *J. of Math. and Mech.*, 18 (1969), 805–834, p. 834, for some related results.

3. (Turán) Find a positive measure $d\alpha(x)$ on $[-1, 1]$ for which $\int_{-1}^{1} |f(x) - L_n^f(x)|^p d\alpha(x) \to 0$ fails for some continuous function for all $p > 2$. (Askey) I conjecture that the measures associated with Pollaczek polynomials have this property. These measures vanish so rapidly at $x = \pm 1$ that $\int_{-1}^{1} \frac{|\log w(x)|\, dx}{(1-x^2)^{\frac{1}{2}}}$ diverges, $d\alpha(x) = w(x)dx$.

4. Prove that $\int_{-1}^{1} |L_n^f(x, \alpha, \beta) - f(x)|^p (1-x)^\gamma (1+x)^\delta\, dx \to 0$ for all continuous functions if $\alpha \geqq \beta > -1$ and

(i) if $\alpha > -\frac{1}{2}$ then $p < \min\big(4(\gamma+1)/(2\alpha+1),\ 4(\delta+1)/(2\beta+1)\big)$,

(ii) if $-1 < \alpha \leqq -\frac{1}{2}$ then $p < \infty$, $\gamma \geqq \alpha$, $\delta \geqq \beta$.

This is known for $\gamma = \alpha$, $\delta = \beta$; $\gamma = \delta = 0$, $p = 2$; $\gamma = \delta = 0$, $p = 1$. Condition (i) is best possible.

5. Let $f(x) \sim \sum_{n=0}^{\infty} a_n P_n^{(\alpha,\beta)}(x)$, $f(x) \geqq 0$ and $\alpha + \beta + 1 \geqq 0$. Prove that the $(C, \alpha + \beta + 2)$ means are non-negative. For $\alpha = \beta = -\frac{1}{2}$, $\alpha = \beta = 0$, $\alpha = -\beta = \frac{1}{2}$ this was shown by Fejér, and for $\alpha = \beta > -\frac{1}{2}$ it was shown by Kogbetliantz.

6. Let $d\alpha(x)$ be a positive measure on $[-1, 1]$ and $p_n(x)$ the corresponding orthonormal polynomials. For which p do we have $\int_{-1}^{1} |f(x) - S_n^f(x)|^p\, d\alpha(x) \to 0$, where $S_n^f(x)$ is the n-th partial sum of the orthogonal series in $p_n(x)$. For

$$d\alpha(x) = (1-x)^{\alpha_1} \prod_{i=2}^{j-1} |x - x_i|^{\alpha_i} (1+x)^{\alpha_j} dx,\ \alpha_1,\ \alpha_j \geqq -\frac{1}{2},\ \alpha_i \geqq 0,$$

$i = 2, 3, \ldots, j-1$, $-1 < x_{j-1} < \ldots < x_2 < 1$, I conjecture that the correct range is

$$4(1+\alpha_i)/(2\alpha_i + 3) < p < 4(1+\alpha_i)/(2\alpha_i + 1),\quad i = 1, \ldots, j,$$

$$2(1+\alpha_i)/(\alpha_i + 2) < p < 2(1+\alpha_i)/\alpha_i,\quad i = 2, \ldots, j-1.$$

Some case with $d\alpha(x)$ a set of point masses should be worked out to see if this influences the range of p, or whether it is only the zeros of $(1-x^2)^{\frac{1}{2}} w(x)\, dx$ that matter; $w(x)$ the derivative of the absolutely continuous part of $d\alpha(x)$.

R. DEVORE

1. Let $C^*[-\pi, \pi]$ denote the space of 2π-periodic continuous functions and $\|\,.\,\|$ the supremum norm on $[-\pi, \pi]$. If (L_n) is a sequence of positive operators such that $L_n(f)$ is a trigonometric polynomial of degree $\leqq n$ for each f and n and if (L_n) satisfies the following conditions

$$\|1 - L_n(1)\| = o\left(\frac{1}{n^2}\right) \qquad (n \to \infty)$$

$$\|\sin x - L_n(\sin t, x)\| = O\left(\frac{1}{n^2}\right) \qquad (n \to \infty)$$

$$\|\cos x - L_n(\cos t, x)\| = O\left(\frac{1}{n^2}\right) \qquad (n \to \infty)$$

then

$$\|f - L_n(f)\| = o\left(\frac{1}{n^2}\right) \qquad (n \to \infty)$$

is equivalent to f is constant on $[-\pi, \pi]$.

2. Give an example of a sequence of positive operators (L_n) such that for each $f \in C([-1, 1]\, L_n(f))$ is an algebraic polynomial of degree $\leqq n$ and

(i) $$\|1 - L_n(1)\| = o\left(\frac{1}{n^2}\right) \qquad (n \to \infty)$$

(ii) $$\|x - L_n(t, x)\| = O\left(\frac{1}{n^2}\right) \qquad (n \to \infty)$$

(iii) $$\|x^2 - L_n(t^2, x^2)\| = O\left(\frac{1}{n^2}\right) \qquad (n \to \infty)$$

where $\|\,.\,\|$ is the supremum norm on $[-1, 1]$.

G. FREUD

1. We recently proved the following result: Let $f \in \mathrm{Lip}\,\alpha$, $0 < \alpha < 1$ a 2π-periodic and let

$$\mathfrak{M}_0(f) = \{x : f(x+h) - f(x) = o(|h|^\alpha)\}$$

$$\mathfrak{M}_1(f) = \{x : f(x+h) - f(x-h) = o(|h|^\alpha)\}$$

$$\mathfrak{M}_2(f) = \{x : f(x+h) + f(x-h) - 2f(x) = o(|h|^\alpha)\}$$

further let $\tilde{f}$ be the harmonic conjugate of f, so that by Privalov's theorem $\tilde{f} \in \mathrm{Lip}\,\alpha$.

THEOREM. *Each two of the sets* $\mathfrak{M}_0(f)$; $\mathfrak{M}_1(f)$, $\mathfrak{M}_2(f)$, $\mathfrak{M}_0(\tilde{f})$, $\mathfrak{M}_1(\tilde{f})$, $\mathfrak{M}_2(\tilde{f})$ *differ by a set of measure zero at most* (G. FREUD, *Studia Mathematica*, 1969). (Proved with the aid of trigonometric approximation.)

Problem 1a: Extend the result to the sets

$$\mathfrak{M}_k(f) = \{x : \Delta_h^k f(x) = o(|h|^\alpha)\}$$

Problem 1b: The operator $f \to \tilde{f}$ is a special singular integral. Extend the result to singular integrals of more general type.

Problem 1c: Extend the result to functions of several variables.

2. Let $e^{Q(x)}$ be a weight function on the whole real axis, let us denote by Π_n the set of polynomials of degree at most n and for an arbitrary on $(-\infty, +\infty)$ continuous f let

$$\varepsilon_n(Q; f) = \inf_{p_n \in \Pi_n} \sup_{-\infty < x < \infty} |f(x) - p_n(x)|\, e^{-Q(x)}.$$

We recently proved that for a continuously differentiable $f(x)$

$$\varepsilon_n(x^2; f) \leq cn^{-1/2}\, \varepsilon_n(x^2; f')$$

where $n^{-1/2}$ is the best possible order of decrease. The emphasis of this result is on the fact, that no restriction concerning the rapidity of increase of $f(x)$ for $x \to \pm\infty$ is supposed.

Previous results on more general $Q(x)$ from M. M. Dzrbasian *(Dokl. A. N. USSR*, **84** (1952), pp. 1123–1126) assume the uniform boundedness of $f(x)$ and those of the speaker (G. Freud, *Acta Math. Ac. Sci. Hung.*, **20** (1969), pp. 223–225) apply only for functions with polynomial growth.

Prove the more general inequality

$$\varepsilon_n(Q; f) \leq \alpha_n\, \varepsilon_n(Q; f')$$

with correct order of α_n.

J. Musielak

1. Let AC_p, $p \geq 1$, be the Banach space of functions f defined in the interval $[a, b]$, $f(a) = 0$, satisfying the following condition: for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for any finite system of non-overlapping subintervals $(a_1, b_1), \ldots, (a_n, b_n)$ of the interval $[a, b]$ the inequality $\sum_{k=1}^{n} (b_k - a_k)^p < \delta$ implies $\sum_{k=1}^{n} |f(b_k) - f(a_k)|^p < \varepsilon$, equipped with the norm $\|f\|_p = \operatorname{Var}_{a \leq x \leq b} f(x)$. Let $\{B_n(f)\}$ be the sequence of Bernstein polynomial of a function $f \in AC_p$. It is known that in case $p = 1$, $\|f - B_n(f)\|_1 \to 0$ as $n \to \infty$. Does the same hold for $p > 1$, i.e. does $\|f - B_n(f)\|_p \to 0$ as $n \to \infty$ for any function $f \in AC_p$, where $p > 1$?

2. Let C be the non-separable Banach space of uniformly almost periodic functions (in the sense of Bohr) on the real line, provided with the norm $\|f\|_C = \sup_{-\infty < x < \infty} |f(x)|$. Find an orthonormal Schauder basis in C.

J. Peetre

1. Does the space C^1 have the interpolation property with respect to the couple $\{C^0, C^2\}$, i.e. is it true that

$$T: \{C^0, C^2\} \to \{C^0, C^2\} \Rightarrow T: C^1 \to C^1\,?$$

It is known that this is true if we substitute Z (Zygmund space) for C^1 (cf. e.g. Lions–Peetre, *Publ. Math. I.H.E.S.* **19** (1964), 5–68). Also the corresponding result is known in the L_p-metric $(1 < p < \infty)$, i.e. W_p^1 has the interpolation property with respect to $\{L_p, W_p^2\}$ (cf. Calderón, *Studia Math.* **24** (1964), 113–190). The proof depends however on the Marcinkiewicz multiplier theorem and does not generalize.

2. Let M_p be the space of Fourier-multipliers on L_p, i.e. $a \in M_p$ if and only if $f \in L_p \Rightarrow F^{-1}$ a $Ff \in L_p$ where F denotes the Fourier transform. Choose a "partition of unity" χ_k $(k = 0, \pm 1, \pm 2, \ldots)$ of the form $\chi_k(\xi) = \chi(\xi/2^k)$ where χ is a function whose support is contained in the interval $(2^{k-1}, 2^{k+1})$. It follows easily from a result of Hardy–Littlewood *(Quart J. Math.* **12** (1941), 221–256) that

$$\left(\sum_{k=-\infty}^{\infty} \|\chi_k a\|_{M_p}^q\right)^{1/q} < \infty \Rightarrow a \in M_p \quad \text{where} \quad \frac{1}{q} = \left|\frac{1}{p} - \frac{1}{2}\right|, \quad 1 < p < \infty.$$

It is possible to replace q by a larger exponent? It follows from e.g. Stein–Zygmund, *(Ann. Math.* **85** (1967), 337–349) that at least $q = \infty$ is not enough.

3. Let E be a, say, finite dimensional vectorspace. Which set-functions f satisfy

$$f(M + N) \leq f(M) + f(N)$$

where $+$ denotes the Minkowsky sum (i.e. $z \in M + N$ if and only if z has a representation of the form $z = x + y$ with $x \in M$ and $y \in N$). I know two trivial solutions: 1° $f(M) = \log_2 \operatorname{card} M$, 2° $f(M) = \dim \overline{M}$ where — denotes the linear hull. They are closely connected with the notions of ε-entropy and n-dimensional width respectively, which explains my interest in the general case.

4. (cf. the second problem posed by Musielak). Does AC_p have the interpolation property with respect to the couple $\{C, AC\}$? A positive answer would in particular solve Musielak's problem. What is the relation of AC_p to the so-called Besov-spaces B_p^{sq} (cf. Besov, *Trudy Mat. Inst. Steklov* **60** (1961), 42–81)? It is known that (Peetre (unpublished); cf. also e.g. Krabbe, *Math. Ann.* **151** (1963), 219–238)

$$B_p^{1/p,p} \cap C \subset AC_p \subset B_p^{1/p,\infty}$$

but are the exponents q involved the best possible?

T. Popoviciu

Considérons un ensemble F du type I_n sur l'intervalle $[a, b]$ et soit $L(x_1, x_2, \ldots, x_n; f \mid x)$ l'élément unique de F qui prend les mêmes valeurs que la fonction $f(x)$ sur les noeuds $x_1, x_2, \ldots, x_n$, supposés distincts.

Trouver tous les ensembles F tel que le quotient

$$\frac{f(x_{n+1}) - L(x_1, x_2, \ldots, x_n; f \mid x_{n+1})}{g(x_{n+1}) - L(x_1, x_2, \ldots, x_n; g \mid x_{n+1})}$$

soit une fonction symétrique des variables (distinctes) $x_1, x_2, \ldots, x_{n+1}$, $g(x)$ étant une fonction F-convexe ou F-concave donnée. La propriété doit avoir lieu pour toute fonction $f(x)$ définie sur l'intervalle $[a, b]$.

Pour la notion d'ensemble du type I_n (ensemble interpolatoire d'ordre n) et pour les notions de fonction F-convexe et de fonction F-concave voir Elena Moldovan «*Sur une généralisation des fonctions convexes*», Mathematica, 1 (24), 1959, 49–80.