CONSIDERATION ON THE "EQUILIBRIUM" ELECTRONS DISTRIBUTION FUNCTION FOR A HOMOGENEOUS, HIGH-FREQUENCY, FULLY IONIZED PLASMA *

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The "equilibrium" electrons distribution function for a homogeneous, high-frequency, fully ionized plasma is

$$f_{0,0}^{0,0} = K_{1,2} \cdot u^{\frac{3A}{3A+1}} \cdot \exp.\left(- \frac{3}{2(3A+1)} \cdot u^2 \right)$$

which have as limit, the global maxwellian electrons distribution function. $f_{0,0}^{0,0}$ is not maxwellian not only to some restrictive physical condition imposed on the plasma (and therefore on the integrated equations) but also to the truncation procedure of the system of equations.

1. INTRODUCTION

In a recent series of papers [1] we showed that the derivation of certain explicit solutions of the Boltzmann equation means in essence to know the "equilibrium" distribution function. The purpose of this paper is to determine the analytical expression of this function for a homogeneous, fully-ionized plasma in an high-frequency electric field.

Following Krusckal and Bernstein [2] we have considered that in the velocity space exist two regions :

— The region in which the collisions are dominated in the energetical distribution of electrons. This means that the collision term is larger that the Lorentz force term and the collisions may be considered as inelastic [3].

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— The conexion region in which the Lorentz force term dominates and the collisions may be considered as elastic. It is true that the residual effect of inelastic collisions persists, especially in the direction of electric field, but the effect of these is very small compared with those of Lorentz forces and in any case may be considered of the same order of magnitude as the neglected terms in the evaluation of explicit solutions of Boltzmann equation [1].

Let us return, now, on the expression of collision frequency :

$$\nu_l \doteq 4\pi N v \left(\frac{\sigma_l}{2l+1} - \sigma_0\right) \tag{1}$$

where N is the ions number, v-the electron velocity and σ_l is the terms of series which is obtained by development of total cross section σ in spherical harmonics [4].

The expression (1) for $l = 2l'$ and $l = 2l' + 1$ becomes :

$$\nu_{2l'} = 4\pi \cdot N \cdot v \left[\frac{\sigma_{2l'}}{4l'+1} - \sigma_0\right] \tag{2}$$

$$\nu_{2l'+1} = 4\pi \cdot N \cdot v \left[\frac{\sigma_{2l'+1}}{4l'+3} - \sigma_0\right] \tag{3}$$

For $l' = 0$, we immediately obtain $\nu_0 = 0$ and $\nu_1 \neq 0$.

The case $\nu_1 \neq 0$ has been studied by us in [1] and it corresponds to the region of velocity space where the Lorentz force term dominates, i.e. to elastic collisions.

The case $\nu_0 = 0$, corresponding to the same region, has no physical significance. Thus for equilibrium, where it is possible to have $E \simeq 0$ or $E \equiv 0$, we are obliged to take also into consideration the inelastic collision term.

The expression of inelastic collision term, or "imparfaitement" Lorentzian in the Jancel terminology [5], has been intensively studied in literature (see f.e. [2], [6], [7] etc.). It is :

$$S_{in}(f_{0,0}^{0,0}) = \frac{m_e}{M \cdot v^2} \cdot \frac{d}{dv}\left(\nu_1 v^3 f_{0,0}^{0,0}\right) + \frac{k^0 T_l}{2Mv^2} \cdot \frac{d}{dv}\left(\nu_1 v^2 \cdot \frac{d}{dv} f_{0,0}^{0,0}\right) \tag{4}$$

or, if we consider the dimensionless coordinates, $u = (v/\bar{v})$, where $\bar{v}$ is mean velocity, we obtain

$$S_{in}(f_{0,0}^{0,0}) = \frac{\delta \bar{\nu}}{2u^2} \cdot \frac{d}{du}\left[u^2 \frac{\nu_1}{\bar{\nu}}\left(u f_{0,0}^{0,0} + \frac{1}{3}\frac{d}{du} f_{0,0}^{0,0}\right)\right] \tag{5}$$

where $f_{0,0}^{0,0}$ is the equilibrium electron distribution function ; $\delta = 2(m_e/M)$; $\bar{\nu}$ is the mean collision frequency.

Taking into consideration (5), the collision term of Boltzmann equation is

$$S = S_{in} + S_{el} = \frac{\delta\bar{\nu}}{2u^2}\cdot\frac{d}{du}\left[u^2\frac{\nu_1}{\bar{\nu}}\left(uf_{0,0}^{0,0} + \frac{1}{3}\frac{d}{du}f_{0,0}^{0,0}\right)\right] +$$

$$+ 4\pi N\cdot\bar{v}\cdot u\left\{\left(\frac{\sigma_{2e'}}{4l'+1} - \sigma_0\right)\cdot f_{2k',n'}^{2l',2m'} + \left(\frac{\sigma_{2l'+1}}{4l'+3} - \sigma_0\right) f_{2k'+1,n'}^{2l'+1,2m'+1}\right\}.$$

The equation for $f_{0,0}^{0,0}$ and its solution are presented in Sec. 2. Some remarks on the obtained results are contained in Sec. 3 and brief conclusions are presented in Sec. 4.

2. EXPRESSION OF $f_{0,0}^{0,0}$

The equation for $f_{0,0}^{0,0}$ is obtained starting from (29). Thus, taking into consideration the inelastic collision term too, we have:

$$S_{in}(f_{0,0}^{0,0}) = \frac{\bar{\nu}\alpha}{6}\frac{d}{du}f_{-1,0}^{1,1} - \frac{\bar{\nu}}{12}\frac{d}{du}f_{-1,0}^{1,-1} \tag{6}$$

where $\alpha = (e^2E_0^2/m_e v^2\bar{\nu}^2)$. But, observing that $f_{-1,0}^{1,1} = f_{-1,0}^{1,-1}$ and taking into account the expression of $f_{-1,0}^{1,1}$ from [1 — v]

$$f_{-1,0}^{1,1} = -\frac{\bar{\nu}\nu_1}{12(\nu_1^2+\omega^2)}\left(\frac{1}{u}f_{0,0}^{0,0} + \frac{d}{du}f_{0,0}^{0,0}\right) \tag{7}$$

and the expression of $S_{in}(f_{0,0}^{0,0})$ from (5), one obtains for $f_{0,0}^{0,0}$ the following equation

$$\frac{d}{du}\left[u^2\left(uf_{0,0}^{0,0} + \frac{1}{3}\frac{d}{du}f_{0,0}^{0,0}\right)\right] + A\cdot u^2\cdot\frac{d}{du}\left[\frac{1}{u}f_{0,0}^{0,0} + \frac{d}{du}f_{0,0}^{0,0}\right] = 0, \tag{8}$$

where:

$$A = \frac{\nu_1\gamma}{24(\nu_1^2+\omega^2)} \tag{9}$$

$$\gamma = \frac{\alpha}{\delta}; \qquad \gamma \ll 1 \tag{10}$$

and $\nu_1 = \bar{\nu}$ for a maxwellian plasma.

But

$$Au^2\frac{d}{du}\left(\frac{1}{u}f_{0,0}^{0,0} + \frac{d}{du}f_{0,0}^{0,0}\right) = \frac{d}{du}\left[Au^2\left(\frac{1}{u}f_{0,0}^{0,0} + \frac{d}{du}f_{0,0}^{0,0}\right) - 2Auf_{0,0}^{0,0}\right]. \tag{11}$$

Replacing, now, (11) in (8) we obtain:

$$\frac{d}{du}\left[u^3 f_{0,0}^{0,0} - Auf_{0,0}^{0,0} + \frac{1}{3}u^2\frac{d}{du}f_{0,0}^{0,0} + Au^2\frac{d}{du}f_{0,0}^{0,0}\right] = 0 \tag{12}$$

that

$$\frac{d}{du}f_{0,0}^{0,0}\cdot\frac{3A+1}{3}u^2 + f_{0,0}^{0,0}(u^3 - Au) = C_1. \tag{12'}$$

The solution of the homogeneous equation ($C_1 = 0$) corresponding to the eq. (12′) is:

$$f_{0,0}^{0,0} = K_1 \exp\left[-\int_0^u \frac{3t}{3A+1}dt + \int_0^u \frac{3A}{2(3A+1)}\cdot\frac{1}{t}dt\right] \tag{13}$$

which may be rewritten:

$$f_{0,0}^{0,0} = K_1\cdot u^{\frac{3A}{3A+1}}\cdot\exp\left(-\frac{3}{2(3A+1)}u^2\right). \tag{14}$$

The constant K_1 may be determined from the normalization condition

$$4\pi\int_0^\infty v^2 f_{0,0}^{0,0}\,dv = n_0. \tag{15}$$

One obtains, thus:

$$K_1 = \frac{n_0}{4\pi\bar{v}^3\int_0^\infty u^{\frac{9A+2}{3A+1}}\exp\left(-\frac{n_3}{2(3A+1)}u^2\right)du}. \tag{16}$$

The integral from denominator is immediately if we observe that:

$$\int_0^\infty u^p e^{-\beta_1 u^2}\,du = \left(\frac{1}{\sqrt{\beta_1}}\right)^{p+1}\int_0^\infty x^p e^{-x^2}\,dx \tag{17}$$

and taking into account [8]

$$\int_0^\infty x^p e^{-x^q}\,dx = \frac{1}{q}\Gamma\left(\frac{p+1}{q}\right) \tag{18}$$

we obtain:

$$K_1 = \frac{n_0}{2\pi\bar{v}^3\left[\frac{2(3A+1)}{3}\right]^{\frac{12A+3}{6A+2}}\cdot\Gamma\left(\frac{12A+3}{2}\right)}. \tag{19}$$

The solution of the inhomogeneous equation (12′) ($C_1 \neq 0$) may be obtained using the method of variation of constants [see e.g. [15]].
By derivation of the equality (14) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}u} f_{0,0}^{0,0} = \left(\frac{\mathrm{d}}{\mathrm{d}u} K_1 \cdot u^{\frac{3A}{3A+1}} + K_1 \cdot \frac{3A}{3A+1} \cdot u^{-\frac{1}{3A+1}} - \right.$$

$$\left. - K_1 \cdot \frac{3}{3A+1} \cdot u^{\frac{6A+1}{3A+1}} \right) \cdot \exp\left(- \frac{3}{2(3A+1)} u^2 \right) \cdot \quad (20)$$

Replacing (20) and (14) in (12′) (with $C_1 \neq 0$) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}u} K_1 = \frac{3C_1}{3A+1} \cdot u^{-\frac{9A+2}{3A+1}} \cdot \exp\cdot\left(\frac{3}{2(3A+1)} u^2 \right) \cdot \quad (21)$$

The solution of the equation (21) will be denoted by

$$K_2(u) = \frac{3C_1}{3A+1} \int_0^u t^{-\frac{9A+2}{3A+1}} \cdot \exp\left(\frac{3t^2}{2(3A+1)} \right) \mathrm{d}t + H \quad (22)$$

with H constant.
Replacing (22) in (14) we obtain the solution for inhomogeneous equation (12′):

$$f_{0,0}^{0,0} = K_2 \cdot u^{\frac{3A}{3A+1}} \cdot \exp\left(- \frac{3}{2(3A+1)} u^2 \right) \quad (23)$$

or

$$f_{0,0}^{0,0} = \frac{3C_1}{3A+1} u^{\frac{3A}{3A+1}} \cdot \exp\left(- \frac{3}{2(3A+1)} u^2 \right) \cdot \int_0^u t^{-\frac{9A+2}{3A+1}} \cdot \exp\left(\frac{3t^2}{2(3A+1)} \right) \mathrm{d}t +$$

$$+ H \cdot u^{\frac{3A}{3A+1}} \cdot \exp\left(- \frac{3}{2(3A+1)} u^2 \right) \cdot \quad (24)$$

The constant H may be get replacing the solution (24) in (15):

$$H = \left\{ n_0 - \frac{12\pi\bar{v}^3 C_1}{3A+1} \int_0^\infty \left[u^{\frac{9A+2}{3A+1}} \exp\left(- \frac{3}{2(3A+1)} u^2 \right) \cdot \right. \right.$$

$$\left. \left. \cdot \int_0^u t^{-\frac{9A+2}{3A+1}} \cdot \exp\left(\frac{3t^2}{2(3A+1)} \right) \mathrm{d}t \right] \mathrm{d}u \right\} \cdot$$

$$\cdot \left[4\pi\bar{v}^3 \int_0^\infty u^{\frac{3A}{3A+1}} \cdot \exp\cdot\left(- \frac{3}{2(3A+1)} u^2 \right) \mathrm{d}u \right]^{-1}$$

or

$$H = \left\{ n_0 - \frac{12\pi\bar{v}^3 C_1}{3A+1} \int_0^\infty \left[u^{\frac{9A+2}{3A+1}} \cdot \exp\left(-\frac{3}{2(3A+1)} u^2\right) \cdot \right.\right.$$

$$\left.\left. \cdot \int_0^u t^{-\frac{9A+2}{3A+1}} \exp\left(\frac{3t^2}{2(3A+1)}\right) \mathrm{d}t \right] \mathrm{d}u \right\} \cdot$$

$$\cdot \left\{ 2\pi\bar{v}^3 \left[\frac{2(3A+1)}{3}\right]^{\frac{12A+3}{6A+2}} \cdot \Gamma\left(\frac{12A+3}{2}\right)\right\}^{-1}. \tag{25}$$

Finally, the solution of (12′) is :

$$f_{0,0}^{0,0} = K_{1,2} \cdot u^{\frac{3A}{3A+1}} \cdot \exp\left(-\frac{3}{2(3A+1)} u^2\right) \tag{26}$$

where

$$K_{1,2} = \begin{cases} K_1 & \text{if } C_1 = 0 \\ K_2(u) & \text{if } C_2 \neq 0 \end{cases}.$$

3. REMARKS

1° For the diagram one can see that the function from (14) has a maximum

$$K_1 \left(\sqrt{\frac{\alpha_1}{2\beta_1}}\right)^{\alpha_1} . \exp\left(\frac{\alpha_1}{2}\right) \text{ for } u = \sqrt{\frac{\alpha_1}{2\beta_1}}$$

where

$$\alpha_1 = \frac{3A}{3A+1} \text{ and } \beta_1 = \frac{3}{2(3A+1)}$$

The diagram is :

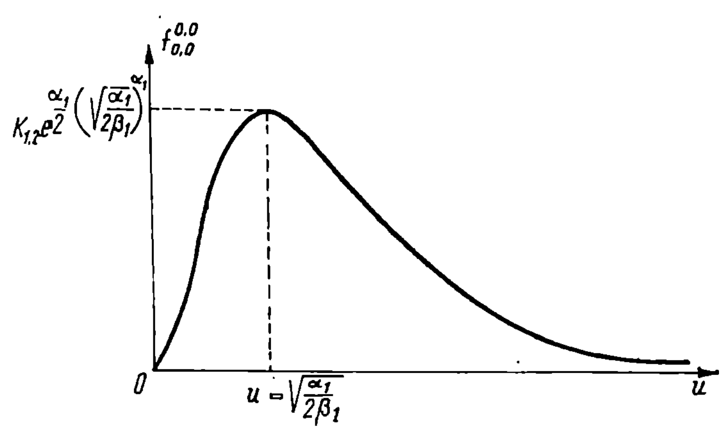

Fig. 1. — The diagram of equilibrium electron distribution function in the velocity space.

which is of the same form as that studied in literature [3], [9].

2°. Because plasma is in a high-frequency electric field, i.e. $\omega \gg \nu_1$, we observe that, allways, we have:

$$\frac{3A}{3A+1} \ll 1 \text{ and } u^{\frac{3A}{3A+1}} \approx 1 \text{ but not} \equiv 1 \tag{27}$$

here we have taken into account (9) and (10).

In this case the expression (14) becomes:

$$f^{0,0}_{0,0} = K_1 . \exp\left(-\frac{3}{2(3A+1)} u^2\right) \tag{28}$$

or

$$f^{0,0}_{0,0} = K_1 \exp\left(-\frac{m_e v^2}{2k^0 T_e\left[1 + \frac{\nu_1 \gamma}{8(\nu^2_1 + \omega^2)}\right]}\right) \tag{29}$$

from where, with [8]:

$$\int_0^\infty x^{2a} . e^{-px^2} \, dx = \frac{(2a-1)!!}{2(2p)^a} \cdot \sqrt{\frac{\pi}{p}} \tag{30}$$

and (15) we obtain:

$$K_1 = n_0 \left[\frac{m_e}{2\pi k^0 T_e\left(1 + \frac{\nu_1 \gamma}{8(\nu^2_1 + \omega^2)}\right)}\right] \cdot \tag{31}$$

Setting:

$$T_{eff} = T'_e = T_e\left(1 + \frac{\nu_1 \gamma}{8(\nu^2_1 + \omega^2)}\right) \tag{32}$$

we obtain for $f^{0,0}_{0,0}$ the expression:

$$f^{0,0}_{0,0} = n_0 \left(\frac{m_e}{2\pi k^0 T'_e}\right)^{3/2} . \exp\left(-\frac{m_e}{2k^0 T'_e} v^2\right) \tag{33}$$

which is the well-known expression of the equilibrium electron distribution function [5], [6], [7], named also, global maxwellian distribution function. The difference of this from "pure" maxwellian distribution is the presence of the effective temperature T'_e. But this latter distribution is obtained immediately where the electrical field is nul, i.e. if

$$E \equiv 0 \Rightarrow E_0 \equiv 0 ; \ \alpha = \gamma \equiv 0 \text{ and } T'_e \equiv T_e \tag{34}$$

then (33) is exactly a maxwellian distribution function done by his author.

Because (14) is a particular case of (23) the above considerations are exactly for this latter case, too.

4. CONCLUSIONS

Up to 1970 the distribution function for electrons at equilibrium was unanimously accepted of the form (33). The results obtained with this distribution were in an acceptable agreement with the experimental data. This is why we utilized it in some previous papers to compare our results with those presented in literature.

In 1970, Wright and Theimer [11] using quasiclasics (or quasi-quantic) considerations showed that the "equilibrium" electron distribution for $E = 0$ is not a maxwellian one but it has a small correction able to explain some desagreements between theory and experimental data.

The distribution of the form (33) may be immediately obtained from the evolution equation for the first two terms of a Hartmann-Margenau type truncation [12] (see [10] too).

The system of differential equations obtained from Boltzmann equation by using a development into spherical harmonics, Fourier series and series in terms of a dimensionless parameter α, [1], [13] lead us to obtain an "equilibrium" electron distribution function for a homogeneous, fully ionized plasma in a high-frequency electric field of the form (14) or (23) which differs from (33). These are approximatively of the same form for $\omega \gg \nu_1$.

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