Nonlinear Analysis, Theory, Methods & Applications, Vol. 10, No. 7, pp. 669–678, 1986.
Printed in Great Britain.

0362-546X/86 $3.00 + .00
Pergamon Journals Ltd.

A NONCONVEX VECTOR MINIMIZATION PROBLEM

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(*Received* 1 *August* 1984; *received for publication* 29 *July* 1985)

Key words and phrases: Ordered regular vector spaces, nonconvex vector minimization, axiom of choice.

INTRODUCTION

IN ORDER to produce a vector minimization principle which contains Ekeland's variational theorem [8] as well as the results of [13, 14], we have to introduce cone valued metrics. Working with these metrics provides noteworthy technical facilities in some application oriented investigations, the most relevant one from our point of view being the fixed point theory. From the results of Eisenfeld and Lakshmikantham [5–7] comes the idea of application of regular cone valued metrics, which will play an important role in this paper.

The normal cone valued metrics induce uniformizable topologies and every uniformizable Hausdorff topology can be induced by a normal and regular cone valued metric according to a result due to Antonovskij, Boltjanskij and Sarymsakov [1]. The above considerations, largely exposed in our preprint [15] will only be summarized in Section 5. Problems concerning a special sort of relativized regularity considered first in [14] will be given in Sections 3 and 4, after showing by some examples in Section 2 the consistency of the notion introduced. The principal result of this paper is theorem 6.1, which constitutes a general nonconvex vector minimization principle containing Ekeland's variational principle [8] and the results of [13, 14]. The principle comprised by theorem 6.1 is in fact a criterion for the above-mentioned relativized regularity of a cone. This is asserted by theorem 6.2. This paper ends with the deduction from theorem 6.1 generalizations of some results due to Eisenfeld and Lakshmikantham [7], among others an ordered vector space variant of the Kirk–Caristi fixed point theorem [10].

The proofs use the axiom of choice. In [16] we have shown that a denumerable variant of this axiom suffices when E is a Fréchet space. This is the case also when $E = R$ with the natural ordering. The relations of this axiom with the fixed point theorem cited above and with other ordering principles of the analysis were considered by Brønsted [2] (see also the assertion in [16, Section 1]).

1. DEFINITIONS

Let E be a vector space over the reals and let K be an acute convex cone in it, i.e. a set having the properties: (i) $K + K \subset K$; (ii) $tK \subset K$ for each nonnegative real number t; (iii) $K \cap (-K) = \{0\}$. We shall refer to K with these properties simply as to a *cone* in E, and shall suppose throughout that $K \neq \{0\}$. Putting $x \leq y$ whenever $y - x$ is in K, we obtain a reflexive, transitive and antisymmetric order relation on E, which is translation invariant and invariant with respect to the multiplication with nonnegative reals. It is called the *order induced* by K or simply the *K-order* in E. The vector space E endowed with an order relation as defined above is called *ordered vector space* and the cone K inducing the order in it is called its *positive cone*. *K bounded*, *K monotone* etc. will mean bounded, monotone etc, with respect to the order induced by K.

The set A in the ordered vector space E with the positive cone K will be called *full* if $A = (A + K) \cap (A - K)$.

Suppose that E is an ordered vector space endowed with a locally convex vector space topology which is Hausdorff. The positive cone K in E is said to be *normal* if the zero element of E has a neighbourhood basis $B(0)$ consisting of full sets.

The cone K_0 in K is called *K bound regular* (*sequentially K bound regular*) if each K_0 increasing and K order bounded net (sequence) in K_0 converges to an element of K_0. If K is itself K bound regular (sequentially K bound regular) then it is called *regular* (*sequentially regular*).

If K_0 is a K bound regular subcone of K then it is obviously a regular cone.

The cone K is called *fully regular* if each of its K increasing and topologically bounded nets is convergent to one of its elements.

A stronger notion of relativized regularity was considered in [11, 1.8.4]. Various concepts of bound regularity were introduced and used in [14].

2. EXAMPLES OF REGULAR CONES

2.1. Let K be a normal cone of the locally convex Hausdorff space E. Suppose that K_0 is subcone of K which is complete and has the property that its linear span is complete, metrizable and does not contain any subspace isomorphic with the Banach space c_0 of the real sequences converging to 0 in its usual norm. According to theorem 1 of McArthur in [12], K_0 is fully regular. Since K is normal, every K_0 increasing K order bounded net in K_0 is topologically bounded too (see for example [17, II, proposition 1.4]) and hence convergent by the full regularity of K_0. The limit of such a net is in K_0, the latter being complete by hypothesis. We have thus shown that K_0 is a K bound regular subcone of K.

2.2. Suppose that B is a complete, convex and bounded subset of the locally convex Hausdorff space E. Suppose B does not contain 0. Let K_0 be the cone generated by B. Then K_0 is a K bound regular subcone of every normal cone K containing it. Indeed, if we consider the subspace sp K_0 of E, then this space can be endowed with a norm considering the unit ball in it the convex circled hull of B. This topology on sp K_0 is finer than that induced from E and sp K_0 is a Banach space with respect to the norm. In this space K_0 admits a plastering and hence it is fully regular according to theorem 1.12 of Krasnosel'skij in [11]. Now, since K is a normal cone containing K_0 the reasoning in 2.1 applies to conclude K_0 is K bound regular.

2.3. Let S be a nonempty set and let R^S be the vector space of all the real functions defined on S, endowed with the topology of the pointwise convergence. This topology is in fact the direct product topology of R^S and converts this space in a locally convex Hausdorff space. The cone R^S_+ of all the nonnegative functions in R^S is normal since the generating family of seminorms p_s on R^S defined by $p_s(x) = |x(s)|$ is monotone on it ([17, II, proposition 1.5]). It is also regular, since the convergence in R^S is the pointwise one and hence the regularity of R^S_+ is a direct consequence of the regularity of R_+, the nonnegative half line.

2.4. Let R^S be the vector space considered at 2.3. Denote by $c_0(S)$ the subspace consisting of the functions x having the property that for every positive real ε the set $\{s \in S : |x(s) > \varepsilon\}$ is finite. Let us define a norm on $c_0(S)$ by putting $\|x\| = \max_s |x(s)|$. Equipped with this norm,

$c_0(S)$ becomes a Banach space ([3, II.2]). The cone $c_0^+(S)$ of the nonnegative functions in $c_0(S)$ is normal and regular.

2.5. Consider the Banach space c of all the convergent sequences of real numbers endowed with the sup norm. Then c^+, the cone of sequences with nonnegative terms is normal. Let c_0 be the subspace in c of the sequences converging to 0. Then $c_0^+ = c^+ \cap c_0$ is a regular cone which is neither c^+ bound regular, nor fully regular.

2.6. Let K be a generating closed normal cone in the barelled space E. The set L^+ of all the linear and continuous operators A with the property that $A(K) \subset K$ forms a regular cone in the space $L(E)$ of all the linear and continuous operators acting in E and equipped with the topology of simple convergence (see [15, 9.1]).

2.7. Let H be a real Hilbert space. The linear and continuous operator A acting in H is called *positive* if $(Ax, x) \geqslant 0$ for every x in H. Let the vector space $L(H)$ of all the linear and continuous operators acting in H be endowed with the topology of simple convergence. Then the set L^+ of all the positive operators forms a regular cone in $L(H)$ ([15, 9.2]).

3. A SUMMATION CRITERION FOR REGULARITY

The reduction of definitions of regularity to criterions which use a denumerable set of terms is very desirable from a technical point of view. In this direction we observe first of all that:

$$\left.\begin{array}{l}\text{a complete cone } K_0 \text{ in } K \text{ is } K \text{ bound regular if and only if}\\ \text{it is sequentially } K \text{ bound regular.}\end{array}\right\} \quad (3.1)$$

Proof of (3.1). The "only if" part is immediate. For the converse implication let us assume that K_0 is sequentially K bound regular but is not K bound regular. Then there exists the K_0 increasing K bounded net $(x_i)_{i\in I}$ in K_0 which is not Cauchy. That is, there exists a neighbourhood U of 0 such that for any i in I there exists x_j and x_k with $j \geqslant i$ and $k \geqslant i$ such that $x_j - x_k \notin U$. We can suppose $x_k - x_j \in K_0$ since (x_i) is K_0 increasing. Fix i and consider x_j and x_k as above. Put $x_1 = x_j$ and $x_2 = x_k$. Starting with x_k instead of x_i, we can determine x_3 and x_4 in (x_i) so as to have $x_3 - x_2 \in K_0$, $x_4 - x_5 \in K_0$ and $x_4 - x_3 \notin U$. Continuing this procedure we obtain a sequence which is K_0 increasing and K bounded, but is not Cauchy, contradicting the hypothesis. ■

The above assertion is used in establishing the following summation criterion:

$$\left.\begin{array}{l}\text{The complete subcone } K_0 \text{ of the cone } K \text{ is } K \text{ bound regular if and}\\ \text{only if for every sequence } (x_n) \text{ of } K_0 \text{ the condition } x_n \notin U\\ \text{for any } n \text{ and for some neighbourhood } U \text{ of 0 implies that the}\\ \text{set } \{s_m : m \in N\} \text{ where } s_m = \sum_{n=1}^{m} x_n, \text{ cannot be } K \text{ order bounded.}\end{array}\right\} \quad (3.2)$$

Proof of (3.2). Suppose that there is a sequence (x_n) in K_0 with $x_n \notin U$ for every n and for some neighbourhood U of 0 such that $\left\{s_m = \sum_{n=1}^{m} x_n : m \in N\right\}$ is K order bounded. Then (s_m) forms a K_0 increasing K bounded sequence which is not convergent, that is, K_0 cannot be sequentially K bound regular.

Assume now that K_0 does not be K bound regular, and hence neither sequentially K bound regular by (3.1). Hence K_0 contains a K_0 increasing K order bounded sequence (z_n) with the property that there exists a neighbourhood U of 0 such that $z_{n+1} - z_n \notin U$ for every n. We can assume $z_1 = 0$. Then the elements $x_n = z_{n+1} - z_n$ form a sequence in K_0 such that

$$\left\{s_m = \sum_{n=1}^{m} x_n = z_{m+1} : m \in N\right\}$$

is K order bounded. This contradicts the condition of the criterion and completes the proof. ■

Summation criteria for various types of regularity were considered by McArthur [12] and by the author in [13, 14]. The summation techniques in some problems concerning the sequentially regular cones had already been used by Krasnosel'skij in [11, theorems 1.6 and 1.7].

4. NEAR TO MINIMUM POINT CRITERION FOR REGULARITY

Let E be a vector space ordered by the cone K. Suppose that M is a set in E and H is a subset of K. The point x in M will be said an *H near to minimum point of M* if

$$(x - H - K) \cap M = \emptyset.$$

Another regularity criterion for a cone can be stated in terms of near to minimality. It is:

$$\begin{array}{l}\text{the complete subcone } K_0 \text{ of the cone } K \text{ in the locally convex space}\\ E \text{ is } K \text{ bound regular if and only if for each nonempty set } H\\ \text{in } K_0 \text{ with the property that } E\backslash H \text{ is a neighbourhood of } 0,\\ \text{every } K \text{ lower bounded subset of } E \text{ has } H \text{ near to minimum points.}\end{array} \tag{4.1}$$

Proof of (4.1). Suppose that M is a set of E which is K lower bounded by b. Assume that H is as in (4.1) and there are no H near to minimum points in M. Then

$$(x - H - K) \cap M \neq \emptyset$$

for each x in M. Let us consider $x_1 \in M$ arbitrarily and choose

$$x_{n+1} \in (x_n - H - K) \cap M.$$

$n = 1, 2, \ldots$. Then we have

$$x_{n+1} \leqslant x_n - h_n,\ n \in N$$

for some h_n in H, where $\leqslant$ denotes the K order in E. Summing these relations from $n = 1$ to $n = m$, we get

$$\sum_{n=1}^{m} h_n \leqslant x_1 - x_{m+1} \leqslant x_1 - b.$$

By the hypothesis on H, there exists a neighbourhood U of 0 in E so as to have $h_n \notin U$ for every n. This, together with the above relation show, via the criterion 3.2, that K_0 cannot be K bound regular.

To prove the "if" part of the assertion, assume that every subset of E which is K bounded from below has H near to minimum points for each H as in 4.1, but K_0 is not sequentially K bound regular. Using the criterion 3.2 again, it follows that there exist the elements x_n in K_0 without some neighbourhood U of 0 such that the sums

$$\left\{\sum_{n=1}^{m} x_n, m \in N\right\}$$

have a K upper bound b. This means that the set

$$M = \left\{-\sum_{n=1}^{m} x_n : m \in N\right\}$$

is K order bounded from below by $-b$. Let x be an arbitrarily chosen element of M. Then $x = -\sum_{n=1}^{m} x_n$ for some m. We have

$$x - x_{m+1} = -\sum_{n=1}^{m+1} x_n \in M$$

and

$$x - x_{m+1} \in x - H \subset x - H - K,$$

where we have denoted by H the set $\{x_n : n \in N\}$. That is, we have got

$$(x - H - K) \cap M \neq \emptyset$$

for every x in M. Since H satisfies the hypothesis in the proposition, this is a contradiction. ■

We shall use criterion 4.1 in the following slightly modified form:

> The complete subcone K_0 of the cone K in the locally convex space E is K bound regular if and only if for each H in K_0 ($H \neq \emptyset$) with the property that $E \setminus H$ is a neighbourhood of 0, every set M in E which has K order bounded K lower sections, i.e. which contains a point z such that the set (4.2)
>
> $$(z - K) \cap M$$
>
> is K order bounded, has H near to minimum points.

5. CONE VALUED METRICS

Let E be an ordered vector space with the positive cone K. A K *metric* r on a nonempty set V is a mapping r of $V \times V$ into K which satisfies for arbitrary elements u, v and z in V the following conditions: (i) $r(u, u) = 0$; (ii) $r(u, v) = 0$ implies $u = v$; (iii) $r(u, v) = r(v, u)$; (iv) $r(u, z) \leq r(u, v) + r(v, z)$, where $\leq$ stands for the K ordering.

If K is a normal cone in the locally convex Hausdorff space E, then the sets

$$V(U, a) = \{u \in V : r(a, u) \in U\}, U \in B(0)$$

with $B(0)$ a neighbourhood basis in E consisting of full sets, and a runs over V, form a neighbourhood basis for a Hausdorff topology on V. The resulting topology is a uniformizable one; it is called *the topology induced by r on V*. Cauchy nets and complexity can be defined in a natural way (for details see [15]). Antonovskij, Boltjanskij and Sarymsakov have shown in [1, 11.4 and 11.5] that a completely regular Hausdorff topological space is K metrizable with K the positive cone R^S_+ of the space R^S considered in 2.3. We have observed that R^S_+ is a normal and regular cone. That is, every completely regular Hausdorff topological space is K metrizable with K a normal and regular cone. Hence by [9, I.15], the Hausdorff topological spaces which are K metrizable by a K metric of this kind are quite those which are uniformizable.

Regular cone valued metrics were considered by Eisenfeld and Lakshmikantham in [5–7] for the case when K is a regular cone of a Banach space. Since criteria using a denumerable set of terms also work for regular cones in nonmetrizable locally convex Hausdorff spaces, we have as an immediate consequence of one of them, the criterion 3.2, the following assertion:

Let V be a set endowed with a K_0 metric r, where K_0 is a K bound regular subcone of the normal cone K in the locally convex Hausdorff space E. Suppose that (v_n) is a sequence in V such that the set

$$\left\{\sum_{n=1}^{m} r(v_{n+1}, v_n) : m \in N\right\} \tag{5.1}$$

is K order bounded. Then (v_n) is Cauchy in the topology on V induced by the K_0 metric r.

6. NONCONVEX MINIMIZATION

Let V be a topological space and let F be an operator from V to the ordered topological vector space E. We shall say that F is *submonotone* if from the conditions:

(i) $\lim_{\nu} v_\nu = v$, where $(v_\nu)_{\nu \in I}$ is a net in V indexed by the totally ordered set I;

(ii) $F(v_\nu) \leq F(v_\mu)$ whenever $\nu \geq \mu$, it follows that

$$F(v) \leq F(v_\nu) \quad \text{for every } \nu \text{ in } I.$$

Observe that submonotonicity is a feeble sort of lower semicontinuity of operators with values in ordered vector spaces. Various related but stronger notions were considered in [5–7] and in [13–15].

The main result of our note is the following.

THEOREM 6.1. Let E be a locally convex Hausdorff space and let K be a closed normal cone in E. Suppose that K_0 is a K bound regular complete subcone of K.

Let (V, r) be a complete K_0 metric space, and let $F: V \to E$ be a submonotone operator with respect to the K ordering.

Suppose that F has K order bounded K lower sections, i.e. that there exists at least an element z in V such that the set

(i) $(F(z) - K) \cap F(V)$ has a K lower bound.

Then for every z with the property (i) and for every positive real ε there is a v so as to have

(ii) $F(z) - F(v) - \varepsilon r(z, v) \in K$

and

(iii) $F(v) - F(w) - \varepsilon r(v, w) \notin K$ whenever $w \in V \backslash \{v\}$.

Let U be a neighbourhood of 0 in E. If $H = K \backslash U \neq \varnothing$, then for a z with the property (i) there exists a u in V such that

(iv) $F(z) - F(u) \in K$

and

(v) $(F(u) - \varepsilon H - K) \cap F(V) = \varnothing$.

For every u with this property there is an element v in V satisfying (iii) and the condition (ii) with u instead of z, and such that

(vi) $r(u, v) \in U$.

Proof. Define the relation $\prec$ on $F(V)$ by putting $F(p) \prec F(q)$ if

$$F(q) - F(p) - \varepsilon r(p, q) \in K.$$

It is straightforward that $\prec$ is reflexive, transitive and antisymmetric, hence an order relation on $F(V)$. Apply Hausdorff's theorem (see [4, I.2.6]) to determine a subset Z in $F(V)$ which is totally ordered with respect to the relation $\prec$, has $F(z)$ as supremum, and is maximal with respect to the set theoretic inclusion. We shall show that Z contains its infimum with respect to $\prec$.

Let us introduce a relation $\leqslant$ in $F^{-1}(Z) = V_0$ by putting $p \leqslant q$ if $F(p) \prec F(q)$. Then V_0 will be totally ordered with respect to $\leqslant$ and the filter of its lower sections is Cauchy. To verify this, let us assume the contrary: there exists a neighbourhood U' of 0 such that for each s in V_0 there are p and q in V_0, $p \leqslant s$ and $q \leqslant s$, such that $r(p, q) \notin U'$. Fix s and let p and q be as above. We can suppose $p \leqslant q$. Put $v_1 = q$, $v_2 = p$. Then $r(v_2, v_1) \notin U'$ and $F(v_1) - F(v_2) - \varepsilon r(v_2, v_1) \in K$. Starting with p instead of s we can continue this procedure. Accordingly we can determine the decreasing sequence (v_n) in V_0 such that

$$r(v_{2k}, v_{2k-1}) \notin U' \quad \text{for every } k. \tag{6.1}$$

From the definition of the relation $\leqslant$ on V_0 we have also

$$F(v_n) - F(v_{n+1}) - \varepsilon r(v_{n+1}, v_n) \in K \quad \text{for each } n.$$

By summing this relation from $n = 1$ to $n = m$, we get

$$F(v_1) - F(v_{n+1}) - \varepsilon \sum_{n=1}^{m} r(v_{n+1}, v_n) \in K.$$

Since the elements $F(v_n)$ are all in the set (i), they have a K lower bound, say y_0. Adding with $F(v_{n+1}) - y_0 \in K$, the above relation yields

$$F(v_1) - y_0 - \varepsilon \sum_{n=1}^{m} r(v_{n+1}, v_n) \in K.$$

The obtained relation shows that the sums

$$\sum_{n=1}^{m} r(v_{n+1}, v_n), m \in N$$

are K order bounded, wherefrom we get, via the assertion (5.1), a contradiction with (6.1).

The obtained contradiction shows that the lower sections of V_0 form a Cauchy filter, which converges by the completeness of V to v.

Since F is submonotone with respect to the K order, we have

$$F(p) - F(v) \in K \tag{6.2}$$

for every p in V_0. Let q be arbitrary in V_0. For every $p \leqslant q$ we have

$$F(q) - F(p) - \varepsilon r(p, q) \in K$$

which by adding to (6.2) yields

$$F(q) - F(v) - \varepsilon r(p, q) \in K.$$

Letting $p \to v$ in this relation, taking into account K is closed, it follows

$$F(q) - F(v) - \varepsilon r(v, q) \in K.$$

that is, $F(v) \prec F(q)$ for each $F(q)$ in Z. Now, since Z is maximal, $F(v)$ must be in Z and it is the infimum of Z with respect to $\prec$.

The last assertion implies also that there does not exist any w in $V\backslash\{v\}$ so as to have $F(w) \prec F(v)$. Thus we have proved the relations (ii) and (iii).

If H is the set defined in the theorem, then by 4.2, with $F(V)$ for M and with εH for H, we conclude the existence in V of u with the properties (iv) and (v).

If we proceed as above taking u in place of z, we can get a v in V so as to have (iii) and (ii) with u instead of z, that is, to have the relation

$$F(v) \in F(u) - \varepsilon r(u, v) - K. \tag{6.3}$$

We assume now that (vi) does not hold. Then we have $r(u, v) \in K\backslash U = H$, that is,

$$F(u) - \varepsilon r(u, v) - K \subset F(u) - \varepsilon H - K.$$

this relation together with (6.3) contradict (v). ∎

We shall show that the principal result of theorem 6.1 consisting in the existence of a v so as to have relation (iii), which may be considered a nonconvex vector minimization principle, is the best possible with respect to the order relation in E, or, in other words, it characterizes the K bound regular subcones. More precisely, we have the following.

THEOREM 6.2. Let K be a closed normal cone of the locally convex Hausdorff space E and let K_0 be a complete subcone of K. Then the minimization principle comprised in the existence of a v satisfying (iii) of theorem 6.1, holds for every K_0 metric space (V, r) and every K submonotone mapping from V to E which has K bounded K lower sections, if and only if K_0 is K bound regular.

Proof. The 'if' part is contained in theorem 6.1. For the converse implication let us suppose that K_0 does not be K bound regular. Then by the criterion (3.1) there exist a neighbourhood U of 0 in E and a K_0 increasing sequence (x_n) in K_0 which is K order bounded and for which $x_{n+1} - x_n \notin U$ for every n. Put $V = \{x_n : n \in N\}$. Define a K_0 metric r on V by putting $r(x_k, x_n) = x_m - x_n$, where $m = \max\{h, k\}$, $n = \min\{h, k\}$. Then V is trivially r complete since it is discrete. Let us define $F: V \to E$ by putting $F(x) = -x$. Then F is K lower bounded (since $V = \{x_n\}$ is K order bounded). Because V is discrete, F is trivially K submonotone.

Put $\varepsilon = 1/2$ and consider x_n to be arbitrary in V. Let $m > n$. Then

$$F(x_n) - F(x_m) - \tfrac{1}{2}r(x_n, x_m) = -x_n + x_m - \tfrac{1}{2}(x_m - x_n) = \tfrac{1}{2}(x_m - x_n) \in K.$$

We have in conclusion for every v $(=x_n)$ in V that there exists some w $(=x_m$ with $m > n)$ in $V \setminus \{v\}$ such that the relation (iii) in theorem 6.1 fails. ∎

We observe the parallelism in form and in content between our theorem and the criterion (4.2). In fact, the nonconvex minimization principle can be considered itself a criterion for K bound regularity.

7. A FIXED POINT THEOREM

In the papers [5–7], Eisenfeld and Lakshmikantham have succeeded in extending various important results in metric fixed point theory for matrices with values in regular cones in Banach spaces. In [7] (Lemma 3.3) they extend the Kirk–Caristi fixed point theorem for metrics with values in regular minihedral cones with nonempty interior in separable Banach spaces. Theorem 6.1 permits us to obtain an essentially extended form of this theorem. The quoted result of [7] is the key of obtaining the principal result of the cited paper, which also can be extended using our following theorem.

THEOREM 7.1. Let E be a locally convex Hausdorff space, K a closed normal cone in E and K_0 a complete K bound regular subcone of K. Let (V, r) be a complete K_0 metric space and let F be an operator from V to E which is K submonotone and has the property that the set

$$(F(z) - K) \cap F(V)$$

is K order bounded for some z in V. If $f: V \to V$ satisfies the condition

$$r(f(u), u) \leq F(u) - F(f(u)) \tag{7.1}$$

for every u in V, then f has a fixed point v such that

$$F(v) \leq F(z) - r(v, z),$$

where $\leq$ stands for the K order.

Proof. Put $\varepsilon = 1$ and apply theorem 6.1 to V and F in the above theorem. Then it follows the existence of a v in V such that $F(v) \leq F(z) - r(v, z)$ and

$$F(v) - F(w) - r(w, v) \notin K \text{ whenever } w \in V \setminus \{v\}. \tag{7.2}$$

On the other hand (7.1) implies

$$F(v) - F(f(v)) - r(f(v), v) \in K.$$

If we would have $f(v) \neq v$, the obtained relation would contradict the relation (7.2). ∎

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