# ON THE SUBDIFFERENTIABILITY OF CONVEX OPERATORS

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## *Introduction*

Since its inception in the papers of Valadier [14] and Levin [5], the theory of vector-valued convex analysis has been concerned with operators having values in order-complete vector lattices. The special interest of these spaces is motivated by the nice subdifferentiability properties of convex operators acting in such spaces. Besides considerable progress in this direction (see [11] and the references therein), there are results concerning more general ordered vector spaces. Zowe [15], and recently Borwein [1], have obtained results when conditions are imposed on the domain space of operators. Another approach is to build up the subgradients using directional minorants (or directional derivatives, when a topology is given). This method, initiated by Fel'dman [4] (see also [8, 2]), has the advantage that conditions are imposed only on the range space. This approach suggests the natural question of how to characterize the ordered (topological) vector spaces in which every convex operator has directional minorants (directional derivatives) at each point and in every direction. Such a characterization can be done in terms of classical ordered (topological) vector spaces, and this is the principal result of the present note. It furnishes, using Fel'dman's theorem, a necessary and sufficient condition for every convex operator with values in an ordered vector space admitting isotone real functionals to have nice subdifferentiability properties.

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## 1. *Prerequisites*

An *ordered vector space* is, by definition, a pair $(Y, K)$, where $Y$ is a real vector space and $K$ is a *cone* in it, that is, a subset having the properties (i) $K+K \subset K$, (ii) $tK \subset K$ for each non-negative real number $t$ and (iii) $K \cap (-K) = \{0\}$. The *order relation induced by $K$ in $Y$* is defined by putting $u \leqslant v$ whenever $v-u \in K$. Then $\leqslant$ is a reflexive, transitive and antisymmetric relation which is invariant with respect to translations and to multiplication by non-negative real numbers. The cone $K$ is called the *positive cone* of the ordered vector space.

The ordered vector space $(Y, K)$ is *Archimedian* if $y \leqslant 0$ whenever $y \leqslant tz$ for some $z \in K$ and all $t > 0$.

The space $(Y, K)$ is said to *admit an isotone functional* if there exists $f\colon Y \to \mathbb{R}$ having the property that for any $u$ and $v$ in $Y$, the relations $u \leqslant v$, $u \neq v$ imply that $f(u) < f(v)$.

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The space $(Y, K)$ is said to have the *monotone sequence* (*monotone net*) property if every decreasing sequence (net) in $Y$ with a lower bound has an infimum.

If we suppose that $Y$ is a topological vector space, then $(Y, K)$ will be called an *ordered topological vector space*. All the topological vector spaces we shall deal with are supposed Hausdorff.

The ordered topological vector space $(Y, K)$ is called *normal* if there is a base of neighbourhoods $V$ of 0 with

$$V = (V-K) \cap (K-V).$$

The ordered topological vector space $(Y, K)$ is called *regular* (*sequentially regular*) if every decreasing net (sequence) having a lower bound has a limit in $Y$. If $(Y, K)$ is regular (sequentially regular) and if $K$ is closed, then $(Y, K)$ has the monotone net (sequence) property by [**12**, Corollary II.3.2]. Observe that in this case $(Y, K)$ is Daniell space. An ordered topological vector space is called (*countably*) *Daniell* if it has the monotone net (sequence) property and every decreasing net (sequence) with a lower bound converges to its infimum.

Let $F$ be an operator from a real vector space $X$ to $(Y, K)$. Then $F$ is called *convex* if

$$F(tx_1+(1-t)x_2) \leqslant tF(x_1)+(1-t)F(x_2)$$

for all $x_1$, $x_2$ in $x$ and all $t$ in $[0, 1]$.

The *directional minorant of $F$ at $x_0$ in the direction $h$* is defined by

$$\nabla F(x_0; h) = \inf_{t>0} t^{-1}(F(x_0+th)-F(x_0))$$

when this infimum exists.

Suppose that $(Y, K)$ is an ordered topological vector space. Then the *directional derivative of $F$ at $x_0$ in the direction $h$* is the limit

$$F'(x_0; h) = \lim_{t \searrow 0} t^{-1}(F(x_0+th)-F(x_0)),$$

if it exists.

Let $x_0$ and $h$ be fixed in $X$. If $F$ is convex, then the operator

$$\phi(t) = t^{-1}(F(x_0+th)-F(x_0)) \qquad (*)$$

is increasing on $\mathbb{R}\backslash\{0\}$ (see for example [**5**] or [**14**]). Hence, when $K$ is closed and $F'(x_0; h)$ exists, $\nabla F(x_0; h)$ also exists (this follows from [**12**, Corollary II.3.2]).

Let $L(X, Y)$ be the space of linear operators from $X$ to $Y$. The set

$$\partial F(x_0) = \{A \in L(X, Y): Ax \leqslant F(x_0+x)-F(x_0) \text{ for all } x \in X\}$$

is called the *subdifferential of $F$ at $x_0$*. The elements of $\partial F(x_0)$ are called *subgradients of $F$ at $x_0$*.

For future reference we shall use Greek letters for alternatives in the following known result concerning the existence of directional minorants and directional derivatives.

PROPOSITION 1. ($\alpha$) *If $(Y, K)$ is an ordered vector space with the monotone sequence property, then each convex operator with values in $(Y, K)$ has a directional minorant in every direction at every point of its domain.*

($\beta$) *If* $(Y, K)$ *is a sequentially regular ordered topological vector space, then each convex operator with values in* $(Y, K)$ *has a directional derivative in every direction at every point of its domain.*

The proof is straightforward. The alternative ($\alpha$) appears, for instance, in [**14**] and in [**1**, Proposition 3.7(a)].

Since ($\beta$) was not explicitly stated in this form (usually the normality of $K$ is also required—see for example [**14**, Theorem 5.1] or [**1**, Proposition 3.7(c)]) we give its proof.

Let $F$ be a convex operator from the vector space $X$ to $(Y, K)$. Let $x_0$ and $h$ be fixed in $X$. The operator $\phi$ defined by (*) is increasing on $\mathbb{R}\setminus\{0\}$. If we assume that $\lim_{t\searrow 0}\phi(t)$ does not exist, then we get a neighbourhood $U$ of 0 in $Y$ for which we can construct a decreasing sequence $(t_n)$ of real numbers converging to 0 such that

$$\phi(t_{2k-1})-\phi(t_{2k})\notin U \quad \text{for each } k.$$

Now, since the sequence $(\phi(t_n))$ is decreasing and $F(x_0)-F(x_0-h)$ is a lower bound for it, we get a contradiction with the hypothesis.

REMARK. For some ordered topological vector spaces, regularity implies normality. McArthur [**7**] has shown that every closed regular cone in a Fréchet space is normal. In the case of locally convex spaces he gave in [**6**] conditions in order that every closed normal cone be regular. There exist ordered normed spaces which are regular but lack normality [**9**].

## 2. *Main results*

We state first our principal results in the form of two theorems.

THEOREM 1. *Let* $(Y, K)$ *be an ordered vector space and let* $X$ *be a vector space of dimension greater than or equal to* 1. *The following assertions are equivalent.*

(i) $(Y, K)$ *has the monotone sequence property.*

(ii) *Each convex operator* $F: X \to (y, K)$ *possesses a directional derivative in every direction at every point.*

(iii) *Each convex operator* $F: X \to (Y, K)$ *possesses a directional minorant at* 0 *in some non-zero direction.*

THEOREM 2. *Let* $(Y, K)$ *be an ordered topological vector space and let* $X$ *be a vector space of dimension at least* 1. *Then the following assertions are equivalent.*

(i) $(Y, K)$ *is sequentially regular.*

(ii) *Each convex operator* $F: X \to (Y, K)$ *possesses a directional derivative in every direction at every point.*

(iii) *Each convex operator* $F: X \to (Y, K)$ *possesses a directional derivative at* 0 *in some non-zero direction.*

We shall follow in the proofs a schema proposed by J. M. Borwein (which simplifies essentially our original version in [**9**]). Consider first some auxiliary results.

PROPOSITION 2. *Let $(a_n)$ and $(y_n)$ be sequences in $\mathbb{R}$ and $Y$, respectively with $a_n > a_{n+1}$ for each $n \in \mathbb{N}$. In order that a convex operator $f: \mathbb{R} \to Y$ such that $f(a_n) = y_n$ exists, it is necessary and sufficient that*

$$\Delta_n = \frac{y_{n+1} - y_n}{a_{n+1} - a_n}$$

*is decreasing with respect to $n$.*

*Proof.* We have $\Delta_n \geqslant \Delta_{n+1}$ if and only if

$$y_{n+1} \leqslant \frac{a_{n+1} - a_{n+2}}{a_n - a_{n+2}} y_n + \frac{a_n - a_{n+1}}{a_n - a_{n+2}} y_{n+2}.$$

This relation shows the necessity of the condition. To verify the sufficiency, define $f_n: \mathbb{R} \to Y$ by

$$f_n(t) = (t - a_n)\Delta_n + y_n,$$

and observe that each $f_n$ is affine and

$$f_n(t) - f_{n+1}(t) = (\Delta_n - \Delta_{n+1})(t - a_{n+1}).$$

Thus

$$f(t) = \max_{k \in \mathbb{N}} f_k(t) = f_n(t) \quad \text{if } t \in [a_{n+1}, a_n]$$

defines a convex operator with $f(a_n) = y_n$.

PROPOSITION 3. *Let $(u_n)$ be a sequence in $K$ with $u_{n+1} \leqslant t_n u_n$ for some $t_n$ in $(0, 1)$. Then one can select a sequence $(a_n)$ in $\mathbb{R}$ decreasing to zero such that the operator $f$ defined on $\{a_n\}$ by $f(a_n) = a_n u_n$ has a convex extension to $\mathbb{R}$.*

*Proof.* Inductively, suppose that $a_1, \ldots, a_n$ have been selected. By Proposition 2 it is sufficient to find $r$ small enough so that

$$\Delta_{n-1} = \frac{a_{n-1} u_{n-1} - a_n u_n}{a_{n-1} - a_n} \geqslant \frac{a_n u_n - r u_{n+1}}{a_n - r} = \Delta_n(r).$$

From the hypothesis, we can find $\delta_n > 0$ such that

$$\Delta_{n-1} = \frac{a_{n-1}}{a_{n-1} - a_n}(u_{n-1} - u_n) + u_n \geqslant (1 + \delta_n) u_n$$

while, since $u_{n+1} \geqslant 0$, for every $r$ with $0 < r < a_n$ we obtain

$$\Delta_n(r) = \frac{r}{a_n - r}(u_n - u_{n+1}) + u_n \leqslant \left(1 + \frac{r}{a_n - r}\right) u_n.$$

Take $r$ sufficiently small in order to have $r/(a_n - r) \leqslant \delta_n$ as well; then $\Delta_{n-1} \leqslant \Delta_n(r)$ and $0 < a_{n+1} < a_n$, as desired.

LEMMA. *If each convex operator $f: X \to (Y, K)$ possesses a directional minorant at $0$ in some non-zero direction, then $(Y, K)$ is Archimedian.*

*Proof.* We shall show that if $(Y, K)$ is not Archimedian, then there exists a convex operator from $\mathbb{R}$ to $(Y, K)$ without directional minorant at $0 \in \mathbb{R}$ in the direction $1 \in \mathbb{R}$. Observe first that $(Y, K)$ is Archimedian if and only if each set of the form $\{tx:t>0\}$ with $x\in K$ has an infimum. Indeed, suppose that $v$ is an infimum of the set $\{tx:t>0\}$, where $x\in K$. Then $v\geqslant 0$ and $sx\geqslant rv$ for arbitrary $s>0$ and $r>0$. If we fix $r=2$, then it follows that $2v$ is a lower bound for the set $\{tx:t>0\}$, thus $v\geqslant 2v$, that is, $v\leqslant 0$. Hence if every set of this kind has an infimum, it must be 0. But then, if $tx\geqslant y$ for some $x\in K$ and every $t>0$, it follows that $y\leqslant 0$, that is, $(Y,K)$ is Archimedian. The converse is immediate. Now assuming that $(Y,K)$ is not Archimedian and considering a set $\{tx:t>0\}$, where $x\in K$, without infimum, we define $f:\mathbb{R}\to(Y,K)$ by putting $f(t)=0$ for $t\leqslant 0$ and $f(t)=t^2x$ for $t>0$. Then $f$ is obviously convex and the set $\{t^{-1}(f(t)-f(0))=tx:t>0\}$ has no infimum, that is, $\nabla f(0;1)$ does not exist.

*Proof of Theorem* 1. Clearly (ii) implies (iii), while (i) implies (ii) by Proposition 1($\alpha$). To show that (iii) implies (i) we argue as follows. Let $(v_n)$ be a decreasing sequence in $K$. Let $u_n=v_n(n+1)/n$. Then $0\leqslant u_n\leqslant(1-1/n^2)u_{n-1}$ for each $n$, and Proposition 3 applies. Let $F(t)=f(|t|)$ with $f$ constructed as in Proposition 3. Then $d=\nabla F(0;1)$ exists by hypothesis and

$$d=\inf_{n\in\mathbb{N}} a_n^{-1}(F(a_n)-F(0))=\inf_{n\in\mathbb{N}} u_n.$$

Since $(Y,K)$ must be Archimedian by the preceding lemma, $d$ also will be an infimum of $(v_n)$. Indeed, we have $d\leqslant(1+1/n)v_m$ for all $n$ and $m$. Fix $m$ for the moment. Then $d-v_m\leqslant v_m/n$ for all $n$ and hence $d\leqslant v_m$, that is, $d$ is a lower bound for $(v_m)$. Every other lower bound $v$ of $(v_n)$ will be a lower bound for $(u_n)$ too, and hence $v\leqslant d$. This completes the proof.

*Proof of Theorem* 2. Clearly (ii) implies (iii), while (i) implies (ii) by Proposition 1($\beta$). For the proof that (iii) implies (i) we consider a decreasing sequence $(v_n)$ in $K$. Then, since $u_n=v_n(n+1)/n$, we can construct by Proposition 3 a convex function $F:R\to Y$ such that $F(0)=0$ and $F(a_n)=a_n u_n$ with some sequence $(a_n)$ decreasing to zero in $\mathbb{R}$. If we consider now

$$\lim_{n\to\infty} a_n^{-1}(F(a_n)-F(0)),$$

which certainly exists, we deduce that $\lim_{n\to\infty}u_n$ exists. But $v_n=(n/(n+1))u_n$ has obviously the same limit, and so Theorem 2 is proved.

## 3. *Application to subdifferentiability*

Let $X$ be a vector space and let $(Y,K)$ be an ordered (topological) vector space. Suppose that $F:X\to(Y,K)$ is a convex operator. We shall say that $F$ is *fully subdifferentiable* at $x_0\in X$ if (i) $\nabla F(x_0;\cdot)$ (or $F'(x_0;\cdot)$) is defined on $X$, (ii) $\partial F(x_0)$ is non-empty, and (iii) for every $h$ in $X$ one has the relation $\nabla F(x_0;h)=\max\{Ah: A\in\partial F(x_0)\}$ (or $F'(x_0;h)=\max\{Ah: A\in\partial F(x_0)\}$).

As a consequence of a result of Fel'dman **[4]** (see also **[8, 2]**) combined with some continuity properties of convex operators in **[3, 1]** we have the following.

PROPOSITION 4. *Let F be a convex operator from the vector space X to the ordered vector space (Y, K) with the monotone net property or to the ordered topological vector space (Y, K) which is Daniell. Then F is fully subdifferentiable at each point of X.*

*If $X$ is a topological vector space, $(Y, K)$ is an ordered normal topological vector space which is Daniell, and if $F$ is continuous at $x_0$, then it is fully subdifferentiable at $x_0$ and all the maps in the definition of the full subdifferentiability are continuous.*

By using Theorems 1 and 2 we can give the following characterization of the ordered vector spaces in which all the convex operators are fully subdifferentiable.

THEOREM 3. *Let $(Y, K)$ be an ordered vector space (respectively an ordered topological vector space with closed positive cone) which admits an isotone functional. Then the following assertions are equivalent.*

(i) *$(Y, K)$ has the monotone sequence property (respectively is countable Daniell).*

(ii) *Every convex operator $F$ from a vector space $X$ to $(Y, K)$ has directional minorants (respectively directional derivatives) in every direction at every point in $X$.*

(iii) *Every convex operator $F$ from a vector space $X$ to $(Y, K)$ is fully subdifferentiable at each point of $X$.*

*Proof.* Condition (i) implies condition (ii) by Proposition 1. To verify that (ii) implies (iii) we observe that in the presence of an isotone functional the monotone sequence property (respectively the countable Daniell property) implies the monotone net property (respectively the Daniell property), the reasoning being similar to that in [**13**, Proposition II.4.9]. Hence Proposition 4 can be used. Finally (iii) implies (i) according Theorem 1 (respectively Theorem 2).

REMARK. In [**4, 2**] it was shown that the condition $\partial F(x) \neq \varnothing$ for each convex operator $F$ with values in $(Y, K)$, called the *subgradient property of* $(Y, K)$, does not imply that the space has the monotone sequence property. In the given counter-examples $(Y, K)$ is not Archimedian. Our results say nothing about the relationship between the monotone sequence property and either the subgradient or the Archimedian properties. (For other problems concerning the subgradient property see [**10**].)

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