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On the two-dimensional inverse problem of dynamics

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The authors extend the deduction of the equations satisfied by the force fields from inertial to rotating frames, when the curves of a certain family are known to be solutions for the equations of motion. Then Drâmbă's equation is obtained as a consequence of this result. The works of Hadamard and Moiseev are proved to be closely related to the inverse problem of dynamics.

Key words: inverse problem of dynamics – inertial frame – rotating frame

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1. Introduction

The inverse problem of dynamics consists in finding the force field (or the potential) which governs the motion of a dynamical system, knowing a given family of orbits. The first outstanding results are due to Newton (1687), who found that the forces generating spirals or ellipses are proportional to the distance or inversely proportional to the square or the cube of the distances.

The problem was reconsidered at the end of the XIXth century by Bertrand (1877) and was generalized by Dainelli (1880) and Jukovski (1890) (more informations on this period for the Russian school are given in Shorokhov (1988)), their results being presented by Whittaker (1904). These are closely related to the research of Hadamard (1897) on the contacts of solutions of a dynamical system with a given family of curves, which was then extended by Moiseev (1934) to systems in a rotating frame. The inverse problem for this last situation was considered by Drâmbă (1963), fact reported by Stavinschi and Mioc (1993); but the paper which gave a real impulse to the field of the inverse problems was that of Szebehely's (1974). An impressive number of papers were ever since dedicated to the subject, a recent synthesis being contained in Bozis (1994).

The aim of this note is to show that Drâmbă's equation can be obtained from a generalization of Dainelli's result to rotating systems and to reveal the importance of the works of Hadamard (1897) and Moiseev (1934).

2. Inverse problems in an inertial frame

We devote this section to the problem considered by Dainelli (1880) and Whittaker (1904, p.93), following the ideas of Dainelli, whose proof is simpler than Whittaker's one using the tangential and normal components of acceleration.

Being given a family of orbits

$$f(x,y) = c = \text{ constant}, \tag{1}$$

one looks for the force field (the force depending only on the position coordinates (x,y)) for which this family is an orbit family of a particle. More precisely, one will find the components $X(x,y)$ and $Y(x,y)$ of the force under whose action a particle of the unit mass will describe the orbit family (1), the motion being governed by the system

$$\begin{aligned} \ddot{x} &= X \\ \ddot{y} &= Y. \end{aligned} \tag{2}$$

Theorem 1. Let $D \subset R^2$ be an open set and $f \in C^2(D)$ such that $f_x^2(x,y) + f_y^2(x,y) \neq 0$ for each $(x,y) \in D$. If the system (2) admits as orbits the curves of the family (1), then

$$\begin{aligned} X &= k\,(f_{xy}f_y - f_{yy}f_x) + (k_x f_y - k_y f_x)\, f_y/2 \\ Y &= k\,(f_{xy}f_x - f_{xx}f_y) - (k_x f_y - k_y f_x)\, f_x/2, \end{aligned} \tag{3}$$

where $k \in C^1(D, R_+)$ is an arbitrary function.

Proof. Differentiating in (1) we obtain

$$\dot{x} f_x + \dot{y} f_y = 0, \tag{4}$$

hence $\dot{x}$ and $\dot{y}$ will be given by

$$\begin{aligned} \dot{x} &= \pm\sqrt{k}\, f_y \\ \dot{y} &= \mp\sqrt{k}\, f_x, \end{aligned} \tag{5}$$

the sign depending on the sense of the motion on the orbit. Differentiating again we have

$$\begin{aligned} \ddot{x} &= k\,(f_{xy}f_y - f_{yy}f_x) + (k_x f_y - k_y f_x)\, f_y/2 \\ \ddot{y} &= k\,(f_{xy}f_x - f_{xx}f_y) - (k_x f_y - k_y f_x)\, f_x/2. \end{aligned} \tag{6}$$

Using the equations (2) we obtain the relations (3). □

Remark 1. For $k = K^2F^2$ one gets the values of X and Y given by Dainelli (1880), for $k = -u$ those given by Whittaker (1904), and for $k = g^2$ those given by Broucke and Lass (1977), where the relations (3) are also given as

$$\begin{aligned} X &= \tfrac{1}{2}(kf_y^2)_x - \tfrac{1}{2}\frac{f_x}{f_y}(kf_y^2)_y \\ Y &= \tfrac{1}{2}(kf_x^2)_y - \tfrac{1}{2}\frac{f_y}{f_x}(kf_x^2)_x. \end{aligned}$$

Now we shall derive the partial differential equation obtained by Szebehely (1974) for the potential function V as a direct consequence of Dainelli's result.

The problem is to determine the potential energy U or the potential function $V = -U \in C^1(D)$ for which the equations of the motion of a unit mass particle are

$$\begin{aligned} \ddot{x} &= V_x \\ \ddot{y} &= V_y, \end{aligned} \tag{7}$$

knowing a given family of orbits (1).

The system (7) admits the energy integral

$$\dot{x}^2 + \dot{y}^2 = 2(V + E(f)), \tag{8}$$

$E(f)$ being constant on every orbit in (1).

Theorem 2. Let $D \subset R^2$ be an open set and $f \in C^2(D)$ such that $f_x^2(x,y) + f_y^2(x,y) \neq 0$ for each $(x,y) \in D$. If the system (7) admits as orbits the curves of the family (1), then the function V satisfies the partial differential equation

$$f_x V_x + f_y V_y + \frac{2(V + E(f))}{f_x^2 + f_y^2}\left(f_{xx}f_y^2 - 2f_{xy}f_x f_y + f_{yy}f_x^2\right) = 0. \tag{9}$$

Proof. Applying theorem (1) we have

$$\begin{aligned} V_x &= k\,(f_{xy}f_y - f_{yy}f_x) + (k_x f_y - k_y f_x)\, f_y/2 \\ V_y &= k\,(f_{xy}f_x - f_{xx}f_y) - (k_x f_y - k_y f_x)\, f_x/2, \end{aligned}$$

with $k \in C^1(D, R_+)$ satisfying the relations (5).

Eliminating the expression which contains the partial derivatives of k in the relations above one obtains

$$f_x V_x + f_y V_y = k\left(2f_{xy}f_x f_y - f_x^2 f_{yy} - f_y^2 f_{xx}\right). \tag{10}$$

From the energy integral and the relations (5), the function k is given by

$$k = 2\,(V + E(f))\,/\,(f_x^2 + f_y^2).$$

Replacing this in (10) one obtains the well-known Szebehely's equation (9). □

Many years ago, Hadamard (1897) studied the case when the isoenergetic trajectories of the system (7) with a given potential function V have a contact of order at least two with the curves of the family (1). This means that at the point $(x, y) = (x(t), y(t))$ we have

$$\Phi(t) = 0,\ \Phi'(t) = 0,\ \Phi''(t) = 0, \tag{11}$$

where

$$\Phi(t) = f(x(t),\ y(t)) - c, \tag{12}$$

$(x(t)\,, y(t))$ being a solution of the system (7). In fact, Hadamard studied more general systems in curvilinear coordinates, but applying his result to (7), we get the following statement.

Theorem 3. In the conditions of theorem (2), if the solutions of the system (7) have a contact of order at least two with the curves of the family (1), then the following relation is true

$$f_x V_x + f_y V_y + \frac{2(V+E)}{f_x^2 + f_y^2}\left(f_{xx} f_y^2 - 2 f_{xy} f_x f_y + f_{yy} f_x^2\right) = 0. \tag{13}$$

Proof. The third equality in (11) is

$$f_x \ddot{x} + f_y \ddot{y} + f_{xx} \dot{x}^2 + 2 f_{xy} \dot{x}\dot{y} + f_{yy} \dot{y}^2 = 0.$$

Substituting $\ddot{x}$ and $\ddot{y}$ from the equations (7), $\dot{x}, \dot{y}$ from the second equality in (11), and taking into account the energy integral

$$\dot{x}^2 + \dot{y}^2 = 2(V + E),$$

we obtain the relation (13). □

Remark 2. If the curves of the family (1) are solutions of the system (7), they have obviously a contact of any order at each point, so theorem is a consequence of theorem (3) (it is not essential in theorem (3) that the trajectories are isoenergetic). But Hadamard, who obtained the relation (13) (even in a more general form) between the potential V and the family of curves (1), did not interpreted it explicitly in terms of an inverse problem. Nevertheless, we have the feeling that his result must be mentioned when speaking of the history of the inverse problem.

3. Inverse problems in a rotating frame

Dainelli considered the system (2) which does not depend on $\dot{x}$ and $\dot{y}$, but one can study a similar system corresponding to a rotating frame. Let this system be

$$\begin{aligned} \ddot{x} - 2\dot{y} &= X \\ \ddot{y} + 2\dot{x} &= Y, \end{aligned} \tag{14}$$

where X, Y are functions of (x, y).

Suppose that in the rotating frame xOy a family of planar curves (1) can be traced by the moving particle. We obtain then a result similar to that of Dainelli (1880).

Theorem 4. Let the conditions in theorem (1) be satisfied by the family of curves (1). If the system (14) admits as orbits the curves of the family (1), then X and Y are given by

$$\begin{aligned} X &= \pm 2\sqrt{k} f_x + k\left(f_{xy} f_y - f_{yy} f_x\right) + \left(k_x f_y - k_y f_x\right) f_y / 2 \\ Y &= \pm 2\sqrt{k} f_y + k\left(f_{xy} f_x - f_{xx} f_y\right) - \left(k_x f_y - k_y f_x\right) f_x / 2, \end{aligned} \tag{15}$$

$k \in C^1(D, R_+)$ being an arbitrary function.

Proof. Following the same idea of Dainelli (1880) as in theorem , we obtain the relations (4), (5) and (6). Replacing now in (14) the values of $\ddot{x}, \ddot{y}, \dot{x}$ and $\dot{y}$ obtained as functions of k, f and their partial derivatives, we finally get the relations (15). □

Remark 3. It is clear that in this case the sense of motion on the orbit is significant, because the system is no more reversible. That is why the $\pm$ sign appears in (15), which was not the case in (3) for the reversible system (2).

Remark 4. The relations (15) have the equivalent form

$$X = \pm 2\sqrt{k} f_x + \tfrac{1}{2}(k f_y^2)_x - \tfrac{1}{2}\frac{f_x}{f_y}(k f_y^2)_y$$
$$Y = \pm 2\sqrt{k} f_y + \tfrac{1}{2}(k f_x^2)_y - \tfrac{1}{2}\frac{f_y}{f_x}(k f_x^2)_x.$$

From theorem (4) we can deduce the condition satisfied by the potential energy U or the potential function $V = -U \in C^1(D)$ appearing in a system of the type

$$\begin{aligned} \ddot{x} - 2\dot{y} &= V_x \\ \ddot{y} + 2\dot{x} &= V_y, \end{aligned} \tag{16}$$

which has as orbits the curves in the family (1).

The system (16) admits the energy integral

$$\dot{x}^2 + \dot{y}^2 = 2(V + E(f)). \tag{17}$$

The result for potential systems in rotating frames was obtained by Drâmbă (1963) and by Szebehely and Broucke (1981). Their result may be proved as follows.

Theorem 5. Let the conditions in theorem (1) be satisfied by the family of curves (1). If the system (16) admits as orbits the curves of the family (1), then the function V satisfies the partial differential equation

$$f_x V_x + f_y V_y \mp 2\left[2\left(V + E(f)\right)\right]^{1/2}\left(f_x^2 + f_y^2\right)^{1/2} + \frac{2(V + E(f))}{f_x^2 + f_y^2}\left(f_{xx} f_y^2 - 2 f_{xy} f_x f_y + f_{yy} f_x^2\right) = 0. \tag{18}$$

Proof. From theorem (4) with $X = V_x$, $Y = V_y$ we obtain

$$V_x = \pm 2\sqrt{k} f_x + k\left(f_{xy} f_y - f_{yy} f_x\right) + \left(k_x f_y - k_y f_x\right) f_y / 2$$
$$V_y = \pm 2\sqrt{k} f_y + k\left(f_{xy} f_x - f_{xx} f_y\right) - \left(k_x f_y - k_y f_x\right) f_x / 2.$$

Multiplying the two equations by f_x, respectively f_y, and summing up we get

$$f_x V_x + f_y V_y + k\left(f_{xx} f_y^2 - 2 f_{xy} f_x f_y + f_{yy} f_x^2\right) \mp 2\sqrt{k}\left(f_x^2 + f_y^2\right) = 0.$$

Having in mind that

$$\dot{x} = \pm\sqrt{k} f_y$$
$$\dot{y} = \mp\sqrt{k} f_x,$$

the energy integral (17) gives

$$k = 2(V + E(f))/(f_x^2 + f_y^2),$$

and the relation (18) follows. □

The considerations of Hadamard for inertial frames were extended by Moiseev (1934, 1936) for rotating frames. Moiseev obtained the partial differential equation (18) as a necessary condition for the system (16) to have isoenergetic solutions with a contact of order at least two with the curves of the family (1).

Moiseev's result is the following:

Theorem 6. If the family of curves (1) satisfies the conditions in theorem (1), and the solutions of the system (16) have a contact of order at least two with the curves of the family (1), then the following relation holds

$$f_x V_x + f_y V_y + \frac{2(V + E)}{f_x^2 + f_y^2}(f_{xx} f_y^2 - 2 f_{xy} f_x f_y + f_{yy} f_x^2) \mp 2[2(V + E)]^{1/2}(f_x^2 + f_y^2)^{1/2} = 0. \tag{19}$$

Proof. From the second equality in (11) we get (4), wherefrom

$$\begin{aligned} \dot{x} &= \pm\frac{f_y v}{(f_x^2 + f_y^2)^{1/2}} \\ \dot{y} &= \mp\frac{f_x v}{(f_x^2 + f_y^2)^{1/2}}, \end{aligned} \tag{20}$$

the upper sign corresponding to trajectories of negative direction.

From the last equality in (11), after substituting $\dot{x}$ and $\dot{y}$ from (20), and $\ddot{x}$, $\ddot{y}$ from (7), we obtain

$$f_x V_x + f_y V_y + \frac{v^2}{f_x^2 + f_y^2}(f_{xx} f_y^2 - 2 f_{xy} f_x f_y + f_{yy} f_x^2) \mp 2v(f_x^2 + f_y^2)^{1/2} = 0. \tag{21}$$

Using the energy integral with $\dot{x}$, $\dot{y}$ from (20) one gets

$$v^2 = 2(V + E)$$

and the theorem is proved. □

Remark 5. The function v in (20) (introduced by Moiseev) and k in (5) are related by

$$k = v^2/(f_x^2 + f_y^2),$$

they both being a measure of the velocity of the particle on the curves of the family (1).

Remark 6. As in the case of Hadamard, the conditions in Moiseev's theorem are weaker then those in theorem (5), so the latter is its consequence (again it is not essential to have isoenergetic trajectories). Moiseev himself did not consider the problem as an inverse one, nevertheless his condition (19) is exactly the one which permits to find an unknown potential function when a given family of curves (1) represents a family of trajectories.

Moiseev (1936) used the following notations

$$P = \sqrt{f_x^2 + f_y^2}$$

$$Q = V_x f_x + V_y f_y$$

$$S = f_{xx} f_y^2 - 2 f_{xy} f_x f_y + f_{yy} f_x^2$$

and wrote (19) in a simplified way

$$2S(V + E) + P^2 Q \mp 2nP^3 \sqrt{2(V + E)} = 0. \tag{22}$$

The number n in (22) represents the angular velocity of the rotating frame (which is not necessarily 1), the system considered by Moiseev being

$$\ddot{x} = 2n\dot{y} + V_x$$

$$\ddot{y} = -2n\dot{x} + V_y.$$

Due to the similarity of the relations obtained by Hadamard (1897) and Moiseev (1934, 1936) to the results in the inverse problems of dynamics, the authors propose these papers to be mentioned as milestones in this research area too.

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