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Inhomogeneous potentials producing homogeneous orbits

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We prove that, in general, a given two-dimensional inhomogeneous potential $V(x,y)$ does not allow for the creation of homogeneous families of orbits. Yet, depending on the case at hand, if the given potential satisfies certain conditions, this potential is compatible either with one (or two) monoparametric homogeneous families of orbits or at most with five such families. The orbits are then found on the grounds of the given potential.

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1. Introduction

The two-dimensional inverse problem of dynamics seeks all the potentials $V(x,y)$ which can give rise to a preassigned monoparametric family of curves $f(x,y) = c$, traced by a unit mass material point. If the total energy dependence $E = E(f(x,y))$ is not given in advance, the connection between orbits and potentials is established by a partial differential equation of the second order (Bozis 1984). The equation is linear in $V(x,y)$, of the hyperbolic type with coefficients depending merely on the given orbits.

The above equation, if rearranged adequately, can also serve to face the direct problem, i.e. given a potential to seek all monoparametric families which can be created by this potential, for adequate initial conditions, of course. Indeed, it turns out that for a function $\gamma(x,y) = f_y/f_x$, related to the slope of the given orbits, the second order partial equation is now nonlinear in the unknown function $\gamma(x,y)$. The direct problem then requires the solution of a harder to solve differential equation.

In the framework of the direct problem, one expects that additional information regarding the orbits will generally facilitate its solution. Such information is e.g. the homogeneity of the family of orbits, i.e., the property of the family to include geometrically similar orbits. The case of having homogeneous families produced by homogeneous potentials has been studied by Bozis and Stefiades (1993) and Bozis and Grigoriadou (1993) and led to an ordinary differential equation.

In the present paper we study the following version of the direct problem: in the system of cartesian coordinates Oxy, a purely inhomogeneous potential V is given. Are there any homogeneous families of orbits satisfying the system of differential equations

$$\ddot{x} = -V_x \ ,$$
$$\ddot{y} = -V_y \ ,$$

i.e., created by this potential?

For an affirmative answer we find that:

(i) In general, certain conditions have to be satisfied by the given potential. In this case there exist no more than two homogeneous monoparametric families of orbits consistent with the given potential. They correspond to the common roots of a quadratic and a quintic algebraic equation.

(ii) In some special cases (specifically if the aforementioned quadratic equation becomes a triviality), it may be that there exist at most five solutions although the existence of at least one solution is not guaranteed.

2. The equation of the inverse problem

Consider a monoparametric family of planar curves

$$f(x,y) = c \tag{1}$$

traced in the inertial frame Oxy by a material point of unit mass under the action of the potential $V = V(x,y)$ and introduce the notation

$$\gamma = \frac{f_y}{f_x}. \tag{2}$$

To each function $f(x,y)$ there corresponds one function $\gamma(x,y)$ and, vice versa, to each $\gamma(x,y)$ there corresponds a family (1). Thus, the function $\gamma(x,y)$ replaces the family (1) and allows us to refer to it as the family of orbits.

Compatible pairs of potentials $V(x,y)$ and orbits $\gamma(x,y)$ are related by the partial differential equation (Bozis 1995)

$$\gamma^2\gamma_{xx} - 2\gamma\gamma_{xy} + \gamma_{yy} = h, \tag{3}$$

where

$$h = \frac{\gamma\gamma_x - \gamma_y}{V_x + \gamma V_y}\left(-\gamma_x V_x + (2\gamma\gamma_x - 3\gamma_y)\, V_y + \gamma\,(V_{xx} - V_{yy}) + (\gamma^2 - 1)V_{xy}\right). \tag{4}$$

Comments

1.(i) It can be shown easily that the expression $\gamma\gamma_x - \gamma_y$ appearing in (4) does not become identically equal to zero in a domain $\boldsymbol{D} \subseteq \boldsymbol{R}^2$ but only for families of straight lines, excluded from our study.

(ii) The expression $V_x + \gamma V_y$, which originally is a factor in the left hand side of equation (3), cannot be identically zero in a domain $\boldsymbol{D} \subseteq \boldsymbol{R}^2$ for inhomogeneous potentials $V(x,y)$, because then the numerator of the right hand side must also be zero and this happens only for constant potentials.

2. Equation (3) is nonlinear in the unknown function $\gamma(x,y)$ and is more suitable to answer the direct problem: Given a potential, find families of orbits generated by it. The dynamical system being autonomous, it is understood that the most general solution of equation (3) can depend on no more than two independent arbitrary constants.

3. Homogeneous families produced by inhomogeneous potentials

In what follows we study a version of the direct problem. We assume that the given potential $V(x,y)$ is not homogeneous (i.e. no m exists such that $xV_x + yV_y = mV$) and we try to find homogeneous families of orbits (1).

No matter what the degree of homogeneity of $f(x,y)$ might be the function γ, defined by (2), will then be homogeneous of degree zero, i.e.

$$x\gamma_x + y\gamma_y = 0. \tag{5}$$

So, for a given inhomogeneous $V(x,y)$ we now look for functions $\gamma(x,y)$ satisfying both equations (3) and (5). To this end:

(i) We differentiate equation (5) with respect to x and y, thus obtaining two equations including second order derivatives of the function $\gamma(x,y)$.

(ii) We solve for γ_{xx}, γ_{xy}, γ_{yy} the algebraic system of the above two equations and equation (3). At the same time we express γ_y in terms of γ_x in view of equation (5). In so doing we obtain the system

$$\begin{aligned}
\gamma_{xx} &= \tfrac{1}{\Pi}\left(K\gamma_x^2 + L\gamma_x\right)\\
\gamma_{xy} &= -\tfrac{1}{y\Pi}\left\{xK\gamma_x^2 + (\Pi + xL)\,\gamma_x\right\}\\
\gamma_{yy} &= \tfrac{x}{y^2\Pi}\left\{xK\gamma_x^2 + (2\Pi + xL)\,\gamma_x\right\}
\end{aligned} \tag{6}$$

where

$$\begin{aligned} \Pi &= (x+y\gamma)\,(V_x+\gamma V_y) \\ K &= 2yV_y\gamma+(-yV_x+3xV_y) \\ L &= yV_{xy}\gamma^2+(yV_{xx}-yV_{yy}-2V_y)\,\gamma-(yV_{xy}+2V_x)\,. \end{aligned} \tag{7}$$

From now on we adopt the notation

$$V_{ij}=\frac{\partial^{i+j}V}{\partial x^i\partial y^j}$$

e.g. $V_{01}=V_y$, $V_{12}=V_{xyy}$ etc.

Working out the compatibility conditions $(\gamma_{xx})_y=(\gamma_{xy})_x$ and $(\gamma_{xy})_y=(\gamma_{yy})_x$ for the system (6) and taking into account the system itself as well as equation (5) we come up with one equation which reads

$$\gamma_x=\frac{yA}{(x+y\gamma)\,B}, \tag{8}$$

where

$$A=A_3\gamma^3+A_2\gamma^2+A_1\gamma+A_0 \tag{9}$$

with

$$A_3=-V_{11}V_{01}+x(V_{11}^2-V_{21}V_{01})+y(V_{11}V_{02}-V_{12}V_{01})$$

$$\begin{aligned} A_2 &= V_{02}V_{01}-V_{11}V_{10}-V_{20}V_{01} \\ &+x(V_{12}V_{01}-V_{21}V_{10}-V_{30}V_{01}-V_{11}V_{02}+2V_{20}V_{11}) \\ &+y(V_{03}V_{01}-V_{12}V_{10}-V_{21}V_{01}+V_{20}V_{02}+V_{11}^2-V_{02}^2) \end{aligned}$$

$$\begin{aligned} A_1 &= V_{02}V_{10}-V_{20}V_{10}+V_{11}V_{01} \\ &+x(V_{12}V_{10}-V_{30}V_{10}+V_{21}V_{01}+V_{20}^2-V_{20}V_{02}-V_{11}^2) \\ &+y(V_{03}V_{10}-V_{21}V_{10}+V_{12}V_{01}+V_{20}V_{11}-2V_{11}V_{02}) \end{aligned}$$

$$A_0=V_{11}V_{10}+x(V_{21}V_{10}-V_{20}V_{11})+y(V_{12}V_{10}-V_{11}^2)$$

and

$$B=3\,(x(V_{11}V_{10}-V_{20}V_{01})+y(V_{02}V_{10}-V_{11}V_{01}))\,.$$

Comment: It is of vital importance to observe that the above equation (8) cannot be obtained for homogeneous potentials. In fact, it is an easy matter to show that if $V\,(x,y)$ is homogeneous in x,y, say of degree m, i.e. if $xV_{10}+yV_{01}=mV$, both A and B are identically equal to zero and consequently γ_x becomes indeterminate. The class of potentials for which $B=0$ on a domain (apart from all homogeneous $V(x,y)$, this class also includes inhomogeneous potentials as e.g. inhomogeneous potentials of the form $V(x-c_0y)$, where c_0 is a constant) is excluded from our study.

Some γ_x being necessarily given by equation (8), in view of equation (5) γ_y must be given by

$$\gamma_y=-\frac{xA}{(x+y\gamma)\,B}. \tag{10}$$

The question now arises as to whether γ_x and γ_y, given by (8) and (10), are compatible. Working on the condition $\gamma_{xy}=\gamma_{yx}$, taking into account equations (8) and (10), we obtain after some straightforward but tedious algebra

$$G_3\gamma^3+G_2\gamma^2+G_1\gamma+G_0=0 \tag{11}$$

where

$$G_i=B\,(xA_{i,x}+yA_{i,y}+A_i)-A_i\,(xB_x+yB_y)\,,\quad i=0,1,2,3. \tag{12}$$

The calculations give that

$$G_0=-\frac{V_{10}}{V_{01}}G_3 \quad \textit{and} \quad V_{01}(G_1+G_3)=V_{10}(G_0+G_2). \tag{13}$$

One step further we observe that the left hand side of equation (11) can be factorized as $(V_{10}+\gamma V_{01})\left(H_2\gamma^2+H_1\gamma+H_0\right)$, where

$$\begin{aligned} H_2 &= -\tfrac{G_0}{V_{10}} \\ H_1 &= \tfrac{G_1}{V_{10}} - \tfrac{V_{01}}{V_{10}^2}\, G_0 \\ H_0 &= -H_2. \end{aligned} \tag{14}$$

Since, for any given inhomogeneous $V\,(x,y)$ and any homogeneous $\gamma\,(x,y)$ that we look for, the expression $V_{10}+\gamma V_{01}$ is not zero in a region $\boldsymbol{D} \subseteq \boldsymbol{R}^2$, we obtain the quadratic equation

$$H_2\gamma^2 + H_1\gamma - H_2 = 0. \tag{15}$$

If both H_1 and H_2 are zero, equation (15) is satisfied identically. If only one of the expressions H_2, H_1 is zero, the above equation leads to the non-interesting solutions $\gamma = 0,\ \pm 1$.

Assuming that for the given potential

$$H_2 \neq 0, \quad H_1 \neq 0 \tag{16}$$

equation (15) constitutes a necessary condition so that a solution γ of our problem exists. In general, the solutions of equation (15) could be inhomogeneous, as it happens in the case of the potential $V(x,y) = \frac{y}{x}\sin x + x$, and the problem has a negative answer, i.e. no homogeneous family of orbits is generated by the given inhomogeneous potential.

On the other hand, having expressed by (8) and (10) the first order derivatives γ_x, γ_y in terms of γ itself and of the given potential, we can also express $\gamma_{xx}, \gamma_{xy}, \gamma_{yy}$ in terms of γ and derivatives of $V\,(x,y)$ up to the fourth order. Then, by inserting these into equation (3), we can obtain γ in terms of the potential. To this end we prepare A_x, A_y. Thus, for instance, in view of equations (8) and (10) we write

$$A_x = A_{3x}\gamma^3 + A_{2x}\gamma^2 + A_{1x}\gamma + A_{0x} + + \left(3A_3\gamma^2 + 2A_2\gamma + A_1\right) \frac{y\left(A_3\gamma^3 + A_2\gamma^2 + A_1\gamma + A_0\right)}{B\,(x+y\gamma)}.$$

After some tedious but straightforward algebra, aided by the use of Maple, we write equation (3) as a quintic algebraic equation in γ:

$$R_5\gamma^5 + R_4\gamma^4 + R_3\gamma^3 + R_2\gamma^2 + R_1\gamma + R_0 = 0. \tag{17}$$

We note that each of the coefficients R_i $(i = 0, 1, ..., 5)$ can be expressed as a sum of the form

$$R_i = R_{i30}x^3 + R_{i21}x^2y + R_{i12}xy^2 + R_{i03}y^3 + + R_{i20}x^2 + R_{i11}xy + R_{i02}y^2 + R_{i10}x + R_{i01}y$$

where the $6 \times 9 = 54$ coefficients R_{ijk} $(i = 0, 1, ...5)$ depend merely on derivatives of the given $V\,(x,y)$ up to the fourth order and they are homogeneous polynomials of the fifth degree in the 14 variables V_{40}, $V_{31}, ..., V_{01}$.

In the case when $H_2 \neq 0, H_1 \neq 0$, equations (15) and (17) constitute a system of two algebraic equations which the zero order homogeneous function $\gamma\,(x,y)$ must satisfy. Obviously this happens only for potentials $V\,(x,y)$ for which the pertinent Sylvester's 7×7 determinant (Mishina and Proskuryakov (1965), p. 164), i.e.,

$$\begin{vmatrix} R_5 & R_4 & R_3 & R_2 & R_1 & R_0 & 0 \\ 0 & R_5 & R_4 & R_3 & R_2 & R_1 & R_0 \\ H_2 & H_1 & H_0 & 0 & 0 & 0 & 0 \\ 0 & H_2 & H_1 & H_0 & 0 & 0 & 0 \\ 0 & 0 & H_2 & H_1 & H_0 & 0 & 0 \\ 0 & 0 & 0 & H_2 & H_1 & H_0 & 0 \\ 0 & 0 & 0 & 0 & H_2 & H_1 & H_0 \end{vmatrix} \tag{18}$$

vanishes.

4. Special cases

It is reminded that the 7×7 determinant (18) is to be checked for the given inhomogeneous potential provided that equation (15) is not a triviality. (In fact, in such a case, (18) becomes an identity also, satisfied for any $V(x,y)$).

Therefore, it must first be checked that the coefficients H_2 and H_1 in (15) are not identically equal to zero. Of course, if only one of them is zero, no solution to our problem exists.

Let us then proceed under the assumption that, for the potential at hand, equation (15) becomes a triviality, i.e.

$$H_2 = H_1 = 0. \tag{19}$$

This means that (15) gives no restriction for the function $\gamma(x,y)$ but also gives no information as to its being zero-order homogeneous, as we want it to be.

Yet, γ_x and γ_y are still given by formulae (8) and (10), and the quintic equation (17) may be written down. This equation now stands for the unique necessary condition which γ and V must satisfy in order to be compatible. Thus, if the conditions (19) are satisfied, we expect at most five solutions $\gamma(x,y)$ to our problem but we are not sure in advance that even one of these is acceptable, in the sense that it is zero-degree homogeneous. The quintic equation has to be solved and the homogeneous roots must be selected.

An interesting result which we established with the aid of Maple is the following: The quadratic equation (15) is a triviality if the given potential $V(x,y)$ is of the form

$$V(x,y) = x^m V_1\left(\frac{y}{x}\right) + x^n V_2\left(\frac{y}{x}\right) \tag{20}$$

with $m \neq n$, i.e. if the potential is the sum of *two* terms, each of which is homogeneous, of a different degree of homogeneity, of course. We have also checked that for a potential being the sum of three homogeneous terms the quadratic (15) is no longer a triviality.

In general, of course, an inhomogeneous potential needs not be the sum of certain homogeneous terms and in this case we expect to have only one of H_1 and H_2 equal to zero, so the problem will have no acceptable solution.

5. Examples

A. For the potential

$$V(x,y) = 4x^2 + y^2 + x^3 + 8x^4 - 2x^2y^2 - y^4 \tag{21}$$

one can check that $H_1 \neq 0$ and $H_2 \neq 0$. The common root of (15) and (17) is

$$\gamma = \frac{2x}{y},$$

which is compatible with the potential (21).

B. For the Hénon-Heiles type potential (1964), which is the sum of two homogeneous of degree two and three functions,

$$V(x,y) = \frac{1}{2}x^2 + 8y^2 + x^2y + \frac{16}{3}y^3 \tag{22}$$

we have $H_2 = H_1 = 0$. In this case, it remains only the equation (17), having the homogeneous solution

$$\gamma = -\frac{x}{4y}$$

which is compatible with given potential.

C. For the potential

$$V(x,y) = \frac{y}{x}\exp(x) \tag{23}$$

one has $H_2 = 0$, $H_1 \neq 0$ and the quadratic equation (15) has only the solution $\gamma = 0$, which is not of interest for our purposes.

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