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Programmed motion for a class of families of planar orbits

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Abstract. Taking as a guide the case of the set of monoparametric families $y = h(x) + c$, for which Szebehely's equation can be solved by quadratures for the potential $V(x, y)$ generating the given set of orbits, we propose the following *programmed motion problem*: can we manage so as to have members of the given set inside a preassigned domain $T \subset \mathbb{R}^2$ of the xy plane?

We come to understand that, among the various inequalities by means of which T can be ascribed, the simplest is $b(x, y) \geqslant 0$ where, for each $h(x)$, the function $b(x, y)$ is related to the kinetic energy of the moving point (equations (19)–(21)). We then proceed to show that, in general, if $b(x, y)$ satisfies two conditions (equations (39) and (40)), the answer to our question is affirmative: on the grounds of the given appropriate $b(x, y)$, a function $h(x)$ is found, associated with a certain potential $V(x, y)$ creating members of the family $y = h(x) + c$ inside the region $b(x, y) \geqslant 0$.

Some special cases which stem from the method are studied separately. The limitations and also the promising features of the method developed to face the above inverse problem are discussed.

1. Introduction

The two-dimensional inverse problem of dynamics consists in finding a potential V which generates a family of curves

$$f(x, y) = c \tag{1}$$

in the xy Euclidean space. The roots of the problem are to be found in Newton's *Principia* (1687) where a force law compatible with Kepler's laws was deduced. Interest in this old problem increased after Szebehely [8] presented the partial differential equation

$$f_x V_x + f_y V_y - \frac{2(E(f) - V)}{f_x^2 + f_y^2}(f_{xx} f_y^2 - 2 f_{xy} f_x f_y + f_{yy} f_x^2) = 0, \tag{2}$$

where the subscripts denote partial derivatives. This linear in V partial differential equation is our tool to find the potential $V(x, y)$ which can produce as orbits a preassigned monoparametric family of curves (1), traced in the xy plane, with adequate initial conditions, by a material point of unit mass, with energy dependence

$$E = E(f(x, y)) \tag{3}$$

given in advance.

In spite of its linear character, in practice, equation (2) cannot be solved analytically. Actually, its solvability is directly connected to the possibility of solving a (generally nonlinear) system of ordinary differential equations.

Additional assumptions regarding the form of either the known function $f(x, y)$ or the unknown potential $V(x, y)$ ease the solution of the problem. Thus, for example, for particular sets of functions f (e.g., homogeneous in x, y) the compatibility with homogeneous (Bozis and Grigoriadou [1]) or nonhomogeneous (Bozis *et al* [4]) potentials was already studied.

In this framework, in the present paper a new special case is considered: that of *the set* of monoparametric families of curves with equation

$$y = h(x) + c, \tag{4}$$

where $h(x)$ is a nonlinear ($h''(x) \neq 0$) function of x. We consider functions h defined on an interval where h' has no zeros. For every function $h(x)$ equation (4) stands for a family of (equidistant) curves (not straight lines), shifted parallel to the y-axis. It is shown that all potentials which can generate families of the form (4) can be found by quadratures.

On the other hand, it is known that during the motion of a material point of unit mass along an orbit of the family (1), the inequality

$$B(x, y) \geqslant 0 \tag{5}$$

must be observed, with

$$B = E(f(x, y)) - V(x, y). \tag{6}$$

This means that the motion is allowed along those members (or part of the members) of the family (1) which are lying only inside some regions of the xy plane, limited by the so-called *family boundary curves* (FBC) (Bozis and Ichtiaroglou [2]), which are given by the equation

$$B(x, y) = 0. \tag{7}$$

The function $B(x, y)$ is the kinetic energy (expressed in terms of the position coordinates x, y) of the material point of unit mass, as it moves on any of the orbits (4) in the presence of the potential $V(x, y)$. We shall refer to it here as *the B-function* and keep in mind that it is associated with the family (4), which, of course, can have infinitely many B-functions. The merit of such a function lies in that, by preassigning to (4) a certain B-function, we can manage to have conservative motion inside a preassigned region $T \subset \mathbb{R}^2$ of the xy plane defined by inequality (5). We remind the reader that if force fields (not necessarily conservative) are demanded, in general, there exist such fields to create any preassigned family (1) inside any preassigned region T (Bozis [3]).

The question raised and answered in this paper is the following: can any (positive in $T \subset \mathbb{R}^2$) function stand for a B-function? In other words: are there potentials $V(x, y)$ generating orbits of the form (4) traced with preassigned kinetic energy (6) and, as a consequence, lying inside a preassigned region (5)? Which is the pertinent family (4) and which is the corresponding energy dependence $E = E(f)$?

It turns out that, for every $h(x)$, there exists a simpler (positive in T) function $b(x, y)$, whose positiveness in T implies the inequality (5). Due to its simplicity, we prefer to represent the FBC by $b(x, y) = 0$ and we focus attention on the problem of obtaining compatible pairs of $b(x, y)$ and $h(x)$. We show that, if the given $b(x, y)$ satisfies two conditions, the function $h(x)$ (and, consequently, the family $f(x, y) = y - h(x) = c$) as well as the energy dependence function $E(f)$ and the potential $V(x, y)$ are determined.

2. Determining the potentials which generate a special family of curves

For the case of the family of curves given by (4), equation (2) has the simpler form

$$h'V_x - V_y = \frac{2h''}{1+h'^2}(E(f) - V), \tag{8}$$

the prime denoting the derivative with respect to the x variable, and, according to (1), $f(x, y) = y - h(x)$. The subsidiary system of ordinary differential equations is

$$\frac{\mathrm{d}x}{h'} = \frac{\mathrm{d}y}{-1} = \frac{(1+h'^2)\,\mathrm{d}V}{2h''(E(f) - V)}. \tag{9}$$

The first of equations (9) gives

$$y = c_1 - I(x), \tag{10}$$

where

$$I(x) = \int^x \frac{\mathrm{d}t}{h'(t)}. \tag{11}$$

Equating the third to the first fraction in (9), we obtain a linear equation in V which has to be solved after replacing y appearing in the argument of E by its expression in (10). So the argument of E will be $y - h(x) = c_1 - I(x) - h(x)$, and the equation in V will have the form

$$\frac{\mathrm{d}V}{\mathrm{d}x} + \frac{2h''(x)}{h'(x)(1+h'^2(x))}V - \frac{2h''(x)}{h'(x)(1+h'^2(x))}E(c_1 - I(x) - h(x)) = 0. \tag{12}$$

The solution of the ordinary differential equation (12) is

$$V = \frac{1+h'^2(x)}{h'^2(x)}[c_2 + K(x, c_1)], \tag{13}$$

where

$$K(x, c_1) = \int^x E(c_1 - I(s) - h(s))\frac{2h'(s)h''(s)}{(1+h'^2(s))^2}\,\mathrm{d}s.$$

Integrating by parts the above integral K, we get

$$K(x, c_1) = -\frac{1}{1+h'^2(x)}E(c_1 - I(x) - h(x)) - J(x, c_1) \tag{14}$$

where

$$J(x, c_1) = \int^x E_c(c_1 - I(s) - h(s))\frac{1}{h'(s)}\,\mathrm{d}s \tag{15}$$

and where E_c denotes the derivative of the one-variable function $E = E(c)$ with respect to its argument.

The general solution of the partial differential equation (8) is given by

$$c_2 = A(c_1) \tag{16}$$

with A an arbitrary function of $c_1 = y + I(x)$. So, for the family of curves (4) traced with a preassigned energy dependence $E = E(c)$, the potentials creating it are given by

$$V(x, y) = -\frac{1}{h'^2}\bar{E} + \frac{1+h'^2}{h'^2}(\bar{A} - \bar{J}), \tag{17}$$

where we adopt the notation

$$\begin{aligned} &\bar{E} = E(c = y - h), \qquad \bar{A} = A(c_1 = y + I) \\ &\bar{J} = J(x, c_1 = y + I) \end{aligned} \tag{18}$$

with the functions h and I depending merely on x.

Clearly, if we consider as given a function $h(x)$ and select arbitrarily a function $E(c)$, we can calculate the integrals I and J by quadratures from (11) and (15), respectively.

3. Family boundary curves (B-functions and b-functions)

Real motion of the moving point takes place only on those members of the family of curves (1) or those parts of each member of the family which lie in the region of the xy plane where the inequality (5) is observed.

We proceed to find the pertinent FBC (7) for the specific set of families (4). As the corresponding potentials are given by (17), the function in (6) is found to be given by

$$B(x, y) = \frac{1 + h'^2}{h'^2}(\bar{E} - \bar{A} + \bar{J}). \tag{19}$$

The meaning of (19) is the following: given a family (4), i.e. given a function $h(x)$, after selecting specific arbitrary functions $E(c)$ and $A(c_1)$, we can find the function $B(x, y)$ and draw conclusions regarding the FBC.

Naturally, wherever $B(x, y) \geqslant 0$, with $B(x, y)$ given by (19), it is also

$$b(x, y) \geqslant 0 \tag{20}$$

with

$$b(x, y) = \bar{E} - \bar{A} + \bar{J} \tag{21}$$

and conversely. Thus, together with the B-function, we have a simpler function $b(x, y)$ which can serve to describe the FBC or, if given in advance, to create *programmed motion* inside a region. So, in what follows, we shall represent the preassigned allowed region by inequality (20). The same region, of course, could be represented by $\theta(x, y)b(x, y) \geqslant 0$, where $\theta(x, y)$ is any arbitrary non-negative function in T (Bozis [3]).

On the grounds of the previous analysis we now pose the following question: consider a preassigned *boundary function* $b = b(x, y)$, positive inside a region $T \subset \mathbb{R}^2$ which we want to programme as an allowed region of some orbits (4). Can we find a function $h(x)$, leading to appropriate functions $E(c)$, $A(c_1)$ and $J(x, c_1)$, such that equation (21) is satisfied?

If the answer to the above question is in the affirmative, inside T there exist members of the family (4). These are orbits, traced by the moving point for adequate initial conditions, in the presence of the potential (17) with kinetic energy equal to $B(x, y) = \frac{1+h'^2}{h'^2} b(x, y)$ at each point of the orbit.

4. Programmed motion: analysis

Consider a (positive in a region $T \subset \mathbb{R}^2$) boundary function $b(x, y)$ to account for our goal to obtain orbits of the form (4) inside the region T. For this version of the inverse problem, suppose that there is an affirmative answer, i.e. there exists an appropriate function $h(x)$ for which (21) can be satisfied.

Since the function A depends on x, y through c_1, it is

$$\frac{\bar{A}_y}{\bar{A}_x} = \frac{c_{1,y}}{c_{1,x}}$$

and, in view of (10) and (11), it must be

$$\bar{A}_y = h'\bar{A}_x. \tag{22}$$

We want to replace into (22) the function $\bar{A} = \bar{E} + \bar{J} - b$ given by (21). To this end we prepare, in view of (3), (15) and (18),

$$\bar{E}_x = -h'E_c, \qquad \bar{E}_y = E_c, \qquad \bar{J}_x = \frac{1}{h'}E_c + \frac{1}{h'}J_{c_1}, \qquad \bar{J}_y = J_{c_1}. \tag{23}$$

In view of (23), the condition (22) is now written as

$$h'^2 E_c = b_{01} - h' b_{10} \tag{24}$$

where b_{10} stands for b_x and b_{01} for b_y. This notation is in agreement with the following more general convention which we adopt in what follows: we denote by $b_{mn}(m, n$ positive integers) the partial derivative of $b(x, y)$ m times with respect to x and n times with respect to y.

Since $h'(x) \neq 0$, we denote $\frac{1}{h'}$ by $z = z(x)$ and we write equation (24) as

$$E_c = z^2 b_{01} - z b_{10}. \tag{25}$$

The simpler case of *isoenergetic families* of orbits (4), for which $E_c = 0$, will be treated separately in section 6. In general, we know that E_c depends on x, y through c, given by equation (4): thus, $\frac{c_y}{c_x} = -\frac{1}{h'}$ and we find

$$(E_c)_y + z(E_c)_x = 0. \tag{26}$$

After some straightforward algebra, from (26) and (25) we obtain

$$b_{11} z^2 + (b_{02} - b_{20}) z - b_{11} = (b_{10} - 2b_{01} z) z' \tag{27}$$

where $z' = \frac{\mathrm{d}z}{\mathrm{d}x}$ ($\neq 0$ because $h''(x) \neq 0$).

Condition (27) is free from the energy and (as we restrict ourselves to choose b, and not θb, to represent the FBC) it is necessary and sufficient for our purpose to achieve programmed motion. As it stands, (27) relates the given boundary function $b(x, y)$ (and up to second-order derivatives of it) to the function

$$z = \frac{1}{h'(x)} \tag{28}$$

(and its derivative z').

To come to know z and z', we differentiate twice both members of (27) with respect to y and we obtain the two equations

$$b_{12} z^2 + (b_{03} - b_{21}) z - b_{12} = (b_{11} - 2b_{02} z) z' \tag{29}$$

and

$$b_{13} z^2 + (b_{04} - b_{22}) z - b_{13} = (b_{12} - 2b_{03} z) z'. \tag{30}$$

Assuming that $b_{10} - 2b_{01} z \neq 0$ (the special case will be treated in section 6) and dividing (29) by (27), we obtain the cubic in z algebraic equation

$$\alpha_3 z^3 + \alpha_2 z^2 + \alpha_1 z + \alpha_0 = 0 \tag{31}$$

with

$$\begin{aligned} \alpha_3 &= 2(b_{02} b_{11} - b_{01} b_{12}) \\ \alpha_2 &= 2b_{02}(b_{02} - b_{20}) - 2b_{01}(b_{03} - b_{21}) + b_{10} b_{12} - b_{11}^2 \\ \alpha_1 &= 2(b_{01} b_{12} - b_{02} b_{11}) + b_{10}(b_{03} - b_{21}) - b_{11}(b_{02} - b_{20}) \\ \alpha_0 &= b_{11}^2 - b_{10} b_{12}. \end{aligned} \tag{32}$$

Now dividing (30) by (27) we obtain a second cubic

$$\beta_3 z^3 + \beta_2 z^2 + \beta_1 z + \beta_0 = 0 \tag{33}$$

with

$$\begin{aligned} \beta_3 &= 2(b_{03} b_{11} - b_{01} b_{13}) \\ \beta_2 &= 2b_{03}(b_{02} - b_{20}) - 2b_{01}(b_{04} - b_{22}) + b_{10} b_{13} - b_{11} b_{12} \\ \beta_1 &= 2(b_{01} b_{13} - b_{03} b_{11}) + b_{10}(b_{04} - b_{22}) - b_{12}(b_{02} - b_{20}) \\ \beta_0 &= b_{11} b_{12} - b_{10} b_{13}. \end{aligned} \tag{34}$$

It can be checked easily that β_i can be obtained alternatively by differentiating each α_i ($i = 0, 1, 2, 3$) with respect to y. A problem may arise when all the α_i are zero, so we cannot make use of equations (31) and (33). This case will be treated in section 6.

Equations (31) and (33) are necessary conditions for z inherited from the unique necessary and sufficient condition (27) which we wish to satisfy, if this is possible, in the first place, with a $z = z(x)$. So, if, for a given b, such a z does exist, this has to be the (at least one) common root of (31) and (33). Of course, for such a root to exist, a sixth-order determinant, called the resultant of the two polynomials in (31) and (33) (Mishina and Proskuryakov [7], p 164) must vanish identically. This requirement will lead to a (rather complicated) condition including derivatives of $b(x, y)$ up to the fourth order. Yet, it is understood that we need not write down this condition. Instead, taking for granted that an appropriate $z = z(x)$ to be put in (27) does exist and that this z cannot be anything but the common root of (31) and (33), we proceed to find it as follows.

We multiply (31) by β_3 and (33) by α_3 and subtract. Assuming that $\alpha_2\beta_3 - \alpha_3\beta_2 \neq 0$ (the special case will be treated in section 6), we obtain

$$z^2 = \gamma_1 z + \gamma_0 \tag{35}$$

with

$$\gamma_0 = \frac{\alpha_0\beta_3 - \alpha_3\beta_0}{\alpha_3\beta_2 - \alpha_2\beta_3}, \qquad \gamma_1 = \frac{\alpha_1\beta_3 - \alpha_3\beta_1}{\alpha_3\beta_2 - \alpha_2\beta_3}. \tag{36}$$

In view of (35), we replace z^3 into (31) by $z^3 = (\gamma_1^2 + \gamma_0)z + \gamma_0\gamma_1$ and we find

$$\delta_1 z + \delta_0 = 0 \tag{37}$$

with

$$\begin{aligned} \delta_1 &= \alpha_3(\gamma_1^2 + \gamma_0) + \alpha_2\gamma_1 + \alpha_1 \\ \delta_0 &= \alpha_3\gamma_0\gamma_1 + \alpha_2\gamma_0 + \alpha_0. \end{aligned} \tag{38}$$

We assume that $\delta_1 \neq 0$ (the special case $\delta_1 = 0$ is examined in section 6) and we conclude by writing down the two conditions which the given function $b(x, y)$ must satisfy:

(i) The function z, found from (37), must depend only on the variable x, i.e.

$$\delta_{0,y}\delta_1 = \delta_0\delta_{1,y}. \tag{39}$$

(ii) The function z must satisfy the equation (27), i.e.

$$\delta_1\{\delta_0^2 b_{11} - (b_{02} - b_{20})\delta_0\delta_1 - b_{11}\delta_1^2\} = (\delta_1 b_{10} + 2\delta_0 b_{01})(\delta_0\delta_{1,x} - \delta_{0,x}\delta_1). \tag{40}$$

Clearly, if we expressed the above conditions (39) and (40) in terms of the given function $b(x, y)$ only, there would appear partial derivatives of b up to the fifth order.

5. Programmed motion: synthesis

We want to have monoparametric families of orbits of the form (4) ‘inside’ a preassigned region $T \subset \mathbb{R}^2$ given by inequality (20) with the given function $b(x, y)$ to be defined as in (21).

In view of the analysis in section 4 and aided by a symbolic algebra program (e.g. MATHEMATICA) we proceed as follows:

(i) For the given $b(x, y)$ we prepare its partial derivatives b_{ij} up to the fourth order and, from equations (32), (34), (36) and (38), we find all the functions α_i, β_i ($i = 0, 1, 2, 3$) and γ_k, δ_k ($k = 0, 1$). We make sure that

$$\alpha_2\beta_3 - \alpha_3\beta_2 \neq 0 \qquad \text{and} \qquad \delta_1 \neq 0. \tag{41}$$

(Otherwise we act as in section 6.)

(ii) We prepare the partial derivatives $\delta_{0,x}$, $\delta_{0,y}$, $\delta_{1,x}$, $\delta_{1,y}$ of the two functions $\delta_k(x, y)$ ($k = 0, 1$) and we check the conditions (39) and (40).

After the remark which follows we shall continue with our synthesis by assuming that *we do have* an appropriate b-function at our disposal.

Remark. As a rule, of course, one does not expect the conditions (39) and (40) to be satisfied. Consequently, one cannot programme motion inside $T \subset \mathbb{R}^2$, described by the b-function at hand. Yet, such an appropriate b-function may be hidden! It may correspond to another selection of the function $h(x)$ in equation (4) and, most likely, to different selections of the arbitrary functions E and A, as given by the equations (3) and (16).

One then may try to find a good b-function. Thus, for example:

- If b is given with some free constants, one may try to determine these constants so that the conditions (39) and (40) are satisfied.
- If this cannot be achieved, we may write (39) and (40) for $b \to \theta b(x, y)$ and try to find *even one* (positive in T) particular solution $\theta_0(x, y)$ satisfying these conditions. This assignment is far from being a simple task, of course.

Assuming that the given b is appropriate, we now proceed as follows:

(iii) From (37) and (28) we find z and h' and then, in turn, $h(x)$ up to an additive constant h_0. Then, out of the set of orbits (4), we obtain the specific monoparametric family $f(x, y) = y - h(x) = c$. In general, the two equations (31) and (33) have one common root, so we obtain one family (4) for some members of which we can manage to have them trapped inside the preassigned region (20). To this end we must determine the appropriate potential.

(iv) From equation (25) we now determine E_c and, as we already know the expression for $c = y - h(x)$, we determine uniquely (apart from an additive constant E_0) the energy dependence function (3) $E = E(c)$ and, from the first of equations (18), we come to know $\bar{E}$, into which the constants h_0 and E_0 will enter.

(v) Finally, since the kinetic energy is

$$B = \frac{1 + h'^2}{h'^2} b(x, y) \tag{42}$$

we write down the potential

$$V(x, y) = \bar{E} - B \tag{43}$$

which gives rise to orbits with equation (4) lying inside the region (20).

Comment. In spite of the fact that the two integration constants, h_0 and E_0 enter into (43) through $\bar{E}$, the potential $V(x, y)$ is essentially unique, at least as far as the study of the present family is concerned. Indeed, h_0 and E_0 appear in (43) either purely as an additive constant or through a combination which amounts to a (constant, again) additive arbitrary function $V_0(f)$, where $f = y - h(x) = c$ is the family under consideration. For these reasons we can set both h_0 and E_0 equal to zero and this is actually what we do in the example of section 7.

6. Special cases

6.1. Isoenergetic families ($E_c = 0$)

Actually condition (26)—and consequently (27)—as it is written is valid for isoenergetic families, i.e. families of orbits all traced with the same (constant) value of the energy E_0. Yet the case needs to be treated separately in the sense that, as seen from (24), when $E_c = 0$, from the given b-function we obtain immediately

$$h' = \frac{b_{01}}{b_{10}} \tag{44}$$

which is acceptable provided that

$$\left(\frac{b_{01}}{b_{10}}\right)_y = 0 \tag{45}$$

or, equivalently,

$$b = b(c_1) \tag{46}$$

with $c_1 = y + I$ and $I = \int \frac{b_{10}}{b_{01}}\,\mathrm{d}x$.

Then the B-function (42) is

$$B = \frac{b_{10}^2 + b_{01}^2}{b_{01}^2}\overline{b}$$

where

$$\overline{b} = b(c_1 = y + I(x))$$

and the potential is

$$V = E_0 - \frac{b_{10}^2 + b_{01}^2}{b_{01}^2}\overline{b}. \tag{47}$$

The meaning of the above reasoning is the following: given a boundary function $b(x, y)$ which satisfies (45), we can directly obtain from (44) h' and from (11) $I(x)$, then $c_1 = y + I$ and check that the given b is of the form (46). The family of orbits $y - h(x) = c$ is traced isoenergetically with energy E_0 by the potential (47).

Thus, e.g., given $b = y - x^2$, we find $h' = -\frac{1}{2x}$, $h(x) = -\frac{1}{2}\ln x, x > 0$, $I = -x^2$, $c_1 = y - x^2$, $\overline{b} = y - x^2$ and $V = -(1 + 4x^2)(y - x^2)$.

The isoenergetic family is $y + \frac{1}{2}\ln x = c$, traced with $E_0 = 0$, by the potential V.

6.2. The case $b_{10} - 2b_{01}z = 0$

Having

$$z = \frac{b_{10}}{2b_{01}} \tag{48}$$

we must also zero the left-hand side of equation (27). This leads to

$$b_{11}b_{10}^2 + 2b_{01}b_{10}(b_{02} - b_{20}) = 4b_{01}^2 b_{11}. \tag{49}$$

Since $(\frac{b_{10}}{b_{01}})_y = 0$, we have

$$b = F(y + \iota(x)) \tag{50}$$

where F is an arbitrary function of its argument $y + \iota$ and $\iota = \int \frac{b_{10}}{b_{01}}\,\mathrm{d}x$.

Inserting (50) into (49) we obtain after some algebra

$$\frac{F''}{F'} = -\frac{2\iota''}{\iota'^2 + 2} = k_0 \tag{51}$$

where primes in F and ι denote differentiation with respect to their respective arguments $y + \iota$ and x and where the constant k_0 was put to make equal the two functions of different argument.

Solving the two equations (51) we find

$$b = c_1^* e^{k_0 y} \cos^2 \frac{2c_2^* - k_0 x}{\sqrt{2}}, \tag{52}$$

where c_1^*, c_2^* are integration constants. Then, in turn, from (48) we find z, from (28) we find $h'(x)$ and, integrating it, $h(x)$ and from (4) we find the family

$$y + \frac{1}{k_0} \ln \left(\sin^2 \frac{2c_2^* - k_0 x}{\sqrt{2}} \right) = c. \tag{53}$$

From (25) we find E_c and then the energy

$$E = -\tfrac{1}{2} c_1^* e^{k_0 c} \tag{54}$$

with which each member of the family (53) is traced.

Finally, from (43) we find the potential

$$V(x, y) = -c_1^* e^{k_0 y}. \tag{55}$$

It is worth noticing that, since $b(x, y) \geqslant 0$ everywhere or nowhere (depending on the sign of c_1^*), orbits (53) exist everywhere or nowhere in the xy plane.

6.3. The case $\alpha_i = 0$ $(i = 0, 1, 2, 3)$

In this case we have not at our disposal the polynomial equations (31) and (33), so we are obliged to consider only the differential equation (27). However, if all the coefficients $\alpha_i = 0$ $(i = 0, 1, 2, 3)$ are zero, the function b must have some particular expressions which we are going to find.

Equating to zero α_3 and α_0 we get

$$b_{02} b_{11} - b_{01} b_{12} = 0, \tag{56}$$

$$b_{11}^2 - b_{10} b_{12} = 0. \tag{57}$$

(i) If $b_{11} = 0$, it follows that b is a sum of two functions, one in the variable x and the other in y. Replacing it in the equations $\alpha_1 = 0$ and $\alpha_2 = 0$ we get only two possibilities for b (apart from an additive constant):

$$b = my + g(x), \qquad m \in \mathbb{R}, \quad g \text{ an arbitrary function;} \tag{58}$$

$$b = r(x^2 + y^2) + py + qx, \qquad r, p, q \in \mathbb{R}, \quad r \neq 0. \tag{59}$$

In fact, b can be also given as $b = \frac{a}{k} e^{ky}$, $a, k \in \mathbb{R}$, $k \neq 0$, which is a special case of (60) below.

(ii) If $b_{11} \neq 0$, from (56) we get $(\frac{b_{01}}{b_{11}})_y = 0$, hence $b_{01} = C(x) b_{11}$; from (57), $(\frac{b_{10}}{b_{11}})_y = 0$, hence $b_{10} = D(x) b_{11}$. It follows $\frac{b_{01}}{b_{10}} = \frac{C(x)}{D(x)}$; so $b = U(y + \int \frac{D(x)}{C(x)} \, dx)$. Replacing in (56) we obtain

$$\frac{U''}{U'} = \frac{1}{D(x)} = k,$$

so

$$b = \frac{c_0}{k} e^{ky} e^{kG(x)}, \qquad c_0, k \in \mathbb{R}, \quad k \neq 0, \quad G \text{ an arbitrary function,} \tag{60}$$

this special expression of b also making α_1 and α_2 vanish in equation (31).

For the three cases (58)–(60) we analyse equation (27).

Consider first $b = my + g(x)$. In this case (27) becomes

$$-g''z = (g' - 2mz)z'$$

with the solution $z(x) = \frac{1}{2m}(g' \pm \sqrt{g'^2 - 4mc_0^*})$, giving $E_c = -c_0^*$ and $E(c) = -c_0^* c$.

For b given by (59), equation (27) has no nonconstant solution z independent of y.

For b given by (60) we obtain from equation (27) a differential equation in z with coefficients depending only on x, having the form

$$kG'z^2 + (k - kG'^2 - G'')z - kG' = (G' - 2z)z'. \tag{61}$$

As an example consider $G(x) = G_0 = \text{const}$ for which $G' = G'' = 0$. Then (61) becomes $kz = -2zz'$, so $z = -\frac{k}{2}x$ and $h = -\frac{2}{k}\ln x$. Hence for $b = \frac{d_0}{k}e^{ky}$, where $d_0 = c_0 e^{kG_0}$, we have $E_c = \frac{d_0 k^2}{4}e^{ky}x^2 = \frac{d_0 k^2}{4}e^{kc}$ and $E = \frac{d_0 k}{4}e^{kc}$, $V = -\frac{d_0}{k}e^{ky}$.

6.4. The case $\alpha_2\beta_3 - \alpha_3\beta_2 = 0$

Apparently, in this case, the task of finding the common roots of equations (31) and (33) becomes easier. Suppose then that we do have such a (y-independent) root $z = z(x)$. We proceed and check if the condition (27) is valid. If so, with this appropriate z we find successfully: the function $h(x)$ (from equation (28)), the energy dependence $E = E(f)$ (from (25)), the function B (from (42)) and, finally, the potential $V(x, y)$ (from equation (43)).

As an example consider as given the b-function on $(-\infty, 0)$ or $(0, +\infty)$:

$$b = \frac{1}{x^4} + \frac{x^2}{2} - \frac{2y}{x^2}. \tag{62}$$

From (32) and (34) we find: $\alpha_3 = 0$, $\alpha_2 = \frac{32}{x^6}$, $\alpha_1 = 16\frac{2+x^6}{x^9}$, $\alpha_0 = \frac{16}{x^6}$ and $\beta_3 = \beta_2 = \beta_1 = \beta_0 = 0$.

Out of the two roots of the quadratic $\alpha_2 z^2 + \alpha_1 z + \alpha_0 = 0$, only the root $z = -\frac{x^3}{2}$ satisfies the condition (27). This z leads to $h = \frac{1}{x^2}$, $y = \frac{1}{x^2} + c$, $E = c^2$ and to the potential

$$V(x, y) = \tfrac{1}{8}(6(2y^2 - x^2) - (x^4 - 2y)^2). \tag{63}$$

It is understood that only those members or parts of members of the family

$$y - \frac{1}{x^2} = c \tag{64}$$

are actually traced by the potential (63) which lies in the nonshaded region of figure 1,

$$y \leqslant \frac{x^4}{4} + \frac{1}{2x^2},$$

where the curves with $c = -3, -2, \ldots, 2$ are displayed.

6.5. The case $\delta_1 = 0$

If $\delta_1 = 0$ and $\delta_0 \neq 0$, equation (37) has no solution, hence we cannot determine a family of orbits in the region $b(x, y) \geqslant 0$. If $\delta_1 = 0$ and $\delta_0 = 0$, equation (37) is an identity and one can use, for example, equation (35) to determine the common root of (31) and (33).

Thus, given $b = e^{x+y} + xy$, we have $\alpha_2\beta_3 - \alpha_3\beta_2 \neq 0$, so that the case cannot be treated as in subsection 6.4. For this specific function b (for which the equation $b = 0$ is equivalent with $y = -\text{Lambert W}(\frac{e^x}{x})$ on the interval $x > 0$), one has $\delta_1 = \delta_0 = 0$. Equation (35) becomes $z^2 = 1$, hence in this case a nonlinear h cannot be obtained.

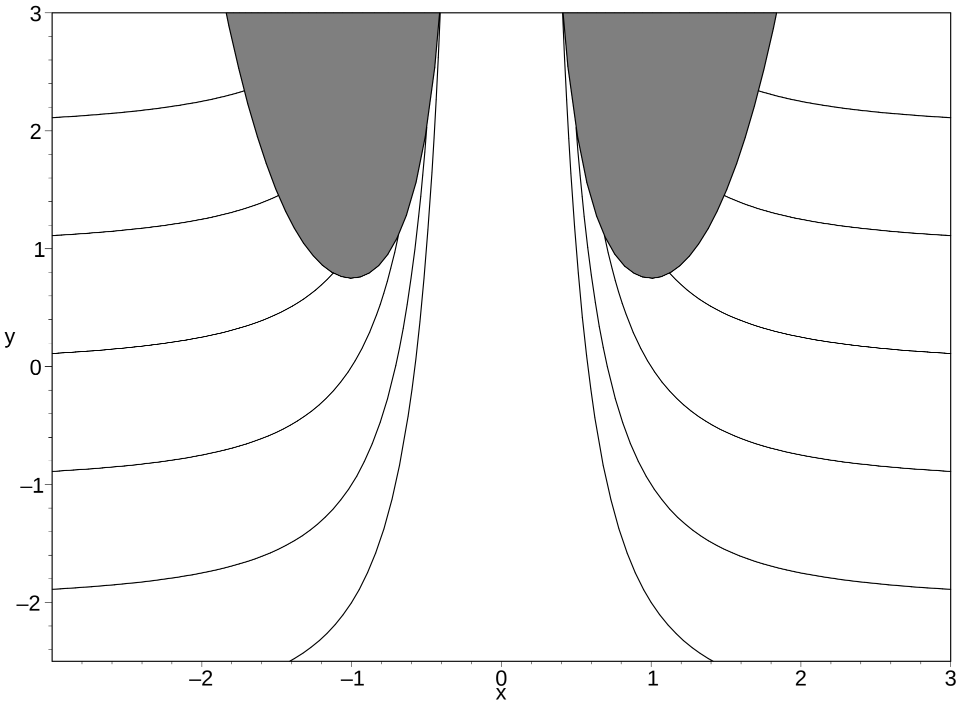

Figure 1. The twelve (symmetric in pairs with respect to the y-axis) unbounded curves $y = 1/x^2 + c$ (for $c = -3, -2, -1, 0, 1, 2$) are real orbits, created by the potential (63) and traced in the nonshaded region of the figure.

7. An example for the general case

For $x > 0$, let us start with the b-function

$$b = b_2 y^2 + b_1 y + b_0 \tag{65}$$

with

$$\begin{aligned} b_2 &= -\frac{3}{x} \\ b_1 &= \frac{3}{x^2}(x^4 + 1) \\ b_0 &= -\frac{1}{x^3}\left(3x^4\left(\frac{x^4}{5} + 1\right) + 1\right). \end{aligned} \tag{66}$$

We want to check if families of the form (4) exist inside the (nonshaded) region of figure 2, given by $b(x, y) \geqslant 0$, which we engage as the allowed region for such families.

According to the procedure exposed in section 5, we write down the equations (31) and (33) and see first if they possess any y-independent common roots. The coefficients α_i, β_i, found from (32) and (34), are (apart from unimportant multiplicative factors)

$$\begin{aligned} \alpha_3 &= -2x^4(3x^4 - 1) \\ \alpha_2 &= x^2(3x^2y^2 - 2x(3x^4 + 1)y - 3(3x^8 - 5x^4 + 1)) \\ \alpha_1 &= x^2(3x^4 + 1)y^2 - 2x(3x^8 + 1)y - (3x^{12} - 16x^8 + 6x^4 - 1) \\ \alpha_0 &= x^2(x^2y^2 + 2x(x^4 - 1)y + (3x^8 - 3x^4 + 1)) \end{aligned}$$

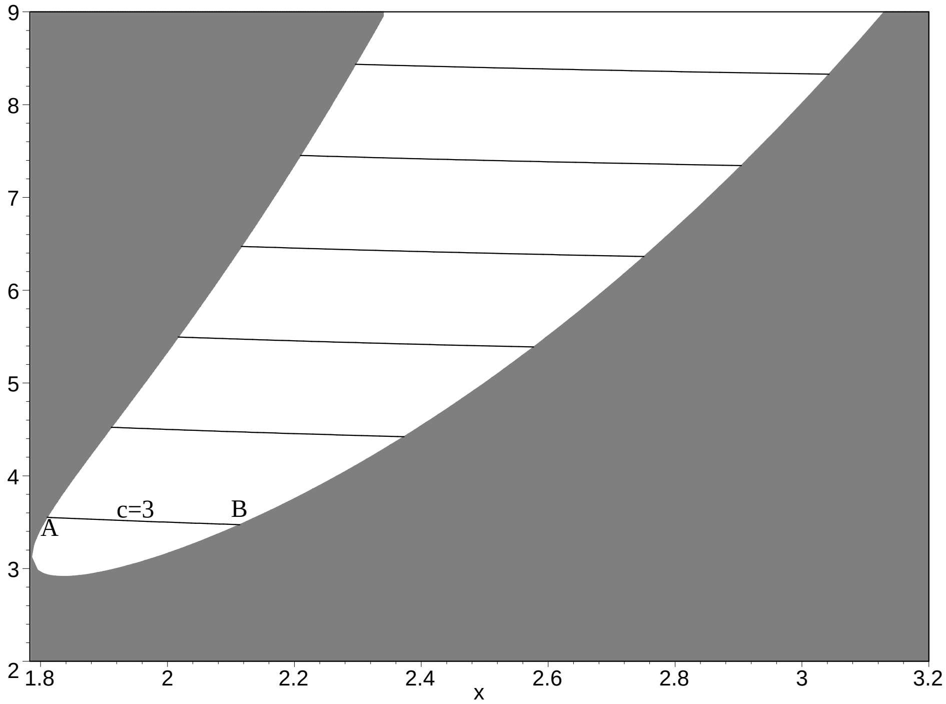

Figure 2. The six bounded curves $y = 1/x + c$ (for $c = 3, 4, 5, 6, 7, 8$) in the nonshaded region of the figure are real orbits (librational motion), created by the potential (68).

and

$$\beta_3 = 0$$
$$\beta_2 = x^2(-3xy + (3x^4 + 1))$$
$$\beta_1 = -x(3x^4 + 1)y + (3x^8 + 1)$$
$$\beta_0 = -x^2(xy + (x^4 - 1)).$$

Out of the two roots $z = -x^2$ and $z = \frac{xy+(x^4-1)}{x^2(-3xy+3x^4+1)}$ of equation (33), only the first root is y-independent and this root happens to satisfy the equation (31) also. In this case $\alpha_2\beta_3 - \alpha_3\beta_2 \neq 0$ and the root $z = -x^2$ can be obtained directly from (37), conditions (39) and (40) being fulfilled. So, we continue with $z = -x^2$ and check the second condition (equation (27)) for our problem to admit of an affirmative answer. For the case at hand, (27) is indeed satisfied.

Now, from (28) we find $h = \frac{1}{x}$ and from (4) we find the family of orbits

$$y - \frac{1}{x} = c, \tag{67}$$

inside the nonshaded region of figure 2 being traced the curves with $c = 3, 4, \ldots, 8$.

From (25) we find $E_c = \frac{3(xy-1)^2}{x^2}$ and, having c at our disposal, we understand that $E_c = 3c^2$, i.e. $E = c^3$, giving $\bar{E} = (y - \frac{1}{x})^3$.

From (42) we find the kinetic energy B and, finally, from (43) we find the potential

$$V(x, y) = y^3 + 3x^3y^2 - 3x^2(x^4 + 2)y + \tfrac{1}{5}x(3x^8 + 18x^4 + 20). \tag{68}$$

We conclude that, in the presence of the potential (68) and for adequate initial conditions, members of the family (67) can be traced, but only branches (like the arc AB for $c = 3$ in figure 2) on which motion is librational.

8. Concluding remarks

According to Galiullin [5], dynamical systems with programmed (or controlled) motion 'are solved in such a way that the process occurring in these systems satisfies some preset requirements'.

The dynamical system here is as simple as one material point moving in the presence of an autonomous two-dimensional potential $V(x, y)$. In Galiullin's terminology, the 'programme of the motion' consists of two elements: (i) the material point moves on a curve with preassigned equation and (ii) the totality of such motions takes place in a preassigned region of the xy plane.

The equation of the orbits is of the (very special, indeed) form (4). For this form, Szebehely's equation is solvable by quadratures, for any function $h(x)$. For any specific $h(x)$, as many potentials as two arbitrary functions introduce can generate the monoparametric family (4). In fact, to each selection of the arbitrary functions there correspond different families (4) in the sense that their members exist as real orbits inside different regions of the xy plane.

So now, in this paper and for this particular solvable inverse problem, we preassign the allowed region also. We effectuate this, essentially by preassigning a positive function $b(x, y)$, related to the (positive) kinetic energy $B(x, y)$ of the moving point and such that its positiveness implies that the preassigned region is allowed.

To the above additional requirement the system may not respond positively and, in fact, this is what generally happens. We prove, however, that if the given function $b(x, y)$ satisfies certain conditions, there is an affirmative answer to our problem. Once this is assured, in general, we are led to a unique specification of the basic data of the motion: i.e., (i) the equation of the family (4), (ii) the energy dependence function and (iii) the potential $V(x, y)$ creating the family.

It is true that the method developed in sections 4 and 5 and all the pertinent formulae refer exclusively to the set of families of orbits (4) for which we cannot offer a convincing motivation of physical nature but which we selected on grounds of simplicity, i.e. because for such sets of families we can write down an analytic expression (17) for the potentials. Yet, in the framework of the inverse problem of dynamics, questions of this sort are interesting. In fact, the reasoning followed in sections 4 and 5 may be adjusted to include similar solvable cases of Szebehely's equation (Grigoriadou *et al* [6]).

A displeasing feature of the method presented is the inherent difficulty of having infinitely many ways of representing analytically a preassigned region T. Out of all these we dealt only with what we defined as the b-function. We mentioned, however, that more generally we are led to a (hard to deal with) problem of solving a partial differential equation. So or otherwise, the calculations involved even in the present version of the problem are almost impossible to be carried out without the help of a program of symbolic algebra.

Finally, let us note a good feature of equations (39) and (40), which generally constitute the criteria to be satisfied by the given function $b(x, y)$: for homogeneous in x, y b-functions, they are behaving homogeneously. This can be seen easily from the formulae (38), (36), (34) and (32).

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