SPECIAL FAMILIES OF ORBITS IN THE DIRECT PROBLEM OF DYNAMICS

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(Received: 3 January 2003; revised: 30 July 2003; accepted: 11 September 2003)

Abstract. The direct problem of dynamics in two dimensions is modeled by a nonlinear second-order partial differential equation, which is therefore difficult to be solved. The task may be made easier by adding some constraints on the unknown function $\gamma = f_y/f_x$, where $f(x, y) = c$ is the monoparametric family of orbits traced in the *xy* Cartesian plane by a material point of unit mass, under the action of a given potential $V(x, y)$. If the function γ is supposed to verify a linear first-order partial differential equation, for potentials V satisfying a differential condition, γ can be found as a common solution of certain polynomial equations.

The various situations which can appear are discussed and are then illustrated by some examples, for which the energy on the members of the family, as well as the region where the motion takes place, are determined. One example is dedicated to a Hénon–Heiles type potential, while another one gives rise to families of isothermal curves (a special case of orthogonal families). The connection between the inverse/direct problem of dynamics and the possibility of detecting integrability of a given potential is briefly discussed.

Key words: integrability, inverse and direct problem of dynamics, special families of orbits

1. Introduction

The direct problem of dynamics consists in finding families of orbits $f(x, y) = c$ traced in the *xy* Cartesian plane by a material point of unit mass, under the action of a given potential V. Throughout the paper, the subscripts will denote partial derivatives, and $'$ the derivative of functions of one variable.

Any family of orbits is determined by the 'slope function' $\gamma = f_y/f_x$. There are two equations relating the functions V, γ (and their derivatives), which have appeared in relation with the inverse problem of dynamics in inertial systems, that is, find all potentials which can give rise to a given family of orbits:

(i) the first-order equation in V given by Szebehely (1974), which is associated with the energy dependence on the family f;
(ii) the free of energy second-order linear equation in V (Bozis, 1984).

Celestial Mechanics and Dynamical Astronomy **88:** 245–257, 2004.

These equations are usually used in the framework of the inverse problem. However they can be rearranged in order to face the direct problem (Bozis, 1995). In particular, the first-order partial differential equation in the unknown function $\gamma = \gamma(x, y)$ (Eq. (8)) can be used for direct problem considerations only for unknown families of orbits which are considered *a priori* as isoenergetic, for example with all their members traced with energy $E = 0$. The reason is that, when $f(x, y)$ is unknown, so is the energy dependence function $E = E(f)$ in (8). In the absence of any information on the energy dependence, the second-order equation is *sine qua non* for the direct problem. Of course, due to its nonlinearity in γ, it is difficult to be solved. For this reason, in several papers additional information on the families of orbits was used in order to obtain solutions of the direct problem. Homogeneous families produced by inhomogeneous potentials were studied by Bozis et al. (1997), as well as families of orbits with $\gamma = \gamma(x)$, corresponding to families $f(x, y) = y + h(x) = c$ (Bozis et al., 2000); in these two cases γ was found as the common root of some algebraic equations in γ, with coefficients depending on V and on its derivatives.

The additional condition satisfied by γ may be put in the terms of a first-order differential equation. Indeed, if f is homogeneous of degree m, then $f(x, y) = x^m F(9y/x)$ and

$$\gamma = \frac{F'(y/x)}{(mF(y/x) - F'(y/x) \cdot (y/x))}$$

is homogeneous of degree 0. This happens if and only if

$$x\gamma_x + y\gamma_y = 0.$$

For the family $f(x, y) = y + h(x)$ the corresponding γ is given by $\gamma = 1/h'(x)$ and satisfies the equation

$$\gamma_y = 0.$$

More generally, we may look for γ in a family of functions, that is, $\gamma = G(g(x, y))$ with G arbitrary and g given in advance. Then $\gamma_x = G'(g)g_x$, $\gamma_y = G'(g)g_y$ and we obtain the partial differential equation

$$g_y\gamma_x - g_x\gamma_y = 0.$$

This is a special case of a more complicated dependence (Courant and Hilbert, 1962, Chapter I) which provides a quasilinear equation in γ.

In what follows we consider a given potential V and study the existence and the construction of solutions γ of the direct problem of dynamics, under the hypothesis that γ satisfies an equation of the form

$$a(x, y)\gamma_x + b(x, y)\gamma_y = 0. \tag{1}$$

We may suppose $b \neq 0$ and denote by $r = a/b$ (the case $b = 0$, i.e. of functions γ depending only on the variable y, is similar to that studied by Bozis et al. (2000)). The above equation has the simpler form

$$r(x, y)\gamma_x + \gamma_y = 0, \tag{2}$$

which we shall use throughout the paper.

Its solutions generalize the families of trajectories described by a homogeneous function f, or by $f(x, y) = y + h(x)$. Geometrically, the trajectories from the two mentioned families can be obtained one from another using an element of a group of simple transformations: geometrical similarity of center O for the first one, translation parallel to the Oy axis for the second one. The relation between Equation (2) and the geometry of the family (3), as well as that between the symmetries of the given potential and the possible symmetry of the family deserves a further study.

In Section 2, we give the basic differential equations mentioned in Section 1 and, using the additional differential relation (2), we indicate how the mathematical handling of the problem is made possible. In Section 3, we obtain two algebraic equations which the required family must satisfy. The resultant of these equations must vanish and this leads to a differential condition which all adequate potentials must satisfy. In Section 4, certain examples are offered and in Section 5 the method is recapitulated. Section 6 is devoted to certain comments regarding the question of possible integrability of the potentials appearing in the inverse or direct problem of dynamics.

2. Partial Differential Equations Satisfied by γ

Let us consider a potential V under the action of which a monoparametric family of orbits

$$f(x, y) = c \tag{3}$$

is described by a material point of unit mass. This family can be represented in a unique way by its 'slope function'

$$\gamma = \frac{f_y}{f_x}. \tag{4}$$

To each γ there corresponds a unique family (3).

The nonlinear second-order differential equation relating potentials and orbits in the form suitable for the direct problem (Bozis, 1995) is

$$\gamma^2\gamma_{xx} - 2\gamma\gamma_{xy} + \gamma_{yy} = h, \tag{5}$$

where

$$h = \frac{\gamma\gamma_x - \gamma_y}{V_y\gamma + V_x}(-\gamma_x V_x + (2\gamma\gamma_x - 3\gamma_y)V_y + \gamma(V_{xx} - V_{yy}) + (\gamma^2 - 1)V_{xy}). \tag{6}$$

On the other hand, Szebehely's equation (1974), as written by Bozis (1983), reads

$$V_x + \gamma V_y + \frac{2(\gamma\gamma_x - \gamma_y)}{1 + \gamma^2}(E(c) - V) = 0, \tag{7}$$

where $E(c)$ is the total energy on each orbit (3) parametrized by the constant c. In order to solve (7) for $E(c)$, the condition $\Gamma = \gamma\gamma_x - \gamma_y \neq 0$ must be imposed, hence it follows also that $V_x + \gamma V_y \neq 0$. The case $\Gamma = 0$ was studied in detail by Bozis and Anisiu (2001) and will not be considered here. If for a given V we can find a solution γ of (5), Equation (7) will allow us to find the energy on the members of the family, namely

$$E = V - \frac{(V_x + \gamma V_y)(1 + \gamma^2)}{2\Gamma}. \tag{8}$$

The curves of the family will be lying in the region defined by the inequality (Bozis and Ichtiaroglou, 1994)

$$\frac{V_x + \gamma V_y}{\Gamma} \leqslant 0. \tag{9}$$

As we have mentioned in Section 1, the special families of orbits we are going to consider are those for which Equation (2) is also satisfied. We differentiate it with respect to x and obtain

$$r\gamma_{xx} + \gamma_{xy} = -r_x\gamma_x. \tag{10}$$

Then we differentiate (2) with respect to y

$$r\gamma_{xy} + \gamma_{yy} = -r_y\gamma_x. \tag{11}$$

The system of Equations (5), (10) and (11) will allow us to obtain the second-order derivatives of γ in terms of γ and its first-order derivatives.

Comment: The function r being given, Equation (2) is equivalent to $\gamma = G(g(x, y))$, where G is an arbitrary function and $g(x, y) = c$ is a solution (not always possible to be found) of the ordinary differential equation $\mathrm{d}y/\mathrm{d}x = 1/r$. The curves $g(x, y) = c$ in the xy plane are isoclinic curves of the various families of orbits (3) which can be traced when (2) is given. If one draws a curve

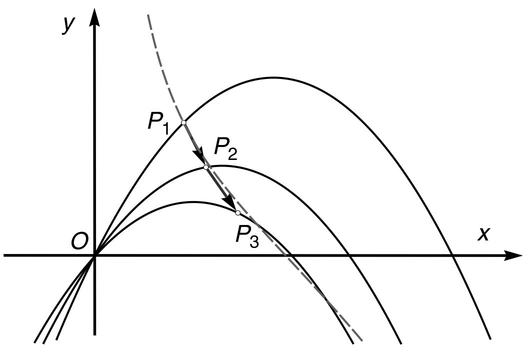

Figure 1. The curve $x^3 + xy = 5/108$ is approximately traced using the isoclinic curves $y = -3x^2 + x/c$, $c = 0.5, 0.7, 0.9$.

$g(x, y) = c_1$, the inclination of the orbit passing through a point P_1 situated on that curve will be $y' = -1/\gamma = -1/G(c_1)$. A small straight line arrow $P_1 P_2$ can be drawn, P_2 being situated on a neighboring isoclinic $c = c_2$; from P_2 we can draw another small segment $P_2 P_3$ with slope $-1/G(c_2)$ and so on. In this way we can construct approximately the entire family corresponding to *one* $\gamma = G(g(x, y))$. For another function G there will be another family (3) which will be constructed on the grounds of the same isoclinic net $g(x, y) = c$.

In Figure 1 an approximation of the dotted curve $x^3 + xy = 5/108$ from the family obtained in Example 1 (Section 4) is given. The procedure described above is applied to the isoclinic curves obtained from $\gamma = g(x, y) = x/(3x^2 + y)$.

3. Algebraic Equations Satisfied by γ

We shall solve the system of Equations (5), (10) and (11) with respect to γ_{xx}, γ_{xy} and γ_{yy}. These second-order derivatives will depend on γ, γ_x, γ_y, on r, r_x, r_y, and, of course, on the first- and second-order derivatives of V. In fact, considering (2), we can express γ_y in terms of γ_x. Denoting by

$$\Pi = (\gamma + r)^2 (V_y \gamma + V_x),$$

$$\begin{aligned} K &= (\gamma + r)(2V_y \gamma + 3r V_y - V_x) \\ &= 2V_y \gamma^2 + (5r V_y - V_x)\gamma + r(3r V_y - V_x), \end{aligned}$$

$$\begin{aligned} L &= V_{xy}\gamma^3 + (V_{xx} - V_{yy} + r V_{xy} - 2r_x V_y)\gamma^2 + \\ &\quad + (r(V_{xx} - V_{yy}) - V_{xy} - 2r_x V_x - (r r_x - r_y) V_y)\gamma - \\ &\quad - (r V_{xy} + (r r_x - r_y) V_x), \end{aligned} \tag{12}$$

the second-order derivatives of γ can be expressed as

$$\gamma_{xx} = \frac{1}{\Pi}(K\gamma_x^2 + L\gamma_x),$$

$$\gamma_{xy} = -\frac{1}{\Pi}\{rK\gamma_x^2 + (\Pi r_x + rL)\gamma_x\},$$

$$\gamma_{yy} = \frac{1}{\Pi}\{r^2 K\gamma_x^2 + (\Pi(r r_x - r_y) + r^2 L)\gamma_x\}. \tag{13}$$

Remark 1. As we have already mentioned, we have $V_y \gamma + V_x \neq 0$. We have also $\gamma + r \neq 0$, because if we suppose that $\gamma = -r$, Equation (2) will imply $\Gamma = 0$, a situation excluded from the present study. Hence all the denominators in (13) are different from zero.

Working with (13) we find that the two compatibility conditions $(\gamma_{xx})_y = (\gamma_{xy})_x$ and $(\gamma_{xy})_y = (\gamma_{yy})_x$ produce one single relation which, after substituting γ_{xx}, γ_{xy} and γ_{yy} given by (13) and γ_y from (2), reduces itself to a first degree algebraic equation in γ_x. This equation has the form

$$(\gamma + r)B\gamma_x = A \tag{14}$$

with A, B polynomials in γ of at most fifth, respectively second degree. The coefficients of $A = A_5\gamma^5 + A_4\gamma^4 + A_3\gamma^3 + A_2\gamma^2 + A_1\gamma + A_0$ are displayed in Appendix A. The polynomial B may be written as

$$B = 3V_x^2 \left(r \left(\frac{V_y}{V_x} \right)_x + \left(\frac{V_y}{V_x} \right)_y \right) (\gamma + r)^2 + (rr_x + r_y)(V_y\gamma + V_x)^2. \tag{15}$$

As indicated in Remark 1, we have $\gamma + r \neq 0$. In what follows we shall consider only potentials V and functions r for which $B \neq 0$.

Comment: Important classes of potentials V for which B is identically null are, in the case when $r = x/y$, those of homogeneous potentials, or of the form $V(x - c_0y)$, c_0 being a constant.

We express γ_x from (14) as

$$\gamma_x = \frac{A}{(\gamma + r)B}, \tag{16}$$

and γ_y in view of (2) as

$$\gamma_y = -\frac{rA}{(\gamma + r)B}. \tag{17}$$

The case of A with identically null coefficients will be considered in the next section.

If A has some coefficients different from zero, we write the compatibility condition $(\gamma_x)_y = (\gamma_y)_x$, in which we replace γ_x by (16) and γ_y by (17); we obtain a first polynomial equation of seventh degree in γ, whose coefficients H_i, $i = 0, 1, \ldots, 7$ contain the derivatives of V up to the fourth order

$$H_7\gamma^7 + H_6\gamma^6 + H_5\gamma^5 + H_4\gamma^4 + H_3\gamma^3 + H_2\gamma^2 + H_1\gamma + H_0 = 0. \tag{18}$$

From (16) and (17) we can express, after differentiation, γ_{xx}, γ_{xy}, γ_{yy} in terms of γ and derivatives of V up to the fourth order. We insert these values in the basic Equation (5), and then the values of γ_x and γ_y from (16) and (17). We are left with a second algebraic equation in γ, this time of 12th degree,

$$R_{12}\gamma^{12} + R_{11}\gamma^{11} + R_{10}\gamma^{10} + R_9\gamma^9 + R_8\gamma^8 + R_7\gamma^7 + \\ + R_6\gamma^6 + R_5\gamma^5 + R_4\gamma^4 + R_3\gamma^3 + R_2\gamma^2 + R_1\gamma + R_0 = 0. \tag{19}$$

The coefficients R_j, $j = 0, 1, ..., 12$ contain the derivatives of V up to the fourth order.

The coefficients in (18) and (19) are too long to be written here, but they can be calculated using symbolic algebra programs (as MATHEMATICA).

The case of potentials V for which both Equations (18) and (19), or at least one of them, have null coefficients will be examined in the next section.

We suppose now that some of the coefficients H_i, $i = 0, 1, \dots, 7$ and some of R_j, $j = 0, 1, \dots, 12$ are different from zero. For a common solution of Equations (18) and (19) to exist, a necessary and sufficient condition is that their resultant, which is equal to their Sylvester determinant of order 19 (Mishina and Proskuryakov, 1965, p. 164) vanishes. This will give the necessary condition

$$\begin{vmatrix} H_7 & H_6 & \dots & H_0 & 0 & 0 & & & \dots & 0 \\ 0 & H_7 & \dots & H_1 & H_0 & 0 & & & \dots & 0 \\ . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & \dots & & & H_7 & H_6 & & \dots & H_0 \\ R_{12} & R_{11} & \dots & & & & R_0 & & \dots & 0 \\ 0 & R_{12} & \dots & & & & & R_0 & \dots & 0 \\ . & . & . & . & . & . & . & . & . & . \\ 0 & \dots & R_{12} & R_{11} & & & & & \dots & R_0 \end{vmatrix} = 0 \tag{20}$$

to be satisfied by the potential V. Example 1 in Section 4 gives a potential V and a function r for which it is possible to find a common solution γ of (18) and (19), which is a solution for the direct problem of dynamics.

4. Special Cases and Examples

The compatibility conditions written for the second-order derivatives of γ as written in (13) gave the first degree algebraic equation in γ_x (14), where we made the assumption that the quadratic polynomial B has nonvanishing coefficients. We shall analyze the special cases mentioned in Section 3.

CASE 1. If the fifth degree polynomial A in (14) has all the coefficients equal to zero, from (16) and (17) we obtain the trivial case $\gamma = \text{const.}$, excluded from the present study because of the condition $\Gamma \neq 0$.

If A has nonvanishing coefficients, we can always obtain the polynomial equations (18) and (19). We proceed then as in Section 3, unless one of the next two cases occurs:

CASE 2. If only one of (18) and (19) has all the coefficients equal to zero, we try to solve the other polynomial equation and then check if the solution satisfies indeed (5). This situation happens for the Hénon–Heiles potential in Example 2 below.

CASE 3. If both (18) and (19) have null coefficients (case illustrated by Example 3), we have no supplementary constraints on γ, but we may use (16). We insert in this equation $\gamma = G(g(x, y))$ and obtain an ordinary first-order differential equation in $G(g)$. To obtain the function G it is necessary to perform a quadrature. This is illustrated by Example 3 but a simple form of the solution cannot be expected in general.

The next examples illustrate the method described in Section 3 and some of the special cases mentioned above.

EXAMPLE 1. For $V(x, y) = -((3x^2 + y)^2 + x^2)\exp(-12y)$, and the auxiliary equation (2) with $r(x, y) = x/(y - 3x^2)$, the two Equations (18) and (19) are of degree 7, respectively 12, and have a unique common solution

$$\gamma = \frac{x}{3x^2 + y},$$

which comes from the family $f = x^3 + xy$. The curves of this family are traced with energy $E = 0$ in the entire plane.

EXAMPLE 2. For the Hénon–Heiles potential $V(x, y) = (1/2)x^2 + 8y^2 + x^2y + (16/3)y^3$, considering (2) with $r(x, y) = x/y$, all the coefficients $H_7, \ldots, H_0$ of Equation (18) are null. In this case no information for γ arises from (18), but Equation (19) has the solution $\gamma = -x/4y$ corresponding to the family $yx^{-4} = c$. This family was found also by Bozis et al. (1997) as an example of a homogeneous family traced under the action of an inhomogeneous potential. The energy on the family is given by $E = -1/24c$ and the allowed region is $(x^2 + 8y^2 + 12y)y \leqslant 0$, in accordance with Anisiu and Pal (1999).

EXAMPLE 3. Let us consider $V(x, y) = -(x^2 + y^2)^2 - 2x^2 + 2y^2$ and the coefficient in (2) $r(x, y) = x(x^2 + y^2 + 1)/y(x^2 + y^2 - 1)$. In this case both Equations (18) and (19) are identically null. Integrating (2) gives $\gamma = G((x^2 - y^2 + 1)/xy)$, hence the function g mentioned in Section 1 is $g(x, y) = (x^2 - y^2 + 1)/xy$. We insert this in Equation (16), which after simplification becomes

$$\gamma_x = \frac{2y(x^2 - y^2 + 1)(\gamma^2 + 1)}{(x^2 + y^2 - 2y + 1)(x^2 + y^2 + 2y + 1)},$$

and get the differential Equation

$$\frac{G'(g)}{G^2(g) + 1} = \frac{2}{g^2 + 4}. \tag{21}$$

We obtain the solution of (21), $G(g) = (g + 2k)/(-kg + 2)$, where k is a constant. It follows that

$$\gamma = \frac{x^2 - y^2 + 1 + 2kxy}{2xy - k(x^2 - y^2 + 1)},$$

so we have in fact a family of functions γ depending on the constant k. For each fixed value of k, the family of functions is given by $f = k((1/3)x^3 - xy^2 + x) + (1/3)y^3 - x^2y - y$. The energy is constant $E = 1$ on each family f and

$$\frac{V_x + \gamma V_y}{\Gamma} = -\frac{2(2xy - k(x^2 + y^2 - 1))^2}{k^2 + 1},$$

hence the curves may be traced in the entire plane.

Remark 2. As we have mentioned in the Comment at the end of Section 2, it may happen that a solution $g(x, y) = c$ of the equation $\mathrm{d}y/\mathrm{d}x = 1/r$ can be found, hence γ from Equation (2) will be of the form $\gamma = G(g(x, y))$, with G an arbitrary function. In this situation, a first thought might be to substitute $\gamma = G(g(x, y))$ in the basic Equation (5), in order to obtain G. The result will be an equation in

$$z = \frac{1}{G'} \tag{22}$$

of the following form

$$z' = Pz + Q, \tag{23}$$

where

$$\begin{aligned} P \;=\; & \frac{1}{(g_x G - g_y)^2 (V_y G + V_x)}((V_y g_{xx} - V_{xy} g_x)G^3 + \\ & + (V_x g_{xx} - 2V_y g_{xy} - (V_{xx} - V_{yy})g_x + V_{xy} g_y)G^2 + \\ & + (-2V_x g_{xy} + V_y g_{yy} + V_{xy} g_x + (V_{xx} - V_{yy})g_y)G + V_x g_{yy} - V_{xy} g_y), \\ Q \;=\; & \frac{-2V_y g_x G + V_x g_x + 3V_y g_y}{(g_x G - g_y)(V_y G + V_x)}. \end{aligned} \tag{24}$$

Generally, it would not be an easy task to obtain z. From (22) we know that z' must depend merely on g, so the condition

$$\frac{(Pz + Q)_y}{(Pz + Q)_x} = \frac{g_y}{g_x}$$

has to be satisfied. This equation will lead us to a polynomial equation in z, where z depends again merely on g, and the condition $z_y/z_x = g_y/g_x$ must also be imposed.

So, if Equation (2) has an easy to obtain solution $\gamma = G(g(x, y))$, it is worth trying to write Equation (22) to see if it happens to have a simple form. Otherwise it is advisable to follow the general procedure which is synthesized in the next section.

5. Synthesis and Remarks

The auxiliary equation (2) (i.e. the coefficient r, respectively the coefficients a and b in (1)) being given in advance, as well as the potential V, one cannot expect always that a family of functions compatible with the given potential will exist. In general, this happens only when V satisfies a differential condition obtained by equating to zero the Sylvester determinant (20) of the polynomials (18) and (19).

In order to obtain a solution γ of Equations (5) and (2), we proceed as follows (the use of a symbolic algebra program is advisable).

We calculate the polynomials A (the coefficients are in Appendix A) and B from (15) and verify that B is not identically null.

If the polynomial A has all its coefficients equal to zero, we have no acceptable solutions for our problem (Case 1).

Let us suppose now that A has coefficients different from zero. We write then γ_x and γ_y as in (16), respectively (17), and from the compatibility condition $(\gamma_x)_y = (\gamma_y)_x$ obtain the seventh degree equation in γ (18). We insert then the second-order derivatives of γ in the basic Equation (5), and then γ_x and γ_y as in (16) and (17). The result will be the 12th degree equation in γ (19). If both the polynomials in (18) and (19) have coefficients different from zero, we calculate their Sylvester determinant: if this is different from zero, there is no γ compatible with V (and r); if it is zero, we find the common roots of the two polynomials and check if they satisfy the basic Equation (5). Thus we can obtain at most seven solutions γ. If only one of (18) and (19) has all the coefficients equal to zero (Case 2 of Section 4) we try to solve the other equation and obtain a number of solutions γ at most equal to the degree of that polynomial equation. If both (18) and (19) have null coefficients, we use Equation (16) as indicated in Case 3, obtaining at most a one parameter family of functions γ.

Remark 3. In Example 3 it can be checked that the families corresponding to $k = m$, respectively to $k = -1/m$ are mutually orthogonal. We mention that the additional linear equation (2) does not prevent us from finding orthogonal families as solutions, because γ and $-1/\gamma$ satisfy (or not) simultaneously the Equation (2). It follows that if Equation (5) admits orthogonal families as solutions, they can be found by the procedure mentioned in Sections 2 and 3.

Remark 4. The pairs of orthogonal families in Example 3 have another special property, namely they are isothermal. As stated in the paper of Puel (1999), two families of orthogonal curves $u(x, y)$ and $v(x, y)$ are *isothermal* if

$$\frac{\|\mathrm{grad}\, v\|^2}{\|\mathrm{grad}\, u\|^2} = \frac{\alpha(u)}{\beta(v)}.$$

Considering the families given by $u(x, y) = m((1/3)x^3 - xy^2 + x) + (1/3)y^3 - x^2y - y$ and $v(x, y) = -(1/m)((1/3)x^3 - xy^2 + x) + (1/3)y^3 - x^2y - y$, we have

$$\begin{aligned} \|\mathrm{grad}\, v\|^2 &= \left(\frac{1}{m^2} + 1\right)((x^2 - y^2 + 1)^2 + 4x^2y^2), \\ \|\mathrm{grad}\, u\|^2 &= (m^2 + 1)((x^2 - y^2 + 1)^2 + 4x^2y^2). \end{aligned}$$

It follows that $\|\mathrm{grad}\, v\|^2/\|\mathrm{grad}\, u\|^2 = 1/m^2$, hence the potential V produces indeed isothermal families of curves.

6. Regarding Integrability

There were several attempts to connect the inverse and direct problems treated on the basis of Szebehely's and Bozis' equations with the important question of integrability. Ichtiaroglou and Meletlidou (1990) identified some families of conic sections or of straight lines whose presence guarantee the integrability of the potentials producing them. On the other hand, Bozis and Meletlidou (1998) presented a method to detect nonintegrability of the potential when it is compatible with a family of geometrically similar orbits.

A potential which produces a 'nice' family of orbits is not necessarily integrable. For example, the so-called homogeneous Hénon–Heiles potential $V_1 = x^2 y + (16/3)y^3$ is compatible with the family of orbits $f_1(x, y) = x^{-4}y$ (with $\gamma_1 = -x/4y$) (Anisiu and Pal, 1999), and it is known to be integrable. It admits, indeed, a second integral of the form $F = \dot{x}^4 + 4x^2 y\dot{x}^2 - (4/3)x^3\dot{x}\dot{y} - 4x^4 y^2 - (2/9)x^6$ (see, e.g. Morales Ruiz, 1999, p. 104). A different potential of the same type, $V_2 = x^2 y + (4/9)y^3$, was shown by Bozis and Meletlidou (1998) to be compatible with the family of hyperbolas $f_2 = 3x^2 - 2y^2$ (with $\gamma_2 = -(2y/3x)$); its nonintegrability was proved on the basis of the inverse problem as well as by applying Yoshida's criterion (Yoshida, 1987).

In the present study we did not assume that the given potential is integrable nor we claimed that the aforementioned in Section 3 differential condition (20) for the potential is a tank of integrable potentials. Due to the simple form of the potentials which illustrate some of the situations which arise during the analysis of the problem in Section 3, we can try to check if they are integrable or not.

The cubic potential in Example 2 has the form $V = \varphi(y) - (1/2)\alpha(y)x^2$ with $\varphi(y) = 8y^2 + (16/3)y^3$ and $\alpha(y) = -2y - 1$. It satisfies the necessary condition of integrability given in Theorem 6.2, Case 2.1 (Morales Ruiz, 1999, p. 123), so we cannot draw any conclusion on its integrability. The same situation appears for the quartic potential in Example 3, which can be written as $V = \varphi(x) - (1/2)\alpha(x)y^2 + O(y^3)$, with $\varphi(x) = -x^4 - 2x^2$ and $\alpha(x) = 4x^2 - 4$. It satisfies also the mentioned necessary condition of integrability, entering this time in Case 1.

The inverse and direct problems of dynamics are based, at a first view, only on geometry in the configuration space; in fact, having a potential and a compatible family of curves, the energy level can be found from Equation (8), and afterwards further information on the geometry in the phase space. A deeper study of the relation between inverse problem and integrability may be in the benefit of both domains.

Appendix A

In what follows we shall use the notation $V_{ij} = \partial^{i+j} V / \partial x^i \partial y^j$. The coefficients of the fifth degree polynomial A in (16) are:

$$A_5 = (V_{11}^2 - V_{21}\,V_{01})r - V_{01}^2\,r_{20} - V_{11}\,V_{01}\,r_{10} - V_{12}\,V_{01} + V_{11}\,V_{02},$$

$$\begin{aligned} A_4 \;=\; & 2(V_{11}^2 - V_{21}\,V_{01})r^2 + (-V_{01}^2\,r_{20} - V_{11}\,V_{01}\,r_{10} - V_{30}\,V_{01} - V_{21}\,V_{10} - \\ & - V_{12}\,V_{01} + 2\,V_{20}\,V_{11} + V_{11}\,V_{02})r - 2\,V_{10}\,V_{01}\,r_{20} + 2\,V_{01}^2\,r_{11} + \\ & + 2\,V_{01}^2\,r_{10}^2 + (-V_{20}\,V_{01} - V_{11}\,V_{10} + V_{02}\,V_{01})r_{10} + V_{11}\,V_{01}\,r_{01} - \\ & - V_{21}\,V_{01} - V_{12}\,V_{10} + V_{03}\,V_{01} + V_{20}\,V_{02} + V_{11}^2 - V_{02}^2, \end{aligned}$$

$$\begin{aligned} A_3 \;=\; & (V_{11}^2 - V_{21}\,V_{01})r^3 + (-2\,V_{30}\,V_{01} - 2\,V_{21}\,V_{10} + V_{12}\,V_{01} + 4\,V_{20}\,V_{11} - \\ & - V_{11}\,V_{02})r^2 + (2\,V_{01}^2\,r_{11} - 2\,V_{10}\,V_{01}\,r_{20} + (-V_{20}\,V_{01} - V_{11}\,V_{10} + \\ & + V_{02}\,V_{01})r_{10} + V_{11}\,V_{01}\,r_{01} - V_{30}\,V_{10} - V_{21}\,V_{01} - V_{12}\,V_{10} + \\ & + 2\,V_{03}\,V_{01} + V_{20}\,V_{02} - 2\,V_{02}^2 + V_{20}^2 + V_{11}^2)r + 4\,V_{10}\,V_{01}\,r_{10}^2 - \\ & - 4\,V_{01}^2\,r_{10}\,r_{01} - V_{10}^2 r_{20} - V_{01}^2\,r_{02} + (-V_{20}\,V_{10} + V_{11}\,V_{01} + \\ & + V_{02}\,V_{10})\,r_{10} + 4\,V_{10}\,V_{01}\,r_{11} + (V_{20}\,V_{01} + V_{11}\,V_{10} - V_{02}\,V_{01})r_{01} - \\ & - V_{21}\,V_{10} + V_{12}\,V_{01} + V_{03}\,V_{10} + V_{20}\,V_{11} - 2\,V_{11}\,V_{02}, \end{aligned}$$

$$\begin{aligned} A_2 \;=\; & (-V_{30}\,V_{01} - V_{21}\,V_{10} + V_{12}\,V_{01} + 2\,V_{20}\,V_{11} - V_{11}\,V_{02})r^3 + \\ & + (2\,V_{20}^2 - V_{02}^2 - 2\,V_{30}\,V_{10} + V_{21}\,V_{01} + V_{12}\,V_{10} + V_{03} V_{01} - \\ & - V_{20}\,V_{02} - V_{11}^2)r^2 + (-V_{10}^2\,r_{20} - V_{01}^2\,r_{02} + 4\,V_{10}\,V_{01}\,r_{11} + \\ & + (-V_{20}\,V_{10} + V_{11}\,V_{01} + V_{02}\,V_{10})r_{10} + (V_{20}\,V_{01} + V_{11} V_{10} - \\ & - V_{02} V_{01})r_{01} + 2\,V_{03}\,V_{10} - V_{21}\,V_{10} + 2\,V_{12}\,V_{01} - 4\,V_{11}\,V_{02} + \\ & + V_{11}\,V_{20})r - 2\,V_{10}\,V_{01}\,r_{02} + 2\,V_{10}^2\,r_{11} + 2\,V_{01}^2\,r_{01}^2 - 8\,V_{10}\,V_{01}\,r_{10}\,r_{01} + \\ & + 2\,V_{10}^2 r_{10}^2 + V_{11}\,V_{10} r_{10} + (V_{20}\,V_{10} - V_{11}\,V_{01} - V_{02}\,V_{10})r_{01} + \\ & + V_{12}\,V_{10} - V_{11}^2, \end{aligned}$$

$$\begin{aligned} A_1 \;=\; & (-V_{30}\,V_{10} + V_{12}\,V_{10} + V_{21}\,V_{01} + V_{20}^2 - V_{11}^2 - V_{20}\,V_{02})r^3 + \\ & + (V_{03}\,V_{10} + V_{21}\,V_{10} + V_{12}\,V_{01} - V_{20}\,V_{11} - 2\,V_{11}\,V_{02})r^2 + \\ & + (-2\,V_{10}\,V_{01}\,r_{02} + 2V_{10}^2\,r_{11} + V_{11}\,V_{10}\,r_{10} + (V_{20}\,V_{10} - V_{11}\,V_{01} + \\ & + V_{02}\,V_{10})r_{01} + 2\,V_{12}\,V_{10} - 2\,V_{11}^2)r + 4\,V_{10}\,V_{01}\,r_{01}^2 - 4\,V_{10}^2\,r_{10}\,r_{01} - \\ & - V_{11}\,V_{10}\,r_{01} - V_{10}^2\,r_{02}, \end{aligned}$$

$$\begin{aligned} A_0 \;=\; & (V_{21}\,V_{10} - V_{20}\,V_{11})r^3 + (V_{12}\,V_{10} - V_{11}^2)r^2 - (V_{10}^2\,r_{02} + \\ & + V_{11}\,V_{10}\,r_{01})r + 2\,V_{10}^2\,r_{01}^2. \end{aligned}$$

Acknowledgement

The research of the first two authors has been partially supported by the Ministry of Education and Research (grant 343/2002-CNCSIS).

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